

## Lower bounds of heights of points on hypersurfaces

by

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*To J. W. S. Cassels with respect and admiration*

**1. Introduction.** Let us first recall Lehmer's conjecture [Le] on lower bounds for the height of an algebraic number which was stated in 1933. Let  $K$  be an algebraic number field of degree  $D$  over  $\mathbb{Q}$ . For any valuation  $v$  we define  $D_v = [K_v : \mathbb{Q}_v]$ , where  $K_v, \mathbb{Q}_v$  are the completions of  $K, \mathbb{Q}$  with respect to  $v$ . For archimedean  $v$  we normalise the valuation by  $|x|_v = |x|^{D_v/D}$  where  $|\cdot|$  is the ordinary complex absolute value. When  $v$  is non-archimedean we take  $|p|_v = p^{-D_v/D}$  where  $p$  is the unique rational prime such that  $|p|_v < 1$ . The height of an algebraic number  $\alpha \in K$  is defined by

$$H(\alpha) = \prod_v \max(1, |x|_v).$$

Because of our normalisation  $H(\alpha)$  does not depend on the choice of the field  $K$  in which  $\alpha$  is contained. We can now state Lehmer's conjecture.

CONJECTURE 1.1. *There exists a number  $c > 1$  such that for any algebraic number  $\alpha$ , not a root of unity and of degree  $D$  we have*

$$H(\alpha)^D \geq c.$$

*Presumably  $c = 1.1762808\dots$ , which is the larger real root of  $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ .*

The best unconditional result so far follows from work of Dobrowolski, Cantor & Straus and Louboutin [Lo], stating that there exists  $\gamma > 0$  such that

$$H(\alpha)^D \geq 1 + \gamma \left( \frac{\log \log D}{\log D} \right)^3.$$

It came as a great surprise when S. Zhang [Zh] showed in 1992 that there does exist a number  $c_1 > 1$  such that

$$H(\alpha)H(1 - \alpha) \geq c_1$$

for all  $\alpha \in \overline{\mathbb{Q}}$  such that  $\alpha \neq 0, 1, \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ . This was proved by using Arakelov intersection theory on  $\mathbb{P}^1$ . It was almost immediately realised by one of us (see [Za]) that an elementary proof could be given which at the same time yields the best possible  $c_1$ , namely  $\sqrt{\eta}$  where  $\eta = (1 + \sqrt{5})/2$ , the golden ratio. The minimum is attained when  $\alpha$  is minus a fifth root of unity. In [Za] there is also a generalisation of the following sort. For any  $K$ -rational point  $P = (P_0 : P_1 : \dots : P_n)$  in  $n$ -dimensional projective space  $\mathbb{P}^n$  we define the height by

$$H(P) = \prod_v \max(|P_0|_v, \dots, |P_n|_v).$$

In particular, the height of an algebraic number  $\alpha$  is nothing but the projective height of  $(1 : \alpha) \in \mathbb{P}^1(K)$ . Then it is shown in [Za] that for any  $(x_0 : x_1 : x_2) \in \mathbb{P}^2(\overline{\mathbb{Q}})$  such that  $x_0 + x_1 + x_2 = 0$ ,  $x_0 x_1 x_2 \neq 0$  and  $(x_0 : x_1 : x_2) \neq (1 : \omega^{\pm 1} : \omega^{\mp 1})$  ( $\omega^3 = 1$ ), we have

$$H(x_0, x_1, x_2) \geq c_2$$

where  $c_2$  is the larger real root of  $x^6 - x^4 - 1$ . The minimum is attained when the  $x_i$  are the roots of  $x^3 + x - 1$ .

Inspired by [Za], H. P. Schlickewei and E. Wirsing [SW] showed the following result. Consider the line  $L : \lambda x + \mu y + \nu z = 0$  in  $\mathbb{P}^2$  with  $\lambda\mu\nu \neq 0$ . Suppose that  $\lambda + \mu + \nu = 0$ . Then, for any two points  $P_1, P_2 \in L(\overline{\mathbb{Q}})$  with non-zero coordinates and such that  $(1 : 1 : 1), P_1, P_2$  are distinct, we have

$$H(P_1)H(P_2) > \exp(1/2400) = 1.00041 \dots$$

This result was applied by Schlickewei [Schl] to estimating numbers of solutions of three term  $S$ -unit equations in a strikingly successful way. Although very useful, the derivation of the Schlickewei–Wirsing result did not look optimal. It is the goal of this paper to remedy this situation and also give a generalisation which encompasses the previous results. We finish the introduction by giving a description of our general setup and main result.

Consider a hypersurface  $S$  of multidegree  $d_1, \dots, d_r$  on  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$  given by a polynomial  $F$  with coefficients in  $\mathbb{Z}$ . Denote the coordinates of  $\mathbb{P}^{n_i}$  by  $\mathbf{x}_i = (x_{i0}, x_{i1}, \dots, x_{in_i})$ . The degree of  $F$  in the variable  $x_{ij}$  is denoted by  $d_{ij}$ . We define  $\tilde{d}_i = -d_i + \sum_j d_{ij}$ .

Choose a subset  $I$  of  $\{i : n_i = 1\}$  and let  $E$  be the set  $\{(i, 0) : i \in I\}$ , to which we refer as *exceptional index pairs*. For any polynomial with coefficients in  $\mathbb{Z}$  we denote by  $\|P\|$  the sum of the absolute values of the coefficients. We define

$$c_{ij} = \left\| \frac{\partial F}{\partial x_{ij}} \right\|, \quad c_F = \max_{(i,j) \notin E} c_{ij}.$$

The advantage of having the exceptional set  $E$  is that the value of  $c_F$  may

be smaller than the one we would get by taking the maximum over all pairs  $(i, j)$ . In the first example in [Za] this enables us to get the optimal lower bound for the product  $H(\alpha)H(1 - \alpha)$ . By  $\delta$  we denote the maximum of the numbers  $\max_{i \in I} (\tilde{d}_i + d_{i,1})/2$  and  $\max_{i \notin I} \tilde{d}_i/(n_i + 1)$ .

**THEOREM 1.2.** *For each point  $(\mathbf{x}_1, \dots, \mathbf{x}_r) \in \mathbb{P}^{n_1}(\overline{\mathbb{Q}}) \times \dots \times \mathbb{P}^{n_r}(\overline{\mathbb{Q}})$  such that  $F(x_{ij}) = 0$ ,  $x_{ij} \neq 0$  for all  $i, j$  and  $F(x_{ij}^{-1}) \neq 0$  we have*

$$H(\mathbf{x}_1)^{n_1+1} \dots H(\mathbf{x}_r)^{n_r+1} \geq \varrho,$$

where  $\varrho$  is the unique real root larger than 1 of  $x^{-2} + c_F^{-1}x^{-\delta} = 1$ .

During the preparation of this paper W. M. Schmidt informed us that in [Schm] he had already proved a theorem very similar to ours in the case where one works in  $(\mathbb{P}^1)^r$ . The logarithm of the lower bound given in [Schm] is  $1/(2^{4f+2r}H)$ , where  $f$  is the total degree of  $F$  and  $H$  the maximum of all coefficients. Although the basic starting point in this paper and [Schm] is the same, we nevertheless found that the principle of our approach and the better values of the constants have some interest.

**2. Applications.** Before proving the theorem we describe a few consequences. First of all consider  $r$  algebraic numbers  $\alpha_1, \dots, \alpha_r$  whose sum is a rational integer  $N$ . We like to interpret the  $r$ -tuple as a point  $(1 : \alpha_1) \times \dots \times (1 : \alpha_r) \in (\mathbb{P}^1)^r$ . For the set  $I$  of our theorem we choose  $\{1, \dots, r\}$ . Letting  $F$  be the homogeneous version of  $x_1 + \dots + x_r - N$  one easily checks that  $c_i = 1$  for all  $i$ . Note that the coefficient  $N$  in  $F$  does not appear in the  $c_i$  because of our choice of  $I$ . So we get  $c_F = 1$ . Moreover,  $n_i = 1$  and  $d_i = 1$  for all  $i$ . Hence  $\delta = 1$ . Thus we find

**COROLLARY 2.1.** *Let  $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}^*$  and  $N \in \mathbb{Z}$  be such that  $\alpha_1 + \dots + \alpha_r = N$  and  $\alpha_1^{-1} + \dots + \alpha_r^{-1} \neq N$ . Then*

$$H(\alpha_1) \dots H(\alpha_r) \geq \sqrt{\eta}$$

where  $\eta$  is the golden ratio.

Note that when  $r \geq 4$  the lower bound is actually attained for the  $r$ -tuple  $-\zeta_5, 1 + \zeta_5, 1, \zeta_{r-2}, \dots, \zeta_{r-2}^{r-3}$  where  $\zeta_k$  denotes a primitive  $k$ th root of unity. When we take for the  $\alpha_i$  the conjugates of an algebraic number  $\alpha$  of degree  $D$  we get the following consequence.

**COROLLARY 2.2.** *Let  $\alpha \in \overline{\mathbb{Q}}^*$  be such that  $\text{trace}(\alpha)$  is integral and  $\text{trace}(\alpha) \neq \text{trace}(\alpha^{-1})$ . Then  $H(\alpha)^D \geq \sqrt{\eta}$ .*

However, this result is already contained in a result of C. Smyth [Sm] which states that  $H(\alpha)^D \geq \theta$  for every non-reciprocal  $\alpha \in \overline{\mathbb{Q}}^*$ . Here  $\theta$  is the real root of  $x^3 - x - 1$ .

We now consider  $r$  algebraic numbers  $\alpha_i$  whose sum is 1 and give a lower bound for  $H(1, \alpha_1, \dots, \alpha_r)$ . The polynomial  $F$  can be written as  $x_1 + \dots + x_r - x_0$  and we have  $c_F = 1$ ,  $d_1 = 1$ . Furthermore,  $\delta = r/(r + 1)$ .

**COROLLARY 2.3.** *For any  $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}^*$  such that  $\alpha_1 + \dots + \alpha_r = 1$  and  $\alpha_1^{-1} + \dots + \alpha_r^{-1} \neq 1$  we have*

$$H(1, \alpha_1, \dots, \alpha_r) \geq \varrho$$

where  $\varrho$  is the real root larger than 1 of  $1 = x^{-2r-2} + x^{-r}$ .

As pointed out in the introduction, this result is optimal when  $r = 2$ . For  $r > 2$  this is not true any more. When  $r = 3$  for example we find the lower bound 1.14613... (which improves the bound  $\exp(1/402) = 1.00249\dots$  from [SW]). However, the lowest height we could find was  $H = 1.15096\dots$  when the  $\alpha_i$  are the zeros of  $x^3 - x^2 + 1$ . On the other hand, the asymptotic behaviour of  $\varrho$  as a function of  $r$  looks optimal. It is not hard to show that  $\varrho^{r+1} \rightarrow \eta$  as  $r \rightarrow \infty$  while on the other hand the zeros  $\alpha_0, \dots, \alpha_r$  of  $x^{r+1} - x - 1$  satisfy  $H(\alpha_0, \dots, \alpha_r)^{r+1} \rightarrow 2$  as  $r \rightarrow \infty$ .

We now consider the Schlickewei–Wirsing result. Suppose we have a line  $L : \lambda x + \mu y + \nu z = 0$  in  $\mathbb{P}^2$  with  $\lambda\mu\nu \neq 0$ . Let  $P_1, P_2, P_3 \in L(\overline{\mathbb{Q}})$  be three distinct points with non-zero coordinates. Letting  $P_i = (x_i : y_i : z_i)$  ( $i = 1, 2, 3$ ) we get the relation

$$\Delta := \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

We want a lower bound of  $H(P_1)H(P_2)H(P_3)$ . Our polynomial  $F$  is now the determinant form  $\Delta$ . First we point out that

$$\tilde{\Delta} := \begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} \neq 0.$$

Suppose  $\tilde{\Delta} = 0$ . Then there exist  $\alpha, \beta, \gamma$ , not all zero, such that  $\alpha x_i^{-1} + \beta y_i^{-1} + \gamma z_i^{-1} = 0$  for  $i = 1, 2, 3$ . Hence  $\alpha y_i z_i + \beta z_i x_i + \gamma x_i y_i = 0$  ( $i = 1, 2, 3$ ). The conic  $C : \alpha yz + \beta zx + \gamma xy = 0$  is reducible if and only if  $\alpha\beta\gamma = 0$ . So, if  $\gamma = 0$  for example, we get  $\alpha x_i + \beta y_i = 0$  for  $i = 1, 2, 3$ . But this contradicts  $\nu \neq 0$ . So  $C$  is an irreducible conic. But then  $P_1, P_2, P_3$  lie both on  $C$  and  $L$ , which is impossible since  $|C \cap L| \leq 2$ . We conclude that  $\tilde{\Delta} \neq 0$ . We can now apply our theorem with  $r = 3$ ,  $n_1 = n_2 = n_3 = 2$ ,  $d_1 = d_2 = d_3 = 1$ ,  $c_F = 2$  and  $I = \emptyset$ .

**COROLLARY 2.4.** *Consider the line  $L : \lambda x + \mu y + \nu z = 0$  in  $\mathbb{P}^2$  with  $\lambda\mu\nu \neq 0$ . Let  $P_1, P_2, P_3 \in L(\overline{\mathbb{Q}})$  be three distinct points with non-zero coordinates.*

Then

$$H(P_1)H(P_2)H(P_3) \geq \varrho,$$

where  $\varrho$  is the real root larger than 1 of  $1 = \varrho^{-6} + (1/2)\varrho^{-2}$ .

The numerical value of  $\varrho$  is 1.09427..., which compares favourably with the value 1.00041... from [SW] or 1.019... from [Schm]. Moreover, this result was applied successfully to equations of the form  $x + y = 1$  with  $x, y$  unknowns in a finitely generated multiplicative group and to multiplicity estimates for binary recurrences in [BS].

**3. Proof of Theorem 1.2.** The proof is based on the following observation, which is a direct generalisation of [Za]. Let  $X$  be a closed subvariety of  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$  defined over  $\mathbb{Q}$ . We denote the coordinates by  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_r)$  with  $\mathbf{x}_i = (x_{i0}, \dots, x_{in_i})$ . Denote by  $X(\mathbb{C})_1$  the intersection of  $X(\mathbb{C})$  with the polydisc  $\{\mathbf{x} : |x_{ij}| \leq 1 \ \forall i, j\}$ . We also give ourselves a collection of multihomogeneous polynomials  $G_k(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$  of multidegrees  $(d_{k1}, \dots, d_{kr})$ .

LEMMA 3.1. *Let  $\nu_k \geq 0$  for all  $k$  and set*

$$(1) \quad w_i = \sum_k \nu_k d_{ki}, \quad \lambda = - \max_{\mathbf{x} \in X(\mathbb{C})_1} \left\{ \sum_k \nu_k \log |G_k(\mathbf{x})| \right\}.$$

Then for any point  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_r) \in X(\overline{\mathbb{Q}})$  with  $\prod_k G_k(\mathbf{x}) \neq 0$  we have

$$(2) \quad \prod_{i=1}^r H(\mathbf{x}_i)^{w_i} \geq e^\lambda.$$

Proof. Suppose that  $\mathbf{x} \in X(K)$  with  $G_k(\mathbf{x}) \neq 0$  for all  $k$ . Here  $K$  is an algebraic number field of degree  $D$ , say. For any valuation  $v$  of  $K$  we let  $D_v = [K_v : \mathbb{Q}_v]$ . Then the inequality

$$\sum_{i=1}^r w_i \log(\max_j |x_{ij}|_v) \geq \sum_k \nu_k \log |G_k(\mathbf{x})|_v + \begin{cases} \frac{D_v}{D} \lambda & \text{if } v \mid \infty, \\ 0 & \text{if } v \nmid \infty, \end{cases}$$

holds for all places  $v$  of  $K$ , because by the homogeneity condition (1) we may assume that  $\max_j |x_{ij}|_v = 1$  for all  $i$  and the inequality follows from the definition of  $\lambda$  if  $v$  is infinite and is straightforward if  $v$  is finite. The lemma follows by summing over all  $v$  and using the product formula. ■

The following lemma saves us a considerable amount of effort in the determination of  $\lambda$  for the sake of the previous lemma.

LEMMA 3.2. *With notations as above, the function  $\Psi := \sum \nu_k \log |G_k(\mathbf{x})|$  assumes a maximum in  $\mathbf{x} \in X(\mathbb{C})_1$  and it is attained at a point all of whose coordinates have absolute value 1 with at most one exception.*

Proof. Since the  $\nu_k$  are positive,  $\Psi$  is bounded from above in  $X(\mathbb{C})_1$ . For  $\varepsilon > 0$  sufficiently small the set  $\mathbf{x} \in X(\mathbb{C})_1$  such that  $\Psi(\mathbf{x}) \geq \log(\varepsilon)$  is

compact and not empty. Hence it is clear that  $\Psi$ , being continuous, assumes a maximum.

Now suppose that  $\Psi$  assumes a maximum at a point  $P$  where at least two coordinates have absolute value  $< 1$ . Call these coordinates  $\xi, \eta$  and denote the values of these coordinates at  $P$  by  $\xi_0, \eta_0$ . Substitute in  $F = 0$  the values of all coordinates of  $P$  except  $\xi, \eta$ . The equation  $F = 0$  reduces to the equation of a curve  $f(\xi, \eta) = 0$  containing the point  $\xi_0, \eta_0$ . By choosing a branch of  $f = 0$  at the point  $\xi_0, \eta_0$  we find locally analytic functions  $\xi(t), \eta(t)$  such that  $\xi(0) = \xi_0, \eta(0) = \eta_0$  and  $f(\xi(t), \eta(t)) = 0$  identically in a neighbourhood of  $t = 0$ . When  $f$  was identically zero anyway, we can choose  $\xi(t), \eta(t)$  arbitrarily. Choose a disk  $D$  in the complex  $t$ -plane around 0 such that  $|\xi(t)|, |\eta(t)| \leq 1$  for all  $t \in D$ . Specialise the arguments in  $\Psi$  to the values of the point  $P$  except for  $\xi$  and  $\eta$  where we substitute  $\xi(t)$  and  $\eta(t)$ . In this way we obtain a function  $\psi(t)$  in  $t \in D$  which assumes a maximum at  $t = 0$ . Notice that  $\psi(t)$  is harmonic in the real and imaginary part of  $t$ . A harmonic function assuming a maximum in the interior of its domain is necessarily constant. Hence  $\psi(t)$  is constant. But in that case we can continue  $\xi(t)$  and  $\eta(t)$  analytically until either one of them hits the unit circle. At that new point the value of  $\Psi$  is again  $\psi(0)$ , i.e. maximal. We continue this procedure for other coordinates, if necessary, until we have found an optimal point all of whose coordinates have absolute value one with at most one exception. ■

LEMMA 3.3. *Let  $\alpha, \beta, \gamma > 0$ . Let  $m$  be the unique minimum of the function*

$$u \log \frac{\gamma u}{u+v} + v \log \frac{v}{u+v}$$

*under the constraints  $u, v \geq 0, \alpha u + \beta v = 1$ . Then  $e^{-m}$  is the unique real root larger than 1 of  $\gamma^{-1}x^{-\alpha} + x^{-\beta} = 1$ .*

Proof. Put  $x = v/(u+v)$  and  $1-x = u/(u+v)$ . Then

$$u = \frac{1-x}{\beta x + \alpha(1-x)}, \quad v = \frac{x}{\beta x + \alpha(1-x)}$$

and  $x \in [0, 1]$ . We must minimize

$$f(x) = \frac{(1-x) \log(\gamma(1-x)) + x \log x}{\beta x + \alpha(1-x)}$$

on  $[0, 1]$ . Differentiate with respect to  $x$ ,

$$f'(x) = \frac{-\beta \log(\gamma(1-x)) + \alpha \log x}{(\beta x + \alpha(1-x))^2}.$$

This vanishes if  $(\gamma(1-x))^\beta = x^\alpha$ . Since  $x$  is strictly increasing and  $1-x$  strictly decreasing there is a unique solution  $x_0 \in ]0, 1[$ . Choose  $\varrho > 0$  such that  $x_0 = \varrho^{-\beta}$ . Then  $\gamma(1-x) = \varrho^{-\alpha}$  and thus we see that  $\varrho$  satisfies

$1 - \varrho^{-\beta} = \gamma^{-1} \varrho^{-\alpha}$ . It remains to verify that  $f(x_0) = -\log \varrho$ , which is straightforward. ■

**Proof of Theorem 1.2.** We apply Lemma 3.1 to the hypersurface  $X$  given by the multihomogeneous polynomial  $F(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$  with multidegrees  $d_1, \dots, d_r$ . For the  $G_k$  we take the coordinates  $x_{ij}$  and the function

$$\tilde{F}(\mathbf{x}) = F(x_{ij}^{-1}) \prod x_{ij}^{d_{ij}}$$

where  $d_{ij}$  is the degree of  $F$  in  $x_{ij}$ . Let  $\mu, \nu_{ij} \geq 0$ . Let  $\Phi(\mathbf{x})$  be the function  $\mu \log |\tilde{F}(\mathbf{x})| + \sum_{i,j} \nu_{ij} \log |x_{ij}|$  on  $X(\mathbb{C})$ . Let  $\tilde{d}_i = -d_i + \sum_j d_{ij}$  be the degree of  $\tilde{F}$  in  $\mathbf{x}_i$  and suppose

$$w_i = \mu \tilde{d}_i + \sum_j \nu_{ij}, \quad \lambda = - \max_{\mathbf{x} \in X(\mathbb{C})_1} \Phi(\mathbf{x}).$$

Then Lemma 3.1 states that (2) holds for all  $\mathbf{x} \in X(\overline{\mathbb{Q}})$  with  $x_{ij} \neq 0$  and  $F(x_{ij}^{-1}) \neq 0$ .

Let us take  $w_i = n_i + 1$  for all  $i$ . Although there are many other choices for the weights  $w_i$ , this choice gives us the particularly simple shape of our main theorem. It remains to choose  $\mu, \nu_{ij}$  in such a way that  $\lambda$  becomes positive and as large as possible. We choose

$$\nu_{ij} = 1 - \frac{\tilde{d}_i}{n_i + 1} \mu \quad \text{if } i \notin I$$

and

$$\nu_{i,0} = 1 - \frac{\tilde{d}_i - d_{i,1}}{2} \mu, \quad \nu_{i,1} = 1 - \frac{\tilde{d}_i + d_{i,1}}{2} \mu \quad \text{if } i \in I.$$

Let us determine  $\max_{\mathbf{x} \in X(\mathbb{C})_1} \Phi(\mathbf{x})$ . By Lemma 3.2 this maximum is attained at a point all of whose coordinates, with possibly one exception, lie on the unit circle. Suppose that  $|x_{i_0 j_0}| \leq 1$  and that  $|x_{ij}| = 1$  for all  $(i, j) \neq (i_0, j_0)$ . Suppose first that  $(i_0, j_0) \notin E$ . Then

$$\begin{aligned} |\tilde{F}(x_{ij})| &= |F(x_{ij}^{-1})| \cdot \left| \prod_{i,j} (x_{ij})^{d_{ij}} \right| = |F(x_{i_0 j_0}^{-1}, \bar{x}_{ij})| \cdot |x_{i_0 j_0}|^{d_{i_0 j_0}} \\ &= |F(x_{i_0 j_0}^{-1}, \bar{x}_{ij}) - F(\bar{x}_{ij})| \cdot |x_{i_0 j_0}|^{d_{i_0 j_0}} \\ &\leq c_{i_0 j_0} |x_{i_0 j_0}^{-1} - \bar{x}_{i_0 j_0}| \max(|x_{i_0 j_0}|^{-1}, |x_{i_0 j_0}|)^{d_{i_0 j_0} - 1} \cdot |x_{i_0 j_0}|^{d_{i_0 j_0}} \\ &= c_{i_0 j_0} (1 - |x_{i_0 j_0}|^2). \end{aligned}$$

Put  $|x_{i_0 j_0}|^2 = \xi$ . We see that the maximum of  $\Phi$  is

$$\max_{\xi \in [0,1]} [\mu \log(c_{i_0 j_0} (1 - \xi)) + (\nu_{i_0 j_0} / 2) \log \xi].$$

This maximum is attained at  $\xi = \nu_{i_0 j_0} / (\nu_{i_0 j_0} + 2\mu)$  and its value is

$$\mu \log \frac{2\mu c_{i_0 j_0}}{\nu_{i_0 j_0} + 2\mu} + \frac{\nu_{i_0 j_0}}{2} \log \frac{\nu_{i_0 j_0}}{\nu_{i_0 j_0} + 2\mu}.$$

Since we have  $\nu_{i_0 j_0} \geq 1 - \delta\mu$ , this maximum is bounded above by

$$(M) \quad \mu \log c_F \frac{2\mu}{(1 - \delta\mu) + 2\mu} + \frac{1 - \delta\mu}{2} \log \frac{1 - \delta\mu}{(1 - \delta\mu) + 2\mu}.$$

We now determine the maximum when  $(i_0, j_0) \in E$ . In particular,  $j_0 = 0$ . So suppose we have  $|x_{i_0, 0}| \leq 1$  and  $|x_{ij}| = 1$  for all other  $i, j$ . Writing down the dependence on  $x_{i_0, 0}, x_{i_0, 1}$  explicitly and putting  $z = x_{i_0, 0} / x_{i_0, 1}$ , we find

$$\begin{aligned} |\tilde{F}(x_{ij})| &= |F(x_{i_0, 0}^{-1}, x_{i_0, 1}^{-1}, x_{ij}^{-1})| \cdot |x_{i_0, 0}|^{d_{i_0, 0}} = |F(1, z, x_{ij}^{-1})| \\ &= |F(1, z, \bar{x}_{ij}) - F(1, 1/\bar{z}, \bar{x}_{ij})| \\ &\leq c_{i_0, 1} |z - 1/\bar{z}| \max(|z|, |z|^{-1})^{d_{i_0, 1} - 1} \\ &= c_{i_0, 1} |1 - |z|^2| \cdot |z|^{-d_{i_0, 1}}. \end{aligned}$$

Put  $\xi = |z|^2 = |x_{i_0, 0}|^2$ . We see that the maximum of  $\Phi$  is

$$\max_{\xi \in [0, 1]} [\mu \log(c_{i_0, 1}(1 - \xi)) - (d_{i_0, 1}\mu/2) \log |\xi| + (\nu_{i_0, 0}/2) \log |\xi|],$$

which equals

$$\mu \log \frac{2c_{i_0, 1}\mu}{\tilde{\nu} + 2\mu} + \frac{\tilde{\nu}}{2} \log \frac{\tilde{\nu}}{\tilde{\nu} + 2\mu}$$

where  $\tilde{\nu} = \nu_{i_0, 0} - d_{i_0, 1}\mu/2$ . Note that by our choice of  $\nu_{i_0, 0}$ ,

$$\tilde{\nu} = 1 - (\tilde{d}_{i_0} + d_{i_0, 1})\mu/2 \geq 1 - \delta\mu.$$

Hence our maximum is again bounded by (M). Now use Lemma 3.3 with  $\alpha = \delta, \beta = 2, \gamma = c_F$  to minimize (M) by letting  $\mu$  vary. The assertion of our theorem follows immediately. ■

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