For $r = 2$, the proposer provided an elementary proof that a 2-coloring of the integer lattice contains n points of the same color whose centroid also has that color. (Note that any n points are the vertices of a simple polygon, not necessarily convex.) Given a 2-coloring, let $a_1, \dots, a_n$ be the vectors of n red points, and let $C = (1/n)\Sigma a_i$ be their centroid. If C is blue, let $b_j = (n+1)a_j - \Sigma a_i$ for $1 \leqslant j \leqslant n$. Then a_j is the centroid of the set obtained from $\{a_1, a_2, \dots, a_n\}$ by replacing a_j by b_j. If any b_j is red, we have the desired red set. Otherwise, $\{b_j\}$ is a blue set with blue centroid C, by straightforward computation. Note that this proof yields the desired monochromatic figure within a very small grid.

Solved also by A. Bialostocki, R. J. Chapman (Great Britain), R. High, L. Piepmeyer (Germany), and B. Reznick.

Some Strange 3-adic Identities

6625 [1990, 252]. *Proposed by Nicholas Strauss, Pontificia Universidade Católica do Rio de Janeiro, Brasil, and Jeffrey Shallit, Dartmouth College.*

If k is a positive integer, let $3^{\nu(k)}$ be the highest power of 3 dividing k. Put

$$r(n) = \sum_{i=0}^{n-1} \binom{2i}{i}$$

for positive integers n. Prove that

(i) $\nu(r(n)) \geqslant 2\nu(n)$,

(ii) $\nu(r(n)) = \nu\left(\binom{2n}{n}\right) + 2\nu(n)$.

Solution by Don Zagier, University of Maryland, College Park, and Max-Planck-Institut für Mathematik, Bonn, Germany. The assertion of the problem may be stated in the form:

$$v\left(\sum_{k=0}^{n-1} \binom{2k}{k}\right) = v\left(n^2 \binom{2n}{n}\right) \quad \text{for all } n \geqslant 1; \tag{1}$$

here, and throughout this solution, $v(\cdot)$ denotes the 3-adic valuation. We give a simple proof of (1) and of various other 3-adic identities related to it.

If we set

$$f(n) = \frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}} \qquad (n \geqslant 1), \tag{2}$$

then (1) says that $f(n)$ is a 3-adic unit for all $n \in \mathbb{N}$. In fact, a calculation of the first few values suggests that in fact

$$f(n) \equiv -1 \pmod 3 \quad \forall n \tag{3}$$

and a more extensive calculation suggests the more precise congruences

$$n \equiv m \pmod{3^j} \Rightarrow f(n) \equiv f(m) \pmod{3^{j+1}}. \tag{4}$$

This says that the function $f\colon \mathbb{N} \to \mathbb{Q} \subset \mathbb{Q}_3$ extends to a 3-adic continuous map $\mathbb{Z}_3 \to -1 + 3\mathbb{Z}_3$. The range studied ($n \leqslant 2200$) permits one to check these congruences for $j \leqslant 7$ (since $3^7 < 2200$) and hence to interpolate $f(n)$ with accuracy

$O(3^8)$. The interpolated values found in this way for negative integers and half-integers are equal, to this accuracy, to simple rational numbers, suggesting the further identities

$$f(-1) = -1, \qquad f(-2) = -\frac{7}{4}, \qquad f(-3) = -4, \ldots, \tag{5}$$

$$f\left(-\frac{1}{2}\right) = -4, \qquad f\left(-\frac{3}{2}\right) = -4, \qquad f\left(-\frac{5}{2}\right) = -\frac{196}{25}, \ldots. \tag{6}$$

We now state a result which includes all of these experimental observations.

Theorem. *The function f extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1 + 3\mathbb{Z}_3$. Its values at negative integers and half-integers are rational numbers, given by*

$$f(-n) = -\frac{(2n-1)!}{n!^2} \sum_{k=0}^{n-1} \frac{k!^2}{(k-1)!} \qquad (n \geqslant 1), \tag{7}$$

$$f\left(-n-\frac{1}{2}\right) = -\frac{2^{4n+2}}{(2n+1)^2\binom{2n}{n}} \sum_{k=0}^{n} 2^{-4k}\binom{2k}{k} \qquad (n \geqslant 0). \tag{8}$$

As a corollary, we get the identities analogous to (1)

$$v\left(\sum_{k=0}^{n-1} \frac{k!^2}{(2k+1)!}\right) = v\left(\frac{n!^2}{(2n-1)!}\right) \qquad (n \geqslant 1), \tag{9}$$

$$v\left(\sum_{k=0}^{n} 2^{-4k}\binom{2k}{k}\right) = v\left((2n+1)^2\binom{2n}{n}\right) \qquad (n \geqslant 0). \tag{10}$$

Proof: Equation (2) implies that $f(n)$ satisfies the recursion relation

$$(2n+1)(2n+2)f(n+1) = 1 + n^2 f(n) \tag{11}$$

for $n \in \mathbb{N}$. If f has an extension to a 3-adic continuous function from $\mathbb{Z}_3$ to $\mathbb{Z}_3$, then this functional equation must hold for all $n \in \mathbb{Z}_3$. Since the left-hand side vanishes at $n = -1$ and $n = -1/2$, we must have $f(-1) = -1$ and $f(-1/2) = -4$; the further values in (7) and (8) then follow by induction on n using the functional equation (11). Thus we need only prove the first statement of the theorem.

Set $g(n) = 2nf(n)$; we show first that g extends to a 3-adic analytic function of n, and then that $g(x)$ is divisible by x. For g the recursion (11) becomes

$$2(2n+1)g(n+1) = 2 + ng(n). \tag{12}$$

Define rational numbers $a_0 = 1, a_1 = -1/2, \ldots$ by requiring that

$$g(n) = \sum_{r=0}^{\infty} a_r \binom{n-1}{r} \tag{13}$$

for $n = 1, 2, \ldots$ (note that the sum is finite for each n). If we show that $v(a_r) \to \infty$ as $r \to \infty$, then (13) will converge 3-adically for all $n \in \mathbb{Z}_3$ and give the desired continuation. Substituting (13) into (12) gives

$$2 + \sum_{r=0}^{n-1} (r+1)a_r\binom{n}{r+1} = \sum_{r=0}^{n} \left[2(2r+1)\binom{n}{r} + 4(r+1)\binom{n}{r+1}\right]a_r.$$

Comparing coefficients of $\binom{n}{r}$ in this gives $2(2r+1)a_r = -3ra_{r-1}$ for $r \geqslant 1$, whence

$$a_r = \frac{(-3)^r r!^2}{(2r+1)!} \qquad (r \geqslant 0). \tag{14}$$

The 3-adic valuation of this does indeed tend to infinity with r (since $v(3^r/(2r+1)!) \geqslant 0$ and $v(r!) \to \infty$), so (13) gives the analytic continuation of g.

Lemma. *The series* $\sum_{r=0}^{\infty}(3^r r!^2/(2r+1)!)$ *converges* 3*-adically to* 0.

We will prove the lemma in a moment. Assuming it, we find

$$\begin{aligned} g(n) &= \sum_{r=0}^{n-1}(-3)^r \frac{r!}{(2r+1)!}(n-1)(n-2)\cdots(n-r) \\ &= \sum_{r=0}^{n-1}\frac{3^r r!^2}{(2r+1)!} - \frac{1}{2}n \\ &\quad + \sum_{r=2}^{n-1}(-3)^r \frac{r!}{(2r+1)!}\big[(n-1)(n-2)\cdots(n-r) - (-1)^r r!\big]. \end{aligned} \tag{15}$$

By the lemma, the first term in (15) has valuation

$$v\left(\sum_{r=0}^{n-1}\frac{3^r r!^2}{(2r+1)!}\right) = v\left(\sum_{r=n}^{\infty}\frac{3^r r!^2}{(2r+1)!}\right) \geqslant 2\frac{n-2}{3} \geqslant v(n) + 1 \qquad (n \geqslant 4)$$

since $v((3^r r!^2)/(2r+1)!) \geqslant 2v(r!) \geqslant 2(r-2)/3$ for all r. Also,

$$(n-1)(n-2)\cdots(n-r) - (-1)^r r!$$

is divisible by n and $(-3)^r r!/(2r+1)!$ is divisible by 3 for all $r \geqslant 2$, so (15) gives

$$g(n) \equiv -\tfrac{1}{2}n \pmod{3^{v(n)+1}},$$

whence $f(n) = g(n)/2n$ is 3-integral and congruent to -1 modulo 3. Thus the theorem is proved.

Proof of Lemma: We have the power series identity

$$\begin{aligned} \sum_{r=0}^{\infty}\frac{r!^2}{(2r+1)!}x^r &= \sum_{r=0}^{\infty}\left(\int_0^1 t^r(1-t)^r\,dt\right)x^r \quad \text{(beta integral)} \\ &= \int_0^1 \frac{dt}{1+xt+xt^2} \\ &= \frac{1}{\sqrt{x^2-4x}}\log\frac{2-x+\sqrt{x^2-4x}}{2-x-\sqrt{x^2-4x}} \\ &= \frac{1}{3\sqrt{x^2-4x}}\log\frac{\left(2-x+\sqrt{x^2-4x}\right)^3/4}{\left(2-x-\sqrt{x^2-4x}\right)^3/4} \\ &= \frac{1}{3\sqrt{x^2-4x}}\log\frac{2-x(3-x)^2+(3-x)(1-x)\sqrt{x^2-4x}}{2-x(3-x)^2-(3-x)(1-x)\sqrt{x^2-4x}} \\ &= \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{2n+1}\,\frac{x^n(x-4)^n(3-x)^{2n+1}(1-x)^{2n+1}}{\left[2-x(3-x)^2\right]^{2n+1}} \end{aligned}$$

in $\mathbb{Q}[[x]]$. Both sides converge 3-adically if $v(x) > 0$, and the right-hand side vanishes for $x = 3$. This completes the proof of the lemma.

Finally, we remark that the computer calculations to $n = 2200$ suggested the further congruence

$$n \equiv m \equiv 0 \pmod{3^j} \Rightarrow f(n) \equiv f(m) \pmod{3^{2j+1}},$$

analogous to (4). If true, this says that the derivative of f at 0 vanishes. From what we have done we find that the Taylor series of f around the origin is given by

$$\begin{aligned} f(n) &= \frac{1}{2n} \sum_{r=0}^{\infty} \frac{3^r r!^2}{(2r+1)!} \left[(1-n)\left(1 - \frac{n}{2}\right) \cdots \left(1 - \frac{n}{r}\right) - 1 \right] \\ &= A + Bn + Cn^2 + \cdots \end{aligned}$$

with

$$A = -\frac{1}{2} \sum_{r=1}^{\infty} \frac{3^r r!^2}{(2r+1)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right),$$

$$B = \frac{1}{2} \sum_{r=2}^{\infty} \frac{3^r r!^2}{(2r+1)!} \sigma_2\left(1, \frac{1}{2}, \ldots, \frac{1}{r}\right),$$

etc. (σ_2 = second elementary symmetric function). The assertion that $f'(0)$ vanishes is thus equivalent to the following statement, which is similar to but more complicated than our lemma above:

Conjecture. *The series* $\sum_{r=0}^{\infty}((3^r r!^2)/(2r+1)!)\sigma_2(1, 1/2, \ldots, 1/r)$ *converges* 3-*adically to* 0.

Another interesting problem would be to evaluate in closed form the 3-adic number A. To thirty 3-adic digits, A equals $\ldots 110000102110002221022212000212$.

Part (i) was solved also by Derek Hacon and Nicholas Strauss.
Part (ii) was solved also by Jean-Paul Allouche and Jeffrey Shallit.

A Convergent Sequence

E 3388 [1990, 428]. *Proposed by Matthew Cook (student), University of Illinois, Urbana, IL, Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and Marcin E. Kuczma, University of Warsaw, Warsaw, Poland.*

Let x_1 and x_2 be arbitrary positive numbers. Suppose we define a sequence $\{x_n\}_{n=1}^{\infty}$ by putting $x_{n+2} = 2/(x_{n+1} + x_n)$ for $n = 1, 2, 3, \ldots$. Prove that the sequence converges.

Solution by David Borwein, University of Western Ontario, London, Ontario, Canada. We first prove that the sequence is bounded. If both x_n and x_{n-1} are between a^{-1} and a, then $a^{-1} \leqslant (x_n + x_{n-1})/2 \leqslant a$, so x_{n+1} is between the same bounds.

Now let $l = \liminf x_n$ and $L = \limsup x_n$. Since L is finite, for any $\varepsilon > 0$ there is an integer n_0 such that $x_n < L + \varepsilon$ for $n > n_0$. Hence $x_{n+2} = 2/(x_{n+1} + x_n) > 1/(L + \varepsilon)$ for $n > n_0$. It follows that $l \geqslant 1/L > 0$. Similarly, $x_n > l - \varepsilon > 0$ for $n > n_1$ implies $x_{n+2} < 1/(l - \varepsilon)$ for $n > n_1$, whence $L \leqslant 1/l$. Therefore $l = 1/L$.