Math. Ann. 195, 321—329 (1972)

Relations among Invariants of Circle Actions on Three-manifolds

DON ZAGIER*

§ 1. Description of Problem and Results

In connection with a problem in the topology of three-dimensional manifolds, Erich Ossa ([3], p. 44) conjectured that the rational functions

$$f_{mn}(t) = \frac{t^m+1}{t^m-1}\,\frac{t^n+1}{t^n-1} \qquad (m \geqq 2n > 0) \tag{1}$$

are linearly independent over the integers. It is the object of this note to show that there are in fact infinitely many linear relations among these functions.

Before giving the precise statement of our result, we shall sketch the topological background. This is only meant as motivation, however; the actual theorem and its proof will be independent of topological ideas.

Recall, then, that one consequence of the G-signature theorem of Atiyah and Singer is the possibility of defining an interesting and important invariant, the "α-invariant," associated to free actions of a compact Lie group G on manifolds which equivariantly bound. For actions of the circle group, the α-invariant is a rational function $\alpha(t)$ in one variable. It was originally defined only if the manifold itself or some number of disjoint copies of it could be expressed as the equivariant boundary of a manifold with an S^1-action, but it was shown by Ossa in [3] that this is always the case. The definition of the α-invariant is such that if M is the boundary of a *free* S^1-manifold X, then $\alpha(t)$ is a constant, equal to the negative of the signature of X. Therefore the rational function $\alpha(t)-\alpha(0)$ is a free equivariant cobordism invariant, and the function $\alpha(t)$ itself is the same for any two manifolds which are equivariantly cobordant by a fixed-point free cobordism of signature zero.

The conjecture of Ossa in its topological form is the converse of this statement for 3-dimensional manifolds, namely that two 3-manifolds with circle group actions having the same α-invariant are necessarily cobordant in this strong sense. An analysis of the equivariant fixpoint-free cobordism group in dimension three ([3]) showed that it is a free

* This work was supported partly by a National Science Foundation grant, and partly by the Sonderforschungsbereich (theoretische Mathematik) of the University of Bonn.

abelian group on additive generators M_{mn} ($m \geqq 2n > 0$), where M_{mn} is a certain 3-manifold admitting an S^1-action with α-invariant precisely $f_{mn}(t)$. Therefore the topological conjecture is in fact equivalent to the linear independence of the functions of Eq. (1).

In § 2 we will obtain several lower bounds for the number of relations among these functions; here we state the sharpest of these as a theorem:

Theorem. *There are infinitely many relations among the functions* (1). *More precisely, if $r(N)$ denotes the number of (linearly independent) relations among the functions $f_{mn}(t)$ $\left(0 < n \leqq \frac{m}{2} \leqq N\right)$, then*

$$r(N) \geqq N^2\left(1 - \frac{6}{\pi^2}\right) + 0(N^{4/3}).$$

Moreover, it is even true that there are infinitely many relations among the functions f_{mn} ($m \geqq \lambda n > 0$), with λ any real number smaller than $\frac{\pi^2}{3}$.

§ 2. Proof of Results

We make two preliminary observations. First, instead of looking for relations over $\mathbb{Z}$, we can look for relations over $\mathbb{C}$. For if

$$\sum_{m,n} a_{mn} f_{mn}(t) \equiv 0 \tag{2}$$

with $a_{mn} \in \mathbb{C}$ (and all but finitely many of the a_{mn} equal to zero), then replacing t by $\bar{t}$ and taking the complex conjugate of (2) we get

$$\sum_{m,n} \bar{a}_{mn} f_{mn}(t) \equiv 0$$

also; adding and subtracting these equations yields

$$\Sigma\,(\operatorname{Re} a_{mn})\, f_{mn}(t) \equiv 0\,,$$
$$\Sigma\,(\operatorname{Im} a_{mn})\, f_{mn}(t) \equiv 0\,.$$

That is, any complex linear relation is a combination of relations over $\mathbb{R}$. Similarly any relation over $\mathbb{R}$ can be expressed as a combination of relations over $\mathbb{Z}$, since if the a_{mn} in (2) are real, we can let $\omega_1, \ldots, \omega_r$ be a basis for the vector space over $\mathbb{Q}$ spanned by the a_{mn} (this vector space is finite dimensional since almost all a_{mn}'s vanish). Then we can write a_{mn} as $\sum_{s=1}^{r} b^s_{mn}\omega_s$ with $b^s_{mn} \in \mathbb{Q}$ to obtain

$$\sum_{s=1}^{r} \omega_s \left[\sum_{m,n} b^s_{mn} f_{mn}(t)\right] \equiv 0\,.$$

For any rational t, the expression in brackets is rational and therefore must vanish since the ω_i's are linearly independent over $\mathbb{Q}$. Therefore the expression in brackets must vanish identically, so that (2) is indeed a linear combination of finitely many relations over $\mathbb{Q}$ (or, multiplying by a common denominator, over $\mathbb{Z}$).

The second remark is that

$$f_{mn}(t) = 1 + 2A_{mn}(t), \tag{3}$$

where

$$A_{mn}(t) = \frac{t^m + t^n}{(t^m - 1)(t^n - 1)}. \tag{4}$$

Any linear relation among the functions f_{mn} yields a relation for the functions A_{mn}, as one sees by writing down Eq. (2) for $t = 0$ and subtracting it from the original equation. Conversely, a relation

$$\sum_{m,n} a_{mn} A_{mn}(t) \equiv 0$$

gives a relation involving the constant function 1 as well as the f_{mn}. Therefore if we find k independent linear relations among the A_{mn}, we are only sure of $k-1$ relations among the f_{mn} (for if some of the k relations involve the function 1, we use one of these to express 1 as a rational linear combination of the f_{mn} and substitute this expression in the others to get $k-1$ relations involving the f_{mn} only). The shift in attention from f_{mn} to A_{mn} (which is essentially $f_{mn}(t) - f_{mn}(0)$) corresponds, as explained in the introductory paragraphs, to studying the equivariant fixed-point free cobordism group rather than the group involving cobordisms with signature zero.

We have thus replaced the problem of counting the linear relations over $\mathbb{Z}$ among the functions f_{mn} by that of counting the relations over $\mathbb{C}$ among the functions A_{mn}. To attack this, let $N \geqq 1$ be a fixed integer, S_N the set of functions

$$S_N = \left\{ A_{mn} \middle| 1 \leqq n \leqq \frac{m}{2} \leqq N \right\}, \tag{5}$$

and V_N the (complex) vector space spanned by the functions of S_N. Since there are N^2 pairs of integers (m, n) with $1 \leqq n \leqq \frac{m}{2} \leqq N$, we have

$$\dim V_N = N^2 - r(N), \tag{6}$$

where $r(N)$ is the number of linearly independent linear relations among the elements of S_N. We wish to get lower bounds for $r(N)$ and in particular to show that it tends to infinity with N.

The idea of the proof is to give an alternate description of the functions of V_N. Any rational function is completely determined up to a constant by the position and nature of its poles. The constant here will produce no difficulties since we are working with the A_{mn}'s rather than with the f_{mn}'s. As to the poles, it is clear that A_{mn} has a simple pole at t if $t^m=1$, $t^n \neq 1$ or if $t^m \neq 1$, $t^n=1$, a double pole if $t^m=t^n=1$, and is regular elsewhere. Since $n \leqq N$ and $m \leqq 2N$, this means that $V_N \subset W_N$, where W_N is the vector space of all rational functions $f(t)$ with $f(\infty)=0$ which have (at most) a double pole at t if $t^k=1$ ($k \leqq N$), at most a simple pole at t if t is a primitive k^{th} root of 1 for $N<k \leqq 2N$, and are regular at t if t is not a k^{th} root of unity for any $k \leqq 2N$. But it is clear that a basis for W_N is $T_N \cup T_N'$, where

$$T_N=\left\{ \frac{1}{t-\zeta} \;\middle|\; \zeta^k=1 \quad \text{for some } k, \quad 1 \leqq k \leqq 2N \right\}, \tag{7a}$$

$$T_N'=\left\{ \frac{1}{(t-\zeta)^2} \;\middle|\; \zeta^k=1 \quad \text{for some } k, \quad 1 \leqq k \leqq N \right\}. \tag{7b}$$

Therefore

$$\begin{aligned} \dim W_N &= |T_N| + |T_N'| \\ &= \sum_{1 \leqq k \leqq 2N} (\text{number of primitive } k^{\text{th}} \text{ roots of } 1) \\ &+ \sum_{1 \leqq k \leqq N} (\text{number of primitive } k^{\text{th}} \text{ roots of } 1) \\ &= \sum_{k=1}^{2N} \varphi(k) + \sum_{k=1}^{N} \varphi(k), \end{aligned} \tag{8}$$

where $\varphi(k)$ is Euler's totient function, defined as the number of residue classes $(\operatorname{mod} k)$ which are prime to k. Using the standard notation

$$\Phi(K)=\sum_{k=1}^{K} \varphi(k)$$

for the sum-function of φ, we have the well-known result (see, for example, [2]) that

$$\Phi(K) \sim \frac{3K^2}{\pi^2} \qquad (K \to \infty), \tag{9}$$

which says that the probability that two "randomly chosen" large numbers are mutually prime is $\prod_p (1-p^{-2})=\zeta(2)^{-1}=\dfrac{6}{\pi^2}$.

Using (8) and (9), we deduce that, for N large,

$$\dim W_N \sim \frac{3}{\pi^2}[(2N)^2+N^2]=\frac{15}{\pi^2}N^2. \tag{10}$$

Unfortunately, this is larger than N^2, and since $\dim V_N \leqq N^2$, the fact that $V_N \subset W_N$ does not yield anything[1]. However, the functions $A_{mn}(t)$ have another property which we can make use of the improve the bound. Namely,

$$A_{mn}\left(\frac{1}{t}\right) = A_{mn}(t). \tag{11}$$

This can be seen directly from the definition of A_{mn}, but the corresponding equality in fact holds generally for α-invariants of circle actions. It follows that $V_N \subset W_N'$, where W_N' is the subspace of W_N consisting of functions f with $f(t^{-1}) = f(t)$.

It seems reasonable that for $f \in W_N'$ the coefficients of $\frac{1}{t-\zeta}$, $\frac{1}{(t-\zeta)^2}$, can be calculated from those of $\frac{1}{t-\zeta^{-1}}$ $\frac{1}{(t-\zeta^{-1})^2}$ in the expansion of f with respect to the basis $T_N \cup T_N'$, and therefore that the dimension of W_N' should be half that of W_N. To see that this is so, we write out the expansion of f with respect to the basis $T_N \cup T_N'$:

$$f(t) = \sum_{k=1}^{2N} \sum_{\zeta(k)} \frac{\alpha(\zeta)}{t-\zeta} + \sum_{k=1}^{N} \sum_{\zeta(k)} \frac{\beta(\zeta)}{(t-\zeta)^2}, \tag{12}$$

where the notation $\zeta(k)$ means that the sum is over all primitive k^{th} roots of unity. If $f \in W_N'$ then (replacing ζ by ζ^{-1}, which also runs over all k^{th} roots of unity)

$$\begin{aligned}
f(t) &= f(t^{-1}) \\
&= \sum_{k=1}^{2N} \sum_{\zeta(k)} \frac{\alpha(\zeta^{-1})}{t^{-1}-\zeta^{-1}} + \sum_{k=1}^{N} \sum_{\zeta(k)} \frac{\beta(\zeta^{-1})}{(t^{-1}-\zeta^{-1})^2} \\
&= \Sigma\Sigma\left[\frac{-t\zeta\alpha(\zeta^{-1})}{t-\zeta}\right] + \Sigma\Sigma\left[\frac{t^2\zeta^2\beta(\zeta^{-1})}{(t-\zeta)^2}\right] \\
&= \Sigma\Sigma\left[\frac{-\zeta^2\alpha(\zeta^{-1})}{t-\zeta} - \zeta\alpha(\zeta^{-1})\right] \\
&\quad + \Sigma\Sigma\left[\frac{\zeta^4\beta(\zeta^{-1})}{(t-\zeta)^2} + \frac{2\zeta^3\beta(\zeta^{-1})}{t-\zeta} + \zeta^2\beta(\zeta^{-1})\right].
\end{aligned}$$

---

[1] It may seem that the number $15/\pi^2$ crept into the proof by accident and that neither it nor the fact that it happens to be larger than one can have any place in a self-respecting proof. However,

$$\frac{15}{\pi^2} = \frac{\pi^2/6}{\pi^4/90} = \frac{\zeta(2)}{\zeta(4)} = \prod_p \left(\frac{1-p^{-4}}{1-p^{-2}}\right) = \prod_p (1+p^{-2}),$$

so that this number is a more natural one than it seems, and the fact that it is greater than one is by no means accidental.

Letting $t \to \infty$ (or setting $t=0$ in (12), where $f(0)=f(\infty)=0$), we find that the terms $-\zeta\alpha(\zeta^{-1})$ and $\zeta^2\beta(\zeta^{-1})$ sum to zero, so that this becomes

$$f(t) = \sum_{k \geqq 1} \sum_{\zeta(k)} \frac{1}{t-\zeta} [-\zeta^2\alpha(\zeta^{-1}) + 2\zeta^3\beta(\zeta^{-1})] + \sum_{k \geqq 1} \sum_{\zeta(k)} \frac{1}{(t-\zeta)^2} [\zeta^4\beta(\zeta^{-1})].$$

Since the expansion (12) is unique, we conclude that

$$\begin{aligned} \alpha(\zeta) &= 2\zeta^3\beta(\zeta^{-1}) - \zeta^2\alpha(\zeta^{-1}), \\ \beta(\zeta) &= \zeta^4\beta(\zeta^{-1}). \end{aligned} \tag{13}$$

For $\zeta=1$ this says that β is arbitrary but $\alpha=\beta$; for $\zeta=-1$ it says that β is arbitrary but $\alpha=-\beta$; for $\zeta \neq \zeta^{-1}$ it says that of the four numbers $\alpha(\zeta)$, $\beta(\zeta)$, $\alpha(\zeta^{-1})$, $\beta(\zeta^{-1})$, only two are linearly independent. We have therefore proved the claim that

$$\dim W'_N = \tfrac{1}{2} \dim W_N. \tag{14}$$

Comparing Eq. (14) with Eq. (10), we find that

$$\dim W'_N \sim \frac{15}{2\pi^2} N^2 \tag{15}$$

for $N \to \infty$. Since $\frac{15}{2\pi^2} < 1$ and $V_N \subset W'_N$, we find on comparison with (6) that we have proved the original conjecture and have obtained both an exact and an asymptotic lower bound for $r(N)$, namely:

$$r(N) \geqq N^2 - \tfrac{1}{2}[\Phi(N) + \Phi(2N)] \qquad (N>1), \tag{16}$$

$$\liminf_{N\to\infty} \frac{r(N)}{N^2} \geqq 1 - \frac{15}{2\pi^2}. \tag{17}$$

Without doing any more work, we can disprove a weaker conjecture (which, to be sure, has no topological significance). For $\lambda \geqq 1$, let the "λ-conjecture" be that the functions $f_{mn}(t)$, $m \geqq \lambda n$, are linearly independent. Ossa himself (in [3]) notes that the 1-conjecture (i.e. m and n unrestricted) is false, since

$$A_{m+n,m}(t) + A_{m+n,n}(t) = A_{m,n}(t).$$

Indeed, this example disproves the λ-conjecture for $\lambda < \frac{1+\sqrt{5}}{2}$ (since for such λ there are infinitely many pairs (m,n) with $m+n > \lambda m > \lambda^2 n$), but since $\frac{1+\sqrt{5}}{2} < 2$ this is not sufficient for the case of interest. Our method proves that the λ-conjecture is false for $\lambda < \frac{\pi^2}{6} + \sqrt{\left(\frac{\pi^2}{6}\right)^2 - 1}$ ≈ 2.951. Indeed, for large N there are approximately $\lambda N^2/2$ pairs (m, n) with $1 \leqq n \leqq \frac{m}{\lambda} \leqq N$, and for such a pair the simple (resp. double) poles of $f_{mn}(t)$ are at k^{th} roots of 1 with $k \leqq \lambda N$ (resp. $k \leqq N$), so that we obtain a positive lower bound for $r(N)$ (N large) if

$$\frac{\lambda N^2}{2} > [\Phi(\lambda N) + \Phi(N)]/2 \sim \frac{3(1+\lambda^2)N^2}{2\pi^2}, \tag{18}$$

i.e. if $\frac{\pi^2 \lambda}{3} > 1 + \lambda^2$.

Even this can be improved by sharper reasoning. For $1 \leqq k \leqq N$, we used as a bound for the number of double poles at primitive k^{th} roots of unity simply the number $\varphi(k)$ of such roots. But since double poles at these points come from (m, n) with $k|m$, $k|n$, $1 \leqq n \leqq \frac{m}{\lambda} \leqq N$, another possible bound is the number of such pairs, or approximately $\frac{\lambda}{2}\left(\frac{N}{k}\right)^2$. For k bigger than about $N^{2/3}$, this is smaller than $\varphi(k)$ (in other words, for the contribution from the double poles with these k we get a better bound by directly counting the functions in V_N rather than using $V_N \subset W'_N$). Therefore we can replace the term $\Phi(N)/2$ in (18) by the expression

$$\begin{aligned}\sum_{k=1}^{N} \min\left(\frac{\varphi(k)}{2}, \frac{\lambda N^2}{k^2}\right) &\leqq \sum_{k=1}^{[N^{2/3}]} \frac{\varphi(k)}{2} + \sum_{k=[N^{2/3}]+1}^{N} \frac{\lambda N^2}{k^2} \\ &\sim \frac{3N^{4/3}}{2\pi^2} + \lambda N^{4/3} \\ &= O(N^{4/3}) = o(N^2),\end{aligned}$$

so that (18) becomes

$$\frac{\lambda N^2}{2} > \frac{3(\lambda^2 + o(1))}{2\pi^2} N^2 \qquad (N \to \infty).$$

Therefore the λ-conjecture is in fact false for all $\lambda < \frac{\pi^2}{3} \approx 3.290$. Of course, it is quite possible that the λ-conjecture is false for all λ, i.e. that there exist relations of the form (2) in which all the f_{mn} that appear have a ratio $m:n$ larger than any preassigned number.

Finally, we should give examples of some of the relations whose existence we have proved. We have the table

K	1	2	3	4	5	6	7	8	9	10	11	12
$\varphi(K)$	1	1	2	2	4	2	6	4	6	4	10	4
$\Phi(K)$	1	2	4	6	10	12	18	22	28	32	42	46

Therefore the right-hand side of Eq. (16) takes on the values 0, 1, 2, 4, and 7 for $N=2, 3, 4, 5$, and 6. In particular there is already at least one relation among the functions of S_3 and at least one more among those of S_4. In fact there are exactly this many (i.e. equality holds in (16) for $N \leqq 4$), these first two relations being

$$3A_{6,2}(t)+2A_{3,1}(t)-A_{6,1}(t)-2A_{4,2}(t)-A_{2,1}(t)\equiv 0$$

and

$$\begin{aligned}2A_{8,3}(t)+4A_{8,2}(t)-2A_{8,1}(t)-3A_{6,2}(t)+A_{6,1}(t)\\ +2A_{4,1}(t)-A_{2,1}(t)\equiv 0\,.\end{aligned}$$

Expressed in terms of $f_{mn}(t)$, the relations are unchanged except that f replaces A and the right-hand sides of the two equations are 1 and 3 instead of both zero. Therefore the first relation involving the f's alone is (the second equation $-3\times$ the first):

$$2f_{83}+4f_{82}-2f_{81}-12f_{62}+4f_{61}+6f_{42}+2f_{41}-6f_{31}+2f_{21}=0\,,$$

or, written out in full,

$$\begin{aligned}0=&\left(\frac{t^8+1}{t^8-1}\,\frac{t^3+1}{t^3-1}\right)+2\left(\frac{t^8+1}{t^8-1}\,\frac{t^2+1}{t^2-1}\right)-\left(\frac{t^8+1}{t^8-1}\,\frac{t+1}{t-1}\right)\\ &-6\left(\frac{t^6+1}{t^6-1}\,\frac{t^2+1}{t^2-1}\right)+2\left(\frac{t^6+1}{t^6-1}\,\frac{t+1}{t-1}\right)+3\left(\frac{t^4+1}{t^4-1}\,\frac{t^2+1}{t^2-1}\right)\\ &+\left(\frac{t^4+1}{t^4-1}\,\frac{t+1}{t-1}\right)-3\left(\frac{t^3+1}{t^3-1}\,\frac{t+1}{t-1}\right)+\left(\frac{t^2+1}{t^2-1}\,\frac{t+1}{t-1}\right).\end{aligned}$$

§ 3. Final Remarks

After the completion of the work described in this paper, it was drawn to my attention by Prof. Hirzebruch that a counterexample to Ossa's conjecture had been previously given by Prof. Gaier of Heidelberg. I would like to thank Prof. Hirzebruch for his suggestion that I should nevertheless publish this analysis as of possible independent interest.

References

1. Atiyah, M. F., Singer, I. M.: The index of elliptic operators III. Ann. of Math. **87**, 564—604 (1968).
2. Hardy, G. H., Wright, E. M.: An introduction to the theory of numbers, 4th ed. Oxford: Clarendon Press 1960.
3. Ossa, E.: Cobordismustheorie von fixpunktfreien und semifreien S^1-Mannigfaltigkeiten. Bonn 1969.

Don B. Zagier
Mathematisches Institut der Universität
D-5300 Bonn, Wegelerstr. 10
Deutschland

(Received April 26, 1971)