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Higher Dimensional Dedekind Sums

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In this paper we will study the number-theoretical properties of the expression

$$d(p; a_1, \ldots, a_n) = (-1)^{n/2} \sum_{k=1}^{p-1} \cot \frac{\pi k a_1}{p} \cdots \cot \frac{\pi k a_n}{p} \tag{1}$$

and of related finite trigonometric sums. In Eq. (1), p is a positive integer, $a_1, \ldots, a_n$ are integers prime to p, and n is even (for n odd the sum is clearly equal to zero).

There are two reasons for being interested in sums of this type. First of all, the case $n=2$ is, up to a factor, the classical Dedekind sum:

$$d(p; a_1, a_2) = -\tfrac{2}{3}(a_1 a_2^{-1}, p), \tag{2}$$

where a_2^{-1} is an inverse of a_2 (mod p) and where (a, p) is the symbol used by Dedekind for his sums (we will never use it to denote the greatest common divisor function) and is always an integer. These sums were originally introduced [3] in connection with the transformation properties, under the modular group, of the Dedekind η-function

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \tag{3}$$

namely

$$\log \eta\left(\frac{az+b}{cz+d}\right) = \log \eta(z) + \tfrac{1}{4} \log\{-(cz+d)^2\} + \pi i \, \frac{a+d-2(d,c)}{12c} \tag{4}$$

with suitably chosen branches of the logarithm. Since then, the Dedekind sums have been extensively studied [15, 16, 4, 12] and have turned up in a variety of other contexts – in connection with the law of quadratic reciprocity [10], with the evaluation of class numbers of quadratic fields [11], and with the problem of generating random numbers [5], and above all in the Hardy-Ramanujan partition formula [7, 13].

The second ground for studying the sum (1) is that it arises in topological situations. The first appearance of trigonometric functions in this context was the index theorem of Hirzebruch [8], which evaluates the signature (a certain invariant of the homology of a differentiable manifold) by means of a formula involving cotangents. This was then generalized by Atiyah and Singer [2] who found the corresponding formula for the equivariant signature (a similar invariant defined for manifolds on which a group acts). This expression is the sum of contributions from the various components of the fixed-point set. In the simplest case of an isolated fixed point x of an action of the cyclic group $G=\mathbb{Z}/p\mathbb{Z}$ on an n-dimensional complex manifold X, with rotation numbers $a_1, \ldots, a_n$ (i.e. a p^{th} root of unity $\zeta \in G$ acts by $\zeta \circ (z_1, \ldots, z_n) = (\zeta^{a_1} z_1, \ldots, \zeta^{a_n} z_n)$ in local coordinates $z_1, \ldots, z_n$ at x), the contribution from the point x to the equivariant signature $\operatorname{Sign}(\zeta, X)$ is $\frac{\zeta^{a_1}+1}{\zeta^{a_1}-1} \cdots \frac{\zeta^{a_n}+1}{\zeta^{a_n}-1}$. It follows that the number $d(p; a_1, \ldots, a_n)$ has a topological interpretation as the contribution from the fixed point x to the "signature defect" (the deviation, for an action of G with fixed points, from the formula

$$\operatorname{Sign}(X/G) = \frac{1}{p} \operatorname{Sign}(X)$$

which is valid for free actions; cf. [20]). Thus the classical Dedekind sums arise in the study of group actions on 2-dimensional complex manifolds, while higher dimensions correspond to the generalized sums (1) with $n>2$; this is the reason for the title of the paper. In the topological situation just described, the manifold obtained by taking the boundary of a G-invariant disc centred at x and passing to its quotient under the group action is a generalized lens space $\mathscr{L}(p; a_1, \ldots, a_n)$, a manifold of dimension $2n-1$. The classical and higher-dimensional Dedekind sums are therefore also related to the properties and classification of the classical (three-dimensional) and higher-dimensional lens spaces, respectively.

We see, then, that the unexpectedly great importance of the classical Dedekind sums justifies a study of their higher-dimensional analogues, and that topological considerations suggest that (1) is the appropriate generalization. Before entering the body of the paper, we only make one comment about the expression (1), namely that the number it defines is rational. Indeed, if we rewrite (1) as

$$d(p; a_1, \ldots, a_n) = \sum_{k=1}^{p-1} \frac{\theta^{k a_1}+1}{\theta^{k a_1}-1} \cdots \frac{\theta^{k a_n}+1}{\theta^{k a_n}-1} \tag{5}$$

$$= \sum_{\substack{\lambda^p=1 \\ \lambda \neq 1}} \frac{\lambda^{a_1}+1}{\lambda^{a_1}-1} \cdots \frac{\lambda^{a_n}+1}{\lambda^{a_n}-1}, \tag{6}$$

where in the first expression θ denotes a primitive p^{th} root of unity, then we see that $d(p; a_1, \ldots, a_n)$ is an element of the cyclotomic field of p^{th} roots of unity which is invariant under all conjugations of the field, and is therefore a rational number. It will be one of our main tasks to evaluate the denominator of $d(p; a_1, \ldots, a_n)$ and in particular to show that it is bounded by a number depending only on n (3 for $n=2$, as one sees from (2), 45 for $n=4$, and so on).

The plan of the paper is as follows. In the first section we treat the case $n=2$, i.e. the classical Dedekind sum. This section serves to illustrate some of the methods and results of the rest of the paper, but none of the results are new and the section can be skipped. In Section 2 we prove a formula giving a rational expression for $d(p; a_1, \ldots, a_n)$. In the following section a reciprocity law is given generalizing a theorem of Rademacher for the classical Dedekind sums which itself extended a reciprocity law of Dedekind relating (q, p) and (p, q). This is then used to estimate the denominators of the higher Dedekind sums. Section 4 contains further miscellaneous properties of $d(p; a_1, \ldots, a_n)$ as well as tables of values for $n=2$ and $n=4$.

The material in this paper is an expanded and self-contained version of the purely number-theoretical part of [18].

§ 1. The Classical Dedekind Sum

The symbol (q, p) defined by Eq. (4) (the formula for the behaviour of $\log \eta(z)$ under modular transformations) was evaluated by Dedekind, who proved the relation

$$(q, p) = 6p \sum_{k=1}^{p-1} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right). \tag{7}$$

Here we use the standard notation

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} \tag{8}$$

where $[x]$ denotes the greatest integer function. We wish to prove that the right-hand side of (7) can be expressed as a cotangent sum, i.e. that Eq. (2) holds. Since clearly

$$d(p; xa_1, \ldots, xa_n) = d(p; a_1, \ldots, a_n) \tag{9}$$

for x an integer prime to p (this is just the invariance under conjugations $\theta \to \theta^x$ noted in connection with (5)), we only need to prove

$$d(p; 1, q) = -\tfrac{2}{3}(q, p). \tag{10}$$

To do this, we use the formula

$$\left(\left(\frac{a}{p}\right)\right) = \frac{-1}{2p} \sum_{j=1}^{p-1} \sin\frac{2\pi ja}{p} \cot\frac{\pi j}{p} \tag{11}$$

$$= \frac{1}{2p} \sum_{\substack{\lambda^p=1 \\ \lambda \neq 1}} \lambda^{-a} \frac{\lambda+1}{\lambda-1}, \tag{12}$$

a result due to Eisenstein [6]. To prove (11), we observe that the function $((x))$ is odd and periodic, so has a Fourier sine expansion:

$$((x)) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin 2\pi k x . \tag{13}$$

Therefore

$$\begin{aligned} \left(\left(\frac{a}{p}\right)\right) &= -\frac{1}{2\pi} \sum_{\substack{k=-\infty \\ p \nmid k}}^{\infty} \frac{1}{k} \sin \frac{2\pi k a}{p} \\ &= -\frac{1}{2\pi} \sum_{j=1}^{p-1} \sin \frac{2\pi j a}{p} \sum_{\substack{k=-\infty \\ k \equiv j \,(\mathrm{mod}\, p)}}^{\infty} \frac{1}{k}, \end{aligned} \tag{14}$$

where $\sum\limits_{-\infty}^{\infty}$ denotes the principal value $\lim\limits_{K\to\infty} \sum\limits_{-K}^{K}$. Combining Eq. (14) with the well-known expansion

$$\sum_{m=-\infty}^{\infty} \frac{1}{m+y} = \pi \cot \pi y \tag{15}$$

gives Eq. (11). An even shorter proof of the Eisenstein formula in the form (12) is to evaluate the right-hand side of (12) at $a=b$ and at $a=b-1$ and subtract:

$$\begin{aligned} \frac{1}{2p} \Sigma(\lambda^{-b} - \lambda^{1-b}) \frac{\lambda+1}{\lambda-1} &= \frac{1}{2p} \Sigma(-\lambda^{-b} - \lambda^{1-b}) \\ &= \begin{cases} \dfrac{1}{p} - \dfrac{1}{2} & \text{if} \quad b \equiv 0, 1 \,(\mathrm{mod}\, p) \\ \dfrac{1}{p} & \text{otherwise} \end{cases} \\ &= \left(\left(\frac{b}{p}\right)\right) - \left(\left(\frac{b-1}{p}\right)\right). \end{aligned}$$

Since both sides of Eq. (12) are odd functions of a, we immediately deduce that they must be equal.

We now substitute Eisenstein's formula into (7), in which we can replace the sum over k by a summation from $k=1$ to $k=p$ since $\left(\left(\frac{p}{p}\right)\right)=0$. This gives

$$(q, p) = \frac{3}{2p} \sum_{k=1}^{p} \sum_{\lambda} \sum_{\theta} \theta^{-k} \lambda^{-kq} \frac{\theta+1}{\theta-1} \frac{\lambda+1}{\lambda-1}, \tag{16}$$

where the sums over λ and θ are to be taken over p^{th} roots of unity other than one. We now invert the order of summation and use

$$\sum_{k=1}^{p} \theta^{-k} \lambda^{-kq} = \begin{cases} p & \text{if} \quad \theta\lambda^q = 1\,, \\ 0 & \text{if} \quad \theta\lambda^q \neq 1\,, \end{cases} \tag{17}$$

obtaining

$$(q, p) = \frac{3}{2} \sum_{\lambda} \frac{\lambda^{-q}+1}{\lambda^{-q}-1} \, \frac{\lambda+1}{\lambda-1}\,, \tag{18}$$

which is equivalent to Eq. (10).

We now wish to prove Dedekind's reciprocity law for (q, p). For convenience we work instead with the quantity

$$f(p, q) = \sum_{k=1}^{p-1} k \left[\frac{kq}{p}\right], \tag{19}$$

which differs from (q, p) only by a constant factor and a polynomial:

$$\begin{aligned} (q, p) &= 6p \sum_{k=1}^{p-1} \left(\frac{k}{p} - \frac{1}{2}\right)\left(\frac{kq}{p} - \left[\frac{kq}{p}\right] - \frac{1}{2}\right) \\ &= 6p \sum_{k=1}^{p-1} \left(\frac{q}{p^2}\, k^2 - \frac{q+1}{2p}\, k + \frac{1}{4}\right) \\ &\quad + 3p \sum_{k=1}^{p-1} \left[\frac{kq}{p}\right] - 6 \sum_{k=1}^{p-1} k \left[\frac{kq}{p}\right] \\ &= \frac{p-1}{2}\,(4pq - 2q - 3p) - 6f(p, q)\,. \end{aligned} \tag{20}$$

Here we have used the trivial relation

$$\sum_{k=1}^{p-1} \left[\frac{kq}{p}\right] = \frac{(p-1)(q-1)}{2} \qquad (p, q \text{ relatively prime})\,. \tag{21}$$

Equation (20) makes it clear that (q, p) is an integer and therefore, in conjunction with Eq. (2), shows that the rational number $d(p; a_1, a_2)$ has denominator at most equal to three.

Since q is prime to p, the numbers $q, 2q, \ldots, (p-1)\,q$ represent all the non-zero residue classes modulo p, and therefore the numbers $kq - p\left[\frac{kq}{p}\right]$ for $k = 1, \ldots, p-1$ are simply the numbers $1, \ldots, p-1$ in some order. Hence

$$\sum_{k=1}^{p-1} \left(kq - p\left[\frac{kq}{p}\right]\right)^2 = \sum_{k=1}^{p-1} k^2\,. \tag{22}$$

Expanding the left-hand side and substituting the value of Σk^2 yields

$$(q^2-1)\,\frac{p(p-1)(2p-1)}{6}-2pqf(p,q)+p^2\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]^2=0\,. \qquad (23)$$

On the other hand, inserting the definition of $[x]$ into the definition of $f(q,p)$ gives

$$\begin{aligned} f(q,p) &= \sum_{j=1}^{q-1} j \sum_{0<k<jp/q} 1 \\ &= \sum_{0<kq<jp<pq} j \\ &= \sum_{k=1}^{p-1} \sum_{j=\left[\frac{kq}{p}\right]+1}^{q-1} j \qquad\qquad (24) \\ &= \sum_{k=1}^{p-1}\left\{\frac{q(q-1)}{2}-\frac{1}{2}\left[\frac{kq}{p}\right]^2-\frac{1}{2}\left[\frac{kq}{p}\right]\right\} \\ &= \frac{(2q-1)(p-1)(q-1)}{4}-\frac{1}{2}\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]^2, \end{aligned}$$

where to obtain the last line we have used (21). Combining Eqs. (23) and (24) yields

$$qf(p,q)+pf(q,p)=\tfrac{1}{12}(p-1)(q-1)(8pq-p-q-1) \qquad (25)$$

or, expressing this in terms of the Dedekind symbol by means of (20),

$$p(p,q)+q(q,p)=\tfrac{1}{2}(p^2+q^2+1-3pq)\,. \qquad (26)$$

This is the desired reciprocity law of Dedekind. The generalization by Rademacher [14] which was mentioned in the introduction is

$$\frac{1}{p}\,d(p;q,r)+\frac{1}{q}\,d(q;p,r)+\frac{1}{r}\,d(r;p,q)=1-\frac{p^2+q^2+r^2}{3pqr} \qquad (27)$$

for mutually coprime integers p, q, r (Eq. (26) is the special case $r=1$, as we see by substituting (10) and using $d(1;p,q)=0$). It is this generalized reciprocity law of Rademacher which will be extended to the higher dimensional sums in Section 3.

To end the section, we will show how (26) can be used to give a simple proof of the quadratic reciprocity law. This has nothing to do with the following sections and can be omitted. We recall that the Legendre-Jacobi symbol $\left(\frac{q}{p}\right)$ is defined for q and p relatively prime integers with p

odd, and is given by Gauss' lemma, namely

$$\left(\frac{q}{p}\right) = (-1)^{N_{q,p}} \qquad (p \text{ odd, } q \text{ and } p \text{ coprime}), \tag{28}$$

where

$$N_{q,p} = \operatorname{Card}\left\{x \mid 1 \leqq x \leqq \frac{p-1}{2},\ qx - p\left[\frac{qx}{p}\right] > \frac{p}{2}\right\}. \tag{29}$$

We can write (28) more conveniently as

$$\left(\frac{q}{p}\right) \equiv 2N_{q,p} + 1 \pmod 4, \tag{30}$$

which determines $\left(\frac{q}{p}\right)$ since $\left(\frac{q}{p}\right) = \pm 1$. But $N_{q,p} \pmod 2$ can be expressed in terms of $f(p,q)$:

$$\begin{aligned}
N_{q,p} &= \sum_{\substack{0<x<p/2 \\ [2qx/p]\,\text{odd}}} 1 \\
&\equiv \sum_{0<x<p/2} \left[\frac{2qx}{p}\right] && \pmod 2 \\
&= \sum_{\substack{0<k<p \\ k \text{ even}}} \left[\frac{kq}{p}\right] \\
&\equiv \sum_{0<k<p} (k-1)\left[\frac{kq}{p}\right] && \pmod 2 \\
&= f(p,q) - \frac{(p-1)(q-1)}{2}.
\end{aligned} \tag{31}$$

Combining this with Eqs. (20) and (30) gives

$$\left(\frac{q}{p}\right) + (q,p) \equiv \frac{p+1}{2} \pmod 4, \tag{32}$$

which determines the Legendre-Jacobi symbol in terms of the Dedekind symbol. We now deduce from the Dedekind reciprocity law (26) that

$$\left(\frac{q}{p}\right) + \left(\frac{p}{q}\right) \equiv \frac{(p-1)(q-1)}{2} + 2 \pmod 4 \tag{33}$$

for p, q mutually prime odd integers. This is the law of quadratic reciprocity.

§ 2. A Rational Expression for $d(p; a_1, \ldots, a_n)$

In this section we give an expression for $d(p; a_1, \ldots, a_n)$ which generalizes the one given for $n=2$ in the last section. Our proof will again be based on the formula of Eisenstein (Eq. (12)).

To simplify the notation, we will write $\sum\limits_{\lambda}$ to denote a summation in which λ ranges over all p^{th} roots of unity, and will use $\sum\limits_{\lambda}'$ for a sum over p^{th} roots of unity other than one. We will repeatedly use the well-known formulas

$$\sum_{\lambda} \lambda^r = \begin{cases} 0 & \text{if } p \nmid r, \\ p & \text{if } p \mid r, \end{cases} \tag{34}$$

$$\sum_{k=1}^{p} \theta^k = \begin{cases} 0 & \text{if } \theta \neq 1, \\ p & \text{if } \theta = 1, \end{cases} \tag{35}$$

where in the second equation θ is an arbitrary p^{th} root of unity. These two formulas are dual to one another in the sense of the duality between a finite group and its character group.

From (34) we get the formula (with $\lambda^p = 1$, a prime to p)

$$\sum_{\theta}' \frac{\theta+1}{\theta-1} \left(\frac{1}{p} \sum_{k=1}^{p} \theta^{-k} \lambda^{ka} \right) = \begin{cases} \dfrac{\lambda^a+1}{\lambda^a-1} & \text{if } \lambda \neq 1, \\ 0 & \text{if } \lambda = 1. \end{cases} \tag{36}$$

Inverting the order of summation and substituting in Eisenstein's formula transforms the left-hand side of (36) into

$$2 \sum_{k=1}^{p} \left(\left(\frac{k}{p} \right)\right) \lambda^{ka}, \tag{37}$$

and the equality of (37) with the right-hand side of (36) can be looked on as the dual of Eisenstein's formula. Substituting it into the definition (Eq. (6)) of $d(p; a_1, \ldots, a_n)$ gives

$$\begin{aligned} d(p; a_1, \ldots, a_n) &= \sum_{\lambda}' \frac{\lambda^{a_1}+1}{\lambda^{a_1}-1} \cdots \frac{\lambda^{a_n}+1}{\lambda^{a_n}-1} \\ &= \sum_{\lambda} \prod_{j=1}^{n} \left\{ 2 \sum_{k=1}^{p} \left(\left(\frac{k}{p} \right)\right) \lambda^{ka_j} \right\} \\ &= 2^n \sum_{0<k_1, \ldots, k_n<p} \left(\left(\frac{k_1}{p} \right)\right) \cdots \left(\left(\frac{k_n}{p} \right)\right) \sum_{\lambda} \lambda^{k_1 a_1 + \cdots + k_n a_n}. \end{aligned}$$

The last sum can now be evaluated using (34), and we obtain:

Theorem. *Let n be even, p a positive integer, and $a_1, \ldots, a_n$ integers prime to p. Then the number $d(p; a_1, \ldots, a_n)$ defined by* (1) *equals*

$$2^n p \sum_{\substack{0<k_1,\ldots,k_n<p \\ p \mid k_1a_1+\cdots+k_na_n}} \left(\left(\frac{k_1}{p}\right)\right) \cdots \left(\left(\frac{k_n}{p}\right)\right). \tag{38}$$

Notice that everything occurring in this expression only depends on the values of the k_i modulo p, so that we could let them run through any complete set of residues (omitting zero if we wish). However, reducing k_i to lie between 0 and p has the advantage that then $\left(\left(\frac{k_i}{p}\right)\right) = \frac{k_i}{p} - \frac{1}{2}$, so that we can write (38) in closed form:

$$p^{n-1} d(p; a_1, \ldots, a_n) = \sum_{\substack{0<k_1,\ldots,k_n<p \\ p \mid k_1a_1+\cdots+k_na_n}} (2k_1 - p) \ldots (2k_n - p). \tag{39}$$

Since the right-hand side of this equation is clearly integral, we have not only found a finite and elementary expression for the trigonometric sum (1), but have also proved that it is a rational number whose denominator is at most p^{n-1}. (Another proof that only prime factors of p can occur in the denominator is as follows: one easily shows that $\cot \frac{\pi k a}{p}$ satisfies an algebraic equation with integer coefficients and leading coefficient p. It follows that $p \cot \frac{\pi k a}{p}$ is an algebraic integer, and therefore that p^n times any term in the sum (1) also is an algebraic integer. Since the whole sum is rational, we conclude that $p^n d(p; a_1, \ldots, a_n)$ is a rational integer.)

By dualizing the above proof, i.e. interchanging the rôles of (34) and (35) and of (12) and (37), one could show that the expression

$$2^n p^{n-1} \sum_{k=1}^{p-1} \left(\left(\frac{ka_1}{p}\right)\right) \cdots \left(\left(\frac{ka_n}{p}\right)\right), \tag{40}$$

which might have been taken instead of (1) as the appropriate generalization of the classical Dedekind sum, is equal to the trigonometric expression

$$(-1)^{n/2} \sum_{\substack{0<k_1,\ldots,k_n<p \\ p \mid k_1a_1+\cdots+k_na_n}} \cot \frac{\pi k_1}{p} \cdots \cot \frac{\pi k_n}{p}. \tag{41}$$

§ 3. The Reciprocity Law and the Denominator of $d(p; a_1, \ldots, a_n)$

In Section 1 we stated without proof a reciprocity law (Eq. (27)) of Rademacher which relates $d(p; q, r)$, $d(q; p, r)$, and $d(r; p, q)$ for relatively prime integers p, q, r. We now formulate and prove the corresponding law for higher sums. This takes the form

$$\sum_{j=0}^{n} \frac{1}{a_j} d(a_j; a_0, \ldots, \hat{a}_j, \ldots, a_n) = \phi_n(a_0, \ldots, a_n), \tag{42}$$

where $a_0, \ldots, a_n$ (n even) are pairwise coprime positive integers, the hat over the a_j denotes its omission from the list, and ϕ_n is a certain rational function in $n+1$ variables. More precisely,

$$\phi_n(a_0, \ldots, a_n) = 1 - \frac{\ell_n(a_0, \ldots, a_n)}{a_0 \ldots a_n}, \tag{43}$$

where $\ell_n(a_0, \ldots, a_n)$ is a polynomial whose first values are:

$$\ell_0(a) = 1, \tag{44a}$$

$$\ell_2(a, b, c) = (a^2 + b^2 + c^2)/3, \tag{44b}$$

$$\begin{aligned} &\ell_4(a, b, c, d, e) \\ &= (5(a^2 + b^2 + c^2 + d^2 + e^2)^2 - 7(a^4 + b^4 + c^4 + d^4 + e^4))/90. \end{aligned} \tag{44c}$$

Since $\ell_n(a_0, \ldots, a_n)$ is even in each variable, symmetric under interchange of the variables, and homogeneous of total degree n, it can conveniently be written

$$\ell_n(a_0, \ldots, a_n) = L_k(p_1, \ldots, p_k), \tag{45}$$

where $k = n/2$ and p_i ($i = 1, \ldots, k$) is the i^{th} elementary symmetric polynomial in $a_0^2, \ldots, a_n^2$. Then the first few polynomials L_k are

$$L_0 = 1, \tag{46a}$$

$$L_1(p_1) = p_1/3, \tag{46b}$$

$$L_2(p_1, p_2) = (-p_1^2 + 7p_2)/45, \tag{46c}$$

$$L_3(p_1, p_2, p_3) = (2p_1^3 - 13p_1p_2 + 62p_3)/945. \tag{46d}$$

Topologists will recognize these as the Hirzebruch L-polynomials [8].

We now state and prove the complete formula.

Theorem. *Let $a_0, \ldots, a_n$ (n even) be pairwise coprime positive integers. Then*

$$\sum_{j=0}^{n} \frac{1}{a_j} d(a_j; a_0, \ldots, \hat{a}_j, \ldots, a_n) = 1 - \frac{\ell_n(n_0, \ldots, a_n)}{a_0 \ldots a_n}, \tag{47}$$

where $\ell_n(a_0, \ldots, a_n)$ is the polynomial defined as the coefficient of t^n in the power series expansion of

$$\prod_{j=0}^{n} \frac{a_j t}{\tanh a_j t} = \prod_{j=0}^{n} \left(1 + \frac{1}{3} a_j^2 t^2 - \frac{1}{45} a_j^4 t^4 + \frac{2}{945} a_j^6 t^6 - \cdots\right). \tag{48}$$

Proof. We will apply the residue theorem to

$$f(z) = \frac{1}{2z} \prod_{j=0}^{n} \frac{z^{a_j}+1}{z^{a_j}-1}. \tag{49}$$

This is a rational function with poles at $z=0$, at $z=\infty$, and at those points of the unit circle for which $z^{a_j}=1$ for some j. Since the a_j's are prime to one another, $z^{a_j}=z^{a_k}=1$ $(j \neq k) \Rightarrow z=1$. Therefore the poles are: a simple pole at $z=0$, a simple pole at $z=\infty$, a pole of order $n+1$ at $z=1$, and, for $j=0, \ldots, n$, a_j-1 simple poles at the points $z^{a_j}=1$, $z \neq 1$. Therefore

$$\begin{aligned} 0 &= \sum_{t \text{ a pole}} \operatorname{res}_{z=t}(f(z)\,dz) \\ &= \left\{\operatorname{res}_0 + \operatorname{res}_\infty + \operatorname{res}_1 + \sum_{j=0}^{n} \sum_{\substack{t^{a_j}=1 \\ t \neq 1}} \operatorname{res}_t\right\}(f(z)\,dz) \\ &= -\frac{1}{2} - \frac{1}{2} + \operatorname{res}_{z=1} f(z)\,dz + \sum_{j=0}^{n} \frac{1}{a_j} \sum_{\substack{t^{a_j}=1 \\ t \neq 1}} \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{t^{a_k}+1}{t^{a_k}-1}. \end{aligned} \tag{50}$$

The last term is precisely the left-hand side of (47). The substitution $z=e^{2t}$ gives

$$\begin{aligned} \operatorname{res}_{z=1}(f(z)\,dz) &= \operatorname{res}_{t=0}(2e^{2t} f(e^{2t})\,dt) \\ &= \operatorname{res}_{t=0}\left\{\prod_{j=0}^{n} (\coth a_j t)\,dt\right\} \\ &= \operatorname{res}_{t=0}\left\{\prod_{j=0}^{n} \frac{a_j t}{\tanh a_j t} \frac{dt}{t^{n+1}}\right\}\Big/ a_0 \ldots a_n \\ &= \ell_n(a_0, \ldots, a_n)/a_0 \ldots a_n. \end{aligned} \tag{51}$$

Putting this into (50) then gives the result desired.

We now turn to the study of the denominators involved. Let

$$\mu_k = \text{denominator of } L_k(p_1, \ldots, p_k)\,; \tag{52}$$

thus from (46) we find

$$\mu_0 = 1\,, \tag{53a}$$

$$\mu_1 = 3\,, \tag{53b}$$

$$\mu_2 = 45\,, \tag{53c}$$

$$\mu_3 = 945\,. \tag{53d}$$

It has been shown [1] that

$$\mu_k = \prod_{\substack{\ell \text{ prime} \\ \ell \text{ odd}}} \ell^{\left[\frac{n}{\ell-1}\right]}. \tag{54}$$

Thus μ_k is always odd, and is divisible by an odd prime ℓ precisely $\left[\frac{n}{\ell-1}\right]$ times, where [ ] denotes the greatest integer function. The reason that we are interested in μ_k is that it is a universal bound for the denominators of the n-dimensional Dedekind sums, i.e. for any p and any a_i we have

$$\mu_k d(p; a_1, \ldots, a_n) \in \mathbb{Z}\,, \quad (k = n/2)\,. \tag{55}$$

We state this as a theorem, slightly sharpening it by combining it with the result of Section 2 (Eq. (39) or the discussion following that equation) that $d(p; a_1, \ldots, a_n)$ can only contain primes in its denominator that divide p:

Theorem. *Let p be a positive integer and $a_1, \ldots, a_n$ (n even) be integers prime to p. Then $d(p; a_1, \ldots, a_n)$ is a rational number whose denominator divides*

$$\prod_{\substack{\ell \text{ prime} \\ \ell > 2 \\ \ell \mid p}} \ell^{\left[\frac{n}{\ell-1}\right]}. \tag{56}$$

Proof. As observed above, it suffices to prove (55). We first assume that $a_1, \ldots, a_n$ are pairwise coprime so that the reciprocity law (47) applies (with $a_0 = p$). We multiply both sides of it by $\mu_k a_0 \ldots a_n$. The right-hand side is then an integer since $\ell_n(a_0, \ldots, a_n) = L_k(p_1, \ldots, p_k)$. Therefore

$$\sum_{j=0}^{n} a_0 \ldots \hat{a}_j \ldots a_n \mu_k \, d(a_j; a_0, \ldots, \hat{a}_j, \ldots, a_n) \in \mathbb{Z}\,. \tag{57}$$

Again by the result of Section 2, the j^{th} summand in (57) has a denominator involving only prime factors of a_j. Since the a_j's are mutually coprime, it follows that each term in (57) is an integer. But, using the same fact yet again, we know that $a_0 \ldots \hat{a}_j \ldots a_n$ is prime to the denominator of $\mu_k d(a_j; a_0, \ldots, \hat{a}_j, \ldots, a_n)$, and it follows that the latter is an integer.

If the a_i's are not prime to one another, the reciprocity law no longer holds. However, the a_i's in (55) are prime to p, so by Dirichlet's theorem on primes in arithmetic progressions they can be replaced by prime numbers a_i' congruent to a_i modulo p. Since the value of $d(p; a_1, \ldots, a_n)$ only depends on the values of the $a_i \pmod p$, it is sufficient to prove (55) for the a_i'. But they can be chosen as arbitrarily large primes, so in particular they can be made prime to one another and then to p, and the theorem now follows from the special case already considered.

The number (56) gives a bound for the denominator of $d(p; a_1, \ldots, a_n)$ which is in general very good, although the actual denominator can be smaller. We devote the rest of the section to illustrations of this.

Let $n = 2k$ be an arbitrary even number, $p = 3$, $a_1 = \cdots = a_n = 1$. Then

$$\begin{aligned} d(3; 1, \ldots, 1) &= (-1)^k \sum_{j=1}^{2} \left(\cot \frac{\pi j}{3}\right)^n \\ &= (-1)^k \left\{\left(\frac{1}{\sqrt{3}}\right)^n + \left(\frac{-1}{\sqrt{3}}\right)^n\right\} \\ &= \frac{\pm 2}{3^k}, \end{aligned} \tag{58}$$

so that the denominator is exactly 3^k, in agreement with (56). With $p = 5$, $a_1 = \cdots = a_n = 1$, we get

$$\begin{aligned} d(5; 1, \ldots, 1) &= (-1)^k \sum_{j=1}^{4} \left(\cot \frac{\pi j}{5}\right)^n \\ &= (-1)^k\, 2 \left\{\left(\sqrt{\frac{5+2\sqrt{5}}{5}}\right)^n + \left(\sqrt{\frac{5-2\sqrt{5}}{5}}\right)^n\right\} \\ &= \frac{\pm 2}{5^k} \{(5+2\sqrt{5})^k + (5-2\sqrt{5})^k\}, \end{aligned} \tag{59}$$

and this has denominator exactly $5^{[k/2]}$, in agreement with the theorem, except when k is an odd multiple of 5. Thus we get an example where the bound (56) is not sharp by taking, for example, $k = 5$:

$$d(5; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,) = -\tfrac{244}{5}. \tag{60}$$

It is interesting to generalize these two examples, taking for p an odd prime and $a_1 = \cdots = a_n = 1$, since this shows how the factor $p^{[n/(p-1)]}$ in Eq. (56) arises. We have

$$d(p; 1, \ldots, 1) = \sum_{\substack{\lambda^p = 1 \\ \lambda \neq 1}} \left(\frac{\lambda+1}{\lambda-1}\right)^n. \tag{61}$$

The number $x = \frac{\lambda+1}{\lambda-1}$ satisfies the equation

$$1 = \lambda^p = \left(\frac{x+1}{x-1}\right)^p \tag{62}$$

or, multiplying out and cancelling,

$$\binom{p}{1} x^{p-1} + \binom{p}{3} x^{p-3} + \cdots + \binom{p}{p-2} x^2 + 1 = 0\,. \tag{63}$$

Therefore $y = x^2$ is a root of the equation

$$f(y) = y^r - c_1 y^{r-1} + \cdots + (-1)^r c_r = 0\,, \tag{64}$$

where $r = \frac{p-1}{2}$ and

$$c_i = \frac{(-1)^i}{p}\binom{p}{2i+1} \qquad (i = 1, \ldots, r)\,. \tag{65}$$

Now (61) can be rewritten

$$\begin{aligned} d(p; 1, \ldots, 1) &= 2 \sum_{f(y)=0} y^k \\ &= 2 s_k(c_1, \ldots, c_r)\,, \end{aligned} \tag{66}$$

where $s_k(c_1, \ldots, c_r)$ is the Newton's polynomial expressing the sum of the k^{th} powers of r numbers in terms of their i^{th} elementary symmetric functions c_i $(i = 1, \ldots, r)$. It is a homogeneous polynomial of total degree k (where c_i is weighted with weight i), and therefore contains c_r at most to the power $\left[\frac{k}{r}\right] = \left[\frac{n}{p-1}\right]$. Since $s_k(c_1, \ldots, c_r)$ has integer coefficients, and since all of the c_i except $c_r = \pm 1/p$ are integers, it follows that $d(p; 1, \ldots, 1)$ has denominator at most equal to $p^{[n/(p-1)]}$.

For small values of n, the bound given by (56) is sharp. Thus for $n = 2$, if we apply the Rademacher reciprocity law to coprime integers p, q, r with p divisible by 3 (in which case q and r cannot be), we get

$$\begin{aligned} &\frac{1}{p}\, d(p; q, r) + (\text{numbers with no 3 in denominator}) \\ &\qquad = -\frac{p^2 + q^2 + r^2}{3pqr}\,, \end{aligned}$$

from which it follows that the denominator of $d(p; q, r)$ is 3 and that

$$3 a_1 a_2\, d(p; a_1, a_2) \equiv \begin{cases} 0 \pmod 3 & \text{if } \; 3 \nmid p\,, \\ 1 \pmod 3 & \text{if } \; 3 \mid p\,. \end{cases} \tag{67}$$

Similarly for $n=4$ we find from the reciprocity law that

$$9a_1 a_2 a_3 a_4 d(p; a_1, a_2, a_3, a_4) \equiv \begin{cases} 0 \pmod 3 & \text{if } 3 \nmid p, \\ 2 \pmod 3 & \text{if } 3 \mid p, \end{cases} \tag{68}$$

(in fact $9d(p; a_1, a_2, a_3, a_4) \equiv \pm 2 \bmod 9$, the sign being such as to make (68) hold) and that

$$5a_1 a_2 a_3 a_4 d(p; a_1, a_2, a_3, a_4) \equiv \begin{cases} 0 \pmod 5 & \text{if } 5 \nmid p, \\ 1 \pmod 5 & \text{if } 5 \mid p. \end{cases} \tag{69}$$

Therefore for $n=2$ and $n=4$ the denominator of $d(p; a_1, \ldots, a_n)$ is equal to the number (56). For $n=6$ we find similarly that the denominator contains a factor $3^3 = 27$ whenever 3 divides p and a factor 7 whenever 7 divides p, but it may not be divisible by 5 even if p is, e.g.

$$d(5; 1, 1, 1, 2, 2, 2) = 35. \tag{70}$$

Thus, starting with $n=6$ the denominator of $d(p; a_1, \ldots, a_n)$ may be strictly less than the number given in the theorem.

§ 4. Tables and Special Properties of the Dedekind Sums

We collect some properties of the higher-dimensional Dedekind sums which are trivial or have been proved in the preceding sections:

(i) $d(p; a_1, \ldots, a_n)$ only depends on $a_i \pmod p$,

(ii) $d(p; a_1. \ldots, a_n)$ is symmetric in the n arguments $a_1, \ldots, a_n$,

(iii) $d(p; -a_1, \ldots, a_n) = -d(p; a_1, \ldots, a_n)$,

(iv) $d(p; xa_1, \ldots, xa_n) = d(p; a_1, \ldots, a_n)$ if x is an integer prime to p,

(v) $$\frac{1}{a_0} d(a_0; a_1, \ldots, a_n) + \frac{1}{a_1} d(a_1; a_0, a_2, \ldots, a_n) + \cdots$$
$$\cdots + \frac{1}{a_n} d(a_n; a_0, \ldots, a_{n-1}) = \phi_n(a_0, \ldots, a_n)$$

for pairwise coprime integers $a_0, \ldots, a_n$, where ϕ_n is the function given in (43),

(vi) $d(p; a_1, \ldots, a_n)$ is rational and its denominator divides (56).

For a given n and each modulus p, there are only a finite number of values of $d(p; a_1, \ldots, a_n)$, since by (i) we can assume that $0 < a_1, \ldots, a_n < p$. The number can be still further reduced by using (i)–(iv) to obtain a_i's with $a_1 = 1 \leqq a_2 \leqq \cdots \leqq a_n \leqq \frac{p}{2}$ (all a_i's prime to p). This cuts down the size of the tables considerably. On the following two pages we give tables of the Dedekind symbol (q, p) for $1 \leqq q \leqq p \leqq 29$ and of the four-dimensional Dedekind sum $d(p; a_1, a_2, a_3, a_4)$ for $p \leqq 16$ and all a_1, a_2, a_3, a_4. To find

Table 1. The Dedekind symbol (q, p)

(1, 2) = 0	(1, 17) = 120	(1, 24) = 253
	(2, 17) = 48	(5, 24) = 53
(1, 3) = 1	(3, 17) = 30	(7, 24) = 19
	(4, 17) = 0	(11, 24) = −37
(1, 4) = 3	(5, 17) = 6	
	(6, 17) = 30	(1, 25) = 276
(1, 5) = 6	(7, 17) = 6	(2, 25) = 120
(2, 5) = 0	(8, 17) = −48	(3, 25) = 60
		(4, 25) = 24
(1, 6) = 10	(1, 18) = 136	(6, 25) = −24
	(5, 18) = 8	(7, 25) = 0
(1, 7) = 15	(7, 18) = −8	(8, 25) = −60
(2, 7) = 3		(9, 25) = 24
(3, 7) = −3	(1, 19) = 153	(11, 25) = −24
	(2, 19) = 63	(12, 25) = −120
(1, 8) = 21	(3, 19) = 27	
(3, 8) = 3	(4, 19) = 33	(1, 26) = 300
	(5, 19) = 33	(3, 26) = 84
(1, 9) = 28	(6, 19) = −27	(5, 26) = 0
(2, 9) = 8	(7, 19) = 9	(7, 26) = 24
(4, 9) = −8	(8, 19) = −9	(9, 26) = 84
	(9, 19) = −63	(11, 26) = −24
(1, 10) = 36		
(3, 10) = 0	(1, 20) = 171	(1, 27) = 325
	(3, 20) = 45	(2, 27) = 143
(1, 11) = 45	(7, 20) = 45	(4, 27) = 73
(2, 11) = 15	(9, 20) = −21	(5, 27) = 35
(3, 11) = 9		(7, 27) = 73
(4, 11) = 9	(1, 21) = 190	(8, 27) = −1
(5, 11) = −15	(2, 21) = 80	(10, 27) = 1
	(4, 21) = 10	(11, 27) = 35
(1, 12) = 55	(5, 21) = −10	(13, 27) = −143
(5, 12) = −1	(8, 21) = 8	
	(10, 21) = −80	(1, 28) = 351
(1, 13) = 66		(3, 28) = 81
(2, 13) = 24	(1, 22) = 210	(5, 28) = 39
(3, 13) = 6	(3, 22) = 42	(9, 28) = −81
(4, 13) = −6	(5, 22) = 18	(11, 28) = −39
(5, 13) = 0	(7, 22) = −42	(13, 28) = −57
(6, 13) = −24	(9, 22) = 18	
		(1, 29) = 378
(1, 14) = 78	(1, 23) = 231	(2, 29) = 168
(3, 14) = 18	(2, 23) = 99	(3, 29) = 108
(5, 14) = 18	(3, 23) = 63	(4, 29) = 42
	(4, 23) = 51	(5, 29) = 78
(1, 15) = 91	(5, 23) = 21	(6, 29) = 78
(2, 15) = 35	(6, 23) = 51	(7, 29) = −42
(4, 15) = 19	(7, 23) = −3	(8, 29) = 24
(7, 15) = −35	(8, 23) = 63	(9, 29) = −18
	(9, 23) = −21	(10, 29) = 108
(1, 16) = 105	(10, 23) = −3	(11, 29) = 24
(3, 16) = 15	(11, 23) = −99	(12, 29) = 0
(5, 16) = −15		(13, 29) = −18
(7, 16) = −9		(14, 29) = −168

Table 2. Four-dimensional Dedekind sums

$d(2;1,1,1,1)=$	0	$d(13;1,1,1,1)=$	572
		$d(13;1,1,1,2)=$	264
$d(3;1,1,1,1)=$	$0\frac{2}{9}$	$d(13;1,1,1,3)=$	148
		$d(13;1,1,1,4)=$	76
$d(4;1,1,1,1)=$	2	$d(13;1,1,1,5)=$	40
		$d(13;1,1,1,6)=$	$-\ 40$
$d(5;1,1,1,1)=$	$7\frac{1}{5}$	$d(13;1,1,2,2)=$	124
$d(5;1,1,1,2)=$	$1\frac{3}{5}$	$d(13;1,1,2,3)=$	72
$d(5;1,1,2,2)=$	$0\frac{4}{5}$	$d(13;1,1,2,4)=$	40
		$d(13;1,1,2,5)=$	20
$d(6;1,1,1,1)=$	$18\frac{2}{9}$	$d(13;1,1,2,6)=$	$-\ 12$
		$d(13;1,1,3,3)=$	60
$d(7;1,1,1,1)=$	38	$d(13;1,1,3,4)=$	28
$d(7;1,1,1,2)=$	14	$d(13;1,1,3,5)=$	8
$d(7;1,1,1,3)=$	2	$d(13;1,1,3,6)=$	8
$d(7;1,1,2,2)=$	6	$d(13;1,1,4,5)=$	$-\ 8$
$d(7;1,1,2,3)=$	2	$d(13;1,1,4,6)=$	8
		$d(13;1,1,5,5)=$	28
$d(8;1,1,1,1)=$	70	$d(13;1,1,5,6)=$	36
$d(8;1,1,1,3)=$	10	$d(13;1,2,3,4)=$	16
$d(8;1,1,3,3)=$	6	$d(13;1,2,3,5)=$	12
		$d(13;1,2,3,6)=$	$-\ 4$
$d(9;1,1,1,1)=$	$118\frac{2}{9}$		
$d(9;1,1,1,2)=$	$49\frac{7}{9}$	$d(14;1,1,1,1)=$	780
$d(9;1,1,1,4)=$	$-\ 1\frac{7}{9}$	$d(14;1,1,1,3)=$	212
$d(9;1,1,2,2)=$	$22\frac{2}{9}$	$d(14;1,1,1,5)=$	84
$d(9;1,1,2,4)=$	$1\frac{7}{9}$	$d(14;1,1,3,3)=$	76
		$d(14;1,1,3,5)=$	12
$d(10;1,1,1,1)=$	$187\frac{1}{5}$		
$d(10;1,1,1,3)=$	$38\frac{2}{5}$	$d(15;1,1,1,1)=$	$1039\frac{19}{45}$
$d(10;1,1,3,3)=$	$20\frac{4}{5}$	$d(15;1,1,1,2)=$	$489\frac{17}{45}$
		$d(15;1,1,1,4)=$	$185\frac{1}{45}$
$d(11;1,1,1,1)=$	282	$d(15;1,1,1,7)=$	$-\ 86\frac{8}{45}$
$d(11;1,1,1,2)=$	126	$d(15;1,1,2,2)=$	$233\frac{1}{45}$
$d(11;1,1,1,3)=$	66	$d(15;1,1,2,4)=$	$86\frac{8}{45}$
$d(11;1,1,1,4)=$	34	$d(15;1,1,2,7)=$	$-\ 31\frac{19}{45}$
$d(11;1,1,1,5)=$	$-\ 14$	$d(15;1,1,4,4)=$	$79\frac{19}{45}$
$d(11;1,1,2,2)=$	58	$d(15;1,1,4,7)=$	$-\ 9\frac{17}{45}$
$d(11;1,1,2,3)=$	30	$d(15;1,2,4,7)=$	$+\ 6\frac{44}{45}$
$d(11;1,1,2,4)=$	14		
$d(11;1,1,2,5)=$	$-\ 2$	$d(16;1,1,1,1)=$	1358
$d(11;1,1,3,3)=$	26	$d(16;1,1,1,3)=$	386
$d(11;1,1,3,4)=$	2	$d(16;1,1,1,5)=$	126
$d(11;1,1,3,5)=$	2	$d(16;1,1,1,7)=$	$-\ 14$
$d(11;1,1,4,5)=$	18	$d(16;1,1,3,3)=$	142
$d(11;1,2,3,4)=$	6	$d(16;1,1,3,5)=$	50
		$d(16;1,1,3,7)=$	$-\ 2$
$d(12;1,1,1,1)=$	$408\frac{2}{9}$	$d(16;1,1,5,7)=$	2
$d(12;1,1,1,5)=$	$11\frac{7}{9}$	$d(16;1,1,7,7)=$	78
$d(12;1,1,5,5)=$	$24\frac{2}{9}$	$d(16;1,3,5,7)=$	14

Table 3. Four-dimensional Dedekind sums

$d(n;1,1,1,1)=n+(n^4-20n^2-26)/45$.		
$d(n;1,1,1,2)=n+(n^4-35n^2-56)/90$.		
$d(n;1,1,1,3)=n+(n^4-60n^2\mp 10n-66)/135$	if	$n\equiv \pm 1 \pmod 3$.
$d(n;1,1,1,4)=n+(n^4-95n^2\mp 90n+4)/180$	if	$n\equiv \pm 1 \pmod 4$.
$d(n;1,1,1,5)=n+(n^4-140n^2\mp 324n+238)/225$	if	$n\equiv \pm 1 \pmod 5$,
$d(n;1,1,1,5)=n+(n^4-140n^2\mp 72n+238)/225$	if	$n\equiv \pm 2 \pmod 5$.
$d(n;1,2,2,2)=n+(n^4-110n^2-251)/360$.		
$d(n;2,2,2,3)=n+(n^4-510n^2\pm 80n-651)/1080$	if	$n\equiv \pm 1 \pmod 3$.
$d(n;1,3,3,3)=n+(n^4\mp 40n^3-300n^2\pm 230n-1106)/1215$	if	$n\equiv \pm 1 \pmod 3$.
$d(n;1,1,2,2)=n+(n^4-50n^2-131)/180$.		
$d(n;1,1,3,3)=n+(n^4-60n^2\pm 80n-426)/405$	if	$n\equiv \pm 1 \pmod 3$.
$d(n;1,1,4,4)=n+(n^4+10n^2\pm 360n-1091)/720$	if	$n\equiv \pm 1 \pmod 4$.
$d(n;1,1,2,3)=n+(n^4-75n^2\pm 20n-216)/270$	if	$n\equiv \pm 1 \pmod 3$.
$d(n;1,1,2,4)=n+(n^4-110n^2-251)/360$.		
$d(n;1,1,2,5)=n+(n^4-155n^2\mp 144n-152)/450$	if	$n\equiv \pm 1 \pmod 5$,
$d(n;1,1,2,5)=n+(n^4-155n^2\mp 72n-152)/450$	if	$n\equiv \pm 2 \pmod 5$.
$d(n;1,1,3,4)=n+(n^4-135n^2\pm 230n-636)/540$	if	$n\equiv \pm 1 \pmod{12}$,
$d(n;1,1,3,4)=n+(n^4-135n^2\pm 310n-636)/540$	if	$n\equiv \pm 5 \pmod{12}$.
$d(n;1,2,3,4)=n+(n^4-150n^2\pm 80n-1011)/1080$	if	$n\equiv \pm 1 \pmod 3$.

a given Dedekind sum in the table may require the repeated application of the properties (i) to (iv); thus

$$\begin{aligned} d(13;7,9,3,2) &= d(13;-6,-4,-10,2) = -d(13;6,4,10,2) \\ &= -d(13;3,2,5,1) = -d(13;1,2,3,5) = -12\,. \end{aligned}$$

For $n=2$, the properties (i)–(v) characterize $d(p;a_1,a_2)$. Indeed, we can use (iv) to get to a number $d(p;q,1)$, then (i) to obtain $0<q<p$, and then (v) with $n=2$, $a_2=1$ (Dedekind reciprocity) to interchange the rôles of p and q; this reduces the modulus and therefore solves the problem by induction (this reduction process is just the Euclidean algorithm, and gets us down to modulus $p=1$ because p and q are coprime). I do not know whether the same holds for $n=4$ (i.e. whether a function d' of variables $p>0$ and $a_1,\dots,a_4$ prime to p which satisfies (i)–(iv) and (v) with right-hand side zero must be identically zero). To the limits of the Tables I made ($p\leqq 16$), this was the case; that is, it was possible to make the table by using (i)–(iv) to manipulate the a_i's so that a_1,a_2,a_3, and a_4 were smaller than p and prime to one another and then using (v) to reduce the computation to one of Dedekind sums with smaller moduli. Thus it was never necessary, except as a check, to have recourse to one of the explicit formulas (1) or (38). The calculation of the Dedekind symbols was also performed by this sort of downward induction; however, here the computations were made by computer[1] rather than by hand (for $1\leqq q\leqq p\leqq 150$).

[1] The IBM 7090 at Bonn.

Besides the values for small n, we have given (Table 3) formulas for $d(n; a, b, c, d)$ for various quadruples (a, b, c, d) as a function of n. If the numbers a, b, c, d are prime to one another, e.g. for the calculation of $d(n; 1, 1, 2, 3)$ with n prime to 6, these formulas come immediately from the reciprocity law. In other cases their derivation requires an induction. We give an example of the calculation. For n odd, set

$$\tau_n \doteq \frac{1}{n} d(n; 1, 1, 2, 4). \tag{71}$$

Then $\tau_n = -\frac{1}{n} d(n; 1, 1, 4, n-2)$, and the numbers $1, 1, 4, n-2, 4$ are pairwise coprime. We then apply the reciprocity law, using (44c):

$$\begin{aligned}
&-\tau_n + d(1; 1, 4, n-2, n) + d(1; 1, 4, n-2, n) + \frac{1}{4} d(4; 1, 1, n-2, n) \\
&\quad + \frac{1}{n-2} d(n-2; 1, 1, 4, n) \\
&= 1 - \frac{5\{1^2+1^2+4^2+(n-2)^2+n^2\}^2 - 7\{1^4+1^4+4^4+(n-2)^4+n^4\}}{360n(n-2)} \\
&= 1 - \frac{5(2n^2-4n+22) - 7(2n^4-8n^3+24n^2-32n+274)}{360n(n-2)} \\
&= 1 - \frac{3n^2-6n+164}{180} + \frac{251}{360(n-2)} - \frac{251}{360n}.
\end{aligned} \tag{72}$$

But $d(1; a_1, \ldots, a_n) = 0$ always, $d(4; 1, 1, n-2, n) = d(4; 1, 1, 1, 3)$ whether n is congruent to 1 or to 3 modulo 4, and

$$\frac{1}{n-2} d(n-2; 1, 1, 4, n) = \frac{1}{n-2} d(n-2; 1, 1, 4, 2) = \tau_{n-2},$$

so the left-hand side of the equation is equal to $-\tau_n + \frac{1}{2} + \tau_{n-2}$, and therefore we obtain (using the initial value $\tau_1 = 1$)

$$\begin{aligned}
\tau_n &= \frac{251}{360n} + \sum_{\substack{k=3 \\ k \text{ odd}}}^{n} \frac{3k^2-6k+74}{180} + \left(\tau_1 - \frac{251}{360}\right) \\
&= \frac{n^4 - 110n^2 + 360n - 251}{360n}.
\end{aligned} \tag{73}$$

It is interesting that this same polynomial appears a second time in the table, giving the identity

$$d(n; 1, 1, 2, 4) = d(n; 1, 2, 2, 2) \quad (n \text{ odd}). \tag{74}$$

This is typical of a series of identities among the Dedekind sums, arising from elementary trigonometric identities. Here we use

$$\cot x \cot^3 2x - \cot^2 x \cot 2x \cot 4x = \tfrac{1}{2}(\cot^2 x - \cot^2 2x), \tag{75}$$

which gives

$$\begin{aligned} d(n;1,2,2,2) - d(n;1,1,2,4) &= -\tfrac{1}{2}\{d(n;1,1) - d(n;2,2)\} \\ &= 0. \end{aligned} \tag{76}$$

Other identities among the Dedekind sums are the following:

(I) For any integers i, j,

$$\begin{aligned} &d(p; i, i+j, a_3, \ldots, a_n) + d(p; j, i+j, a_3, \ldots, a_n) \\ &\quad = d(p; i, j, a_3, \ldots, a_n) + d(p; a_3, \ldots, a_n) \end{aligned} \tag{77}$$

(compare Eq. 9.3 (6) of [8]). This follows from the identity

$$c_i c_{i+j} + c_j c_{i+j} = c_i c_j + 1, \tag{78}$$

where we have written c_i for $(t^i+1)/(t^i-1)$, t an indeterminate. If $n=2$ then the last term $d(p;\ )$ of (77) must be set equal to $p-1$.

(II) The existence of infinitely many linear relations among the functions $c_i c_j$ (with $j \geqq 2i$, or $j \geqq \lambda i$ with any $\lambda < \pi^2/3$) is proved in [17]. Each of these gives infinitely many relations among Dedekind sums of any dimension. Thus from

$$3c_6 c_2 + 2c_3 c_1 - c_6 c_1 - 2c_4 c_2 - c_2 c_1 = 1 \tag{79}$$

we get

$$\begin{aligned} &3d(p; 6, 2, a_3, \ldots, a_n) + 2d(p; 3, 1, a_3, \ldots, a_n) - d(p; 6, 1, a_3, \ldots, a_n) \\ &\quad - 2d(p; 4, 2, a_3, \ldots, a_n) - d(p; 2, 1, a_3, \ldots, a_n) = d(p; a_3, \ldots, a_n). \end{aligned} \tag{80}$$

(III) Finally, a series of identities involving products of more than two cotangents is given by the equation

$$\begin{aligned} &2c_1 c_2 \ldots c_{2k} - 2c_1^2 c_2 \ldots c_{2k} + 2c_1^2 c_2^2 \ldots c_{2k} - \cdots \\ &\quad \cdots + 2(-1)^{k-1} c_1^2 \ldots c_{k-1}^2 c_k c_{k+1} + (-1)^k c_1^2 \ldots c_k^2 = 1. \end{aligned} \tag{81}$$

The first two cases of this, namely

$$2c_1 c_2 - c_1^2 = 1, \quad 2c_1 c_2 c_3 c_4 - 2c_1^2 c_2 c_3 + c_1^2 c_2^2 = 1, \tag{82}$$

give identities for the four-dimensional sums which can be checked using the tables at the end of the section. To prove (81), we define

$$f(y) = 1 + c_1 y + c_1 c_2 y^2 + c_1 c_2 c_3 y^3 + \cdots, \tag{83}$$

considered as a formal power series in y (recall that $c_i = (t^i+1)/(t^i-1)$).

It is easy to check that

$$f(y)=\frac{1-yt}{1+y}f(yt) \tag{84}$$

by comparing the coefficients of y^n on both sides. Moreover, (83) is the only power series with leading coefficient 1 which satisfies (84), and this leads to a proof of the two identities

$$f(y)=\frac{1}{1+y}\prod_{r=1}^{\infty}\frac{1-yt^r}{1+yt^r}, \tag{85}$$

$$f(y)=(1-y^2)^{-1/2}\,e^{c_1y+c_3y^3/3\,+\,c_5y^5/5+\cdots}, \tag{86}$$

from either of which it is clear that

$$f(y)f(-y)=(1-y^2)^{-1}. \tag{87}$$

Substituting (83) into (87) and comparing the coefficients of y^{2k} on both sides gives (81).

We consider one last property of the Dedekind sums. Inspection of the table of four-dimensional sums shows that the numbers $d(p;a,b,c,d)$ and $d(2p;a,b,c,d)$ always have the same fractional part. Of course this in itself is clear from the formulas for these fractional parts (Eqs. (68) and (69)), but in fact much more is true: the difference between the two Dedekind sums in question is not only an integer but is always divisible by p. A similar fact is also true for larger n:

Theorem. *Let p be a positive integer and $a_1,\ldots,a_n$ odd integers prime to p. Then*

$$d(2p;a_1,\ldots,a_n)-d(p;a_1,\ldots,a_n)=p\,t_p(a_1,\ldots,a_n), \tag{88}$$

where $t_p(a_1,\ldots,a_n)$ is the integer

$$\begin{aligned}&\operatorname{Card}\left\{k_1,\ldots,k_n\,|\,0<k_1,\ldots,k_n<p \text{ and } \frac{a_1k_1+\cdots+a_nk_n}{p} \text{ an even integer}\right\}\\ &-\operatorname{Card}\left\{k_1,\cdots,k_n\,|\,0<k_1,\ldots,k_n<p \text{ and } \frac{a_1k_1+\cdots+a_nk_n}{p} \text{ an odd integer}\right\}.\end{aligned} \tag{89}$$

Proof. Let

$$c_i=\operatorname{Card}\{k_1,\ldots,k_n\,|\,0<k_1,\ldots,k_n<p \text{ and } a_1k_1+\cdots+a_nk_n=i\}. \tag{90}$$

In terms of these numbers, Eq. (89) becomes

$$t_p(a_1,\ldots,a_n)=c_0-c_p+c_{2p}-c_{3p}+\cdots. \tag{91}$$

But clearly c_i is the coefficient of t^i in the polynomial

$$\begin{aligned} f(t) &= \sum_{k_1=1}^{p-1} \cdots \sum_{k_n=1}^{p-1} t^{k_1 a_1 + \cdots + k_n a_n} \\ &= \prod_{j=1}^{n} (t^{a_j} + t^{2a_j} + \cdots + t^{(p-1)a_j}) \qquad (92) \\ &= \prod_{j=1}^{n} \frac{t^{a_j} - t^{pa_j}}{1 - t^{a_j}}. \end{aligned}$$

Since $f(t)$ is a polynomial, the function $f\left(\frac{1}{t}\right)$ is meromorphic at $t=0$, so we can rewrite (91) as

$$\begin{aligned} t_p(a_1, \ldots, a_n) &= \operatorname{res}_{t=0} \left\{ (1 - t^p + t^{2p} - t^{3p} + \cdots) \, f\left(\frac{1}{t}\right) \frac{dt}{t} \right\} \\ &= \operatorname{res}_{t=0} \left\{ \frac{f\left(\frac{1}{t}\right)}{1+t^p} \, \frac{dt}{t} \right\}. \end{aligned} \qquad (93)$$

The function in brackets is rational and we can apply the residue theorem. The only poles other than $t=0$ are simple ones at $t^p = -1$, so

$$\begin{aligned} t_p(a_1, \ldots, a_n) &= -\sum_{\lambda^p = -1} \operatorname{res}_{t=\lambda} \left\{ \frac{f\left(\frac{1}{t}\right)}{1+t^p} \, \frac{dt}{t} \right\} \\ &= \frac{1}{p} \sum_{\lambda^p = -1} f\left(\frac{1}{\lambda}\right) \\ &= \frac{1}{p} \sum_{\lambda^p = -1} \prod_{j=1}^{n} \frac{\lambda^{a_j} + 1}{\lambda^{a_j} - 1} \qquad (94) \\ &= \frac{1}{p} \left\{ \sum_{\substack{\lambda^{2p}=1 \\ \lambda \neq 1}} - \sum_{\substack{\lambda^{p}=1 \\ \lambda \neq 1}} \right\} \prod_{j=1}^{n} \frac{\lambda^{a_j} + 1}{\lambda^{a_j} - 1} \\ &= \frac{1}{p} \{ d(2p; a_1, \ldots, a_n) - d(p; a_1, \ldots, a_n) \}, \qquad (95) \end{aligned}$$

as was to be proved.

The formula (94), namely

$$t_p(a_1, \ldots, a_n) = \frac{(-1)^{n/2}}{p} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2p-1} \cot \frac{\pi k a_1}{2p} \cdots \cot \frac{\pi k a_n}{2p}, \qquad (96)$$

is a special case of a more general trigonometric formula (given in full in [9]) which also yields the equation

$$\begin{aligned}
&\operatorname{Card}\left\{k_1, \ldots, k_n \,|\, 0 < k_i < a_i \;\; (i = 1, \ldots, n)\right. \\
&\qquad \left.\text{and } 2j < \frac{k_1}{a_1} + \cdots + \frac{k_n}{a_n} < 2j+1 \text{ for some } j\right\} \\
&- \operatorname{Card}\left\{k_1, \ldots, k_n \,|\, 0 < k_i < a_i \;\; (i = 1, \ldots, n)\right. \qquad (86) \\
&\qquad \left.\text{and } 2j-1 < \frac{k_1}{a_1} + \cdots + \frac{k_n}{a_n} < 2j \text{ for some } j\right\} \\
&\qquad = \frac{(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2N-1} \cot \frac{\pi k}{2N} \cot \frac{\pi k}{2a_1} \cdots \cot \frac{\pi k}{2a_n},
\end{aligned}$$

where $a_1, \ldots, a_n$ (n odd) are arbitrary positive integers and N is a common multiple of the a_i's. This formula is of interest since the number on the left-hand side is the value found by Brieskorn [19] for the signature of the $(2n-3)$-dimensional manifold

$$V = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_1^{a_1} + \cdots + z_n^{a_n} = 0, \, |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

It was this occurrence of trigonometric functions in the formula for an integer arising in a topological context that formed the original motivation of this paper.

Note Added in Proof. A book on the classical (two-dimensional) Dedekind sums has appeared recently (Rademacher-Grosswald, Dedekind sums, Math. Assoc. Amer. 1972).

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