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# Congruences among generalized Bernoulli numbers

by

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For a Dirichlet character $\chi$ modulo $M$, the *generalized Bernoulli numbers* $B_{m,\chi} \in \mathbb{Q}(\chi(1), \chi(2), \ldots)$ $(m = 0, 1, \ldots)$ are defined by the generating function

$$\sum_{a=1}^{M} \frac{\chi(a)te^{at}}{e^{Mt}-1} = \sum_{m=0}^{\infty} B_{m,\chi} \frac{t^m}{m!}. \tag{1}$$

The main interest of these numbers is that they give the values at negative integers of Dirichlet $L$-series: if $L(s,\chi) = \sum_{n\ge1} \chi(n)n^{-s}$ $(\Re(s) > 1)$ is the $L$-series attached to $\chi$, then we have the formula

$$L(1-m,\chi) = -\frac{B_{m,\chi}}{m} \quad (m \ge 1). \tag{2}$$

The number $B_{0,\chi}$ equals $\varphi(M)/M$ ($\varphi$ is Euler's phi-function) if $\chi$ is the principal character and 0 otherwise. If $m \ge 1$, then $B_{m,\chi} = 0$ if $\chi(-1) = (-1)^{m-1}$ (unless $M = m = 1$). For $m > 1$ the converse is also true, by (2) and the functional equation of $L(s,\chi)$, but we will not use this.

We are going to study some objects related to quadratic characters. Let $d$ be the discriminant of a quadratic field, and denote by $\chi_d = \left(\frac{d}{\cdot}\right)$ the associated quadratic character (Kronecker symbol). The numbers $B_{m,\chi_d}/m$ are always integers unless $d = -4$ or $d = \pm p$, where $p$ is an odd prime number such that $2m/(p-1)$ is an odd integer, in which case they have denominator 2 or $p$, respectively (cf. [3] or [6]). We also have the case $d = 1$ for which $\chi_d$ is the trivial character; in this case, the denominator of $B_m/m$ contains exactly those primes $p$ for which $p-1$ divides $m$. Together, these numbers $d$ are the so-called fundamental discriminants (they can also be described as the set of square-free numbers of the form $4n+1$ and 4 times square-free numbers not of this form) and the corresponding characters $\chi_d$ give all primitive quadratic characters.

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In the paper we find some new congruences among the values of Dirichlet $L$-series attached to quadratic characters at negative integers (or equivalently, among the numbers $B_{m,\chi_d}/m$) modulo powers of 2 or 3. For $r \in \mathbb{Z}$ denote by $\mathcal{T}_r$ the set of all fundamental discriminants dividing $r$. For example, for the divisors of 24 we have $\mathcal{T}_1 = \mathcal{T}_2 = \{1\}$, $\mathcal{T}_3 = \mathcal{T}_6 = \{-3, 1\}$, $\mathcal{T}_4 = \{-4, 1\}$, $\mathcal{T}_8 = \{-8, -4, 1, 8\}$, $\mathcal{T}_{12} = \{-4, -3, 1, 12\}$, and $\mathcal{T}_{24} = \mathcal{T}_8 \cup \mathcal{T}_{12} \cup \{-24, 24\}$. If $\chi$ is a character modulo $M$ and $d$ any non-zero integer, then for $m \ge 0$ we set

$$B^{[d]}_{m,\chi} = \prod_{p|d,\, p\,\text{prime}} (1 - \chi(p)p^{m-1}) \cdot B_{m,\chi}$$

(this is just $B_{m,\chi'}$ for the character $\chi'$ modulo $M|d|$ induced by $\chi$, as we shall check below). Finally, we have the *generalized Bernoulli polynomial* defined by

$$B^{[d]}_{m,\chi}(X) = \sum_{n=0}^{m} \binom{m}{n} B^{[d]}_{n,\chi}\, X^{m-n},$$

which has the property $B^{[d]}_{m,\chi}(-X) = (-1)^m \chi(-1)\, B^{[d]}_{m,\chi}(X)$ unless $M = m = d = 1$.

THEOREM. *Let $d$ be a fundamental discriminant and $r$ and $c$ be integers prime to $d$ with $r \mid 24$. Then for any $m \ge 1$ the number*

$$r^{m-1}\varphi(r) \sum_{e\in\mathcal{T}_d} \chi_e(c) B^{[d]}_{m,\chi_e} - \sum_{\tau\in\mathcal{T}_r} \chi_\tau(-d) \sum_{e\in\mathcal{T}_d} \chi_e(rc) B^{[d]}_{m,\chi_{e\tau}}(d) \tag{3}$$

*is an integer divisible by $2^{\nu+\varepsilon} r^{m-1}\varphi(r)m$, where $\nu$ denotes the number of prime factors of $d$ and $\varepsilon = 1$ if $8 \mid d$ and $0$ otherwise.*

Proof. The proof of the theorem falls naturally into three parts.

1. If $\chi$ is a Dirichlet character modulo $M$, we define $\mathcal{L}_\chi(t) = \sum_{n=1}^{\infty} \chi(n)e^{nt}$. The series converges absolutely for $\Re(t) < 0$. From the obvious identity

$$\sum_{n=1}^{M} \chi(n)e^{nt} = (1 - e^{Mt})\mathcal{L}_\chi(t) \tag{4}$$

and the definition (1) we obtain the Laurent expansion

$$\mathcal{L}_\chi(t) = -\sum_{n=0}^{\infty} B_{n,\chi} \frac{t^{n-1}}{n!} \qquad (t \to 0). \tag{5}$$

Comparing coefficients of $t^{m-1}/(m-1)!$ on both sides of (4) gives the identity

$$\sum_{n=1}^{M} \chi(n)n^{m-1} = \frac{1}{m} \sum_{k=1}^{m} \binom{m}{k} B_{m-k,\chi} M^k \qquad (m \ge 1)$$

which can be used to compute the generalized Bernoulli numbers $B_{m,\chi}$ inductively and whose generalization will be the basis for the proof of the theorem.

We mention that the formula (2) for the values of the Dirichlet series $L(s,\chi)$ at negative integers follows formally from (5), since if we ignore all questions of convergence then the "coefficient" of $t^r/r!$ in $\mathcal{L}_\chi(t)$ is $\sum_{n\ge 1}\chi(n)n^r = L(-r,\chi)$. (To prove (2) rigorously one also uses equation (5): write $\Gamma(s)L(s,\chi)$ as a Mellin transform integral $\int_0^\infty \mathcal{L}_\chi(-t)\,t^{s-1}\,dt$, split up the integral into $\int_0^1 + \int_1^\infty$, expand the first term, and compare residues at $s = 1-m$.) Note also that if the character $\chi$ is induced from a character $\chi_1$ modulo some divisor of $M$, then

$$B_{m,\chi} = B_{m,\chi_1}\sum_{d\mid M}\mu(d)\chi_1(d)d^{m-1} = B_{m,\chi_1}\prod_{p\mid M}(1-\chi_1(p)p^{m-1}) = B^{[M]}_{m,\chi_1}.$$

This follows from (2) and (an analytic continuation of) the identity $L(s,\chi) = L(s,\chi_1)\prod_{p\mid M}(1-\chi_1(p)p^{-s})$, or else from (5) and a Möbius inversion argument:

$$\mathcal{L}_\chi(t) = \sum_{\substack{n\ge 1\\(n,M)=1}}\chi_1(n)\,e^{nt} = \sum_{n\ge 1}\chi_1(n)e^{nt}\sum_{d\mid(n,M)}\mu(d) = \sum_{d\mid M}\mu(d)\,\chi_1(d)\,\mathcal{L}_{\chi_1}(dt).$$

2. Now let $N$ be a multiple of $M$ and $r$ an integer prime to $N$. Then

$$\begin{aligned}\sum_{0<n<N/r}\chi(n)\,e^{rnt} &= \sum_{n>0}\chi(n)\,e^{rnt} - \sum_{n>0,\,r\mid n+N}\overline{\chi}(r)\chi(n)\,e^{(n+N)t}\\ &= \sum_{n=1}^{\infty}\chi(n)\,e^{rnt} - e^{Nt}\sum_{n=1}^{\infty}\Big(\frac{\overline{\chi}(r)}{\varphi(r)}\sum_{\psi}\psi(n)\overline{\psi}(-N)\Big)\chi(n)\,e^{nt}\\ &= \mathcal{L}_\chi(rt) - \frac{\overline{\chi}(r)}{\varphi(r)}\,e^{Nt}\sum_{\psi}\overline{\psi}(-N)\,\mathcal{L}_{\chi\psi}(t),\end{aligned}$$

where the sum is over all Dirichlet characters $\psi$ modulo $r$. Comparing coefficients of $t^{m-1}/m!$ $(m\ge 0)$ on both sides and using (5), we find the identity

$$(6)\quad mr^{m-1}\sum_{0<n<N/r}\chi(n)\,n^{m-1} = -B_{m,\chi}\,r^{m-1}+\frac{\overline{\chi}(r)}{\varphi(r)}\sum_{\psi}\overline{\psi}(-N)\,B_{m,\chi\psi}(N).$$

3. Now specialize to the case when $r$ is a divisor of 24. Then the group $(\mathbb{Z}/r\mathbb{Z})^{\times}$ has exponent 2, so all the characters $\psi$ are quadratic. We also restrict to quadratic characters $\chi$. Specifically, we take two coprime fundamental discriminants $K$ and $d$ and let $\chi$ range over the characters $\operatorname{mod} M = |Kd|$ induced by $\chi_{Ke}$ with $e \in \mathcal{T}_d$. Multiplying both sides of (6) by $\varphi(r)\chi_e(c)$ for a fixed integer $c$ prime to $M$ and summing over all such characters, we find

$$\sum_{e\in\mathcal{T}_d}\chi_e(c)\Big(-r^{m-1}\,\varphi(r)\,B^{[d]}_{m,\chi_{Ke}}+\chi_{Ke}(r)\sum_{\tau\in\mathcal{T}_r}\chi_\tau(-N)\,B^{[d]}_{m,\chi_{Ke\tau}}(N)\Big) = m\,r^{m-1}\,\varphi(r)\sum_{\substack{0<n<N/r\\(n,d)=1}}\chi_K(n)\,n^{m-1}\sum_{e\in\mathcal{T}_d}\chi_e(nc),$$

and this is divisible by $mr^{m-1}\varphi(r)2^{\nu+\varepsilon}$ because

$$\sum_{e\in\mathcal{T}_d}\chi_e(nc) = \prod_{p\mid d,\,p>2}\left(1+\left(\frac{nc}{p}\right)\right)\cdot\left(1+\left(\frac{-4}{nc}\right)\right)_{\text{if }4\mid d}\cdot\left(1+\left(\frac{8}{nc}\right)\right)_{\text{if }8\mid d} \equiv 0 \pmod{2^{\nu+\varepsilon}}.$$

To get the theorem, take $N = M = |d|$ and, if $d<0$, use the evenness or oddness of $B^{[d]}_{m,\chi}(X)$ to replace the argument $N$ of the Bernoulli polynomials by $d$.

Remarks. Since $B_{m,\chi}$ is almost always integral, as mentioned at the beginning of the paper, the essential statement of the theorem is a divisibility by a power of 2 and, if $3\mid r$, of 3. For example, for $r=24$ it says that the quotient of (3) by $m$ is divisible by $2^{3m+\nu}3^{m-1}$. These congruences are of the same general type as those of [4], [5], [8], [9] and [11]. In particular, for $r=8$ we get the congruence of [8] which is modulo $2^{3m-1+\nu}m$, and for $r=8$ and $m=1$ or 2 we get the special cases obtained in [5] or [11]. Formulas similar to (6) appear also in [2], [7], [10] and [12].

We also make some remarks about the proof. The theorem (for $r=8$) was found and proved by the first two authors using a different method which required a considerably longer calculation; the third author found the simpler method of proof, presented here, during a visit to the International Banach Center in Warsaw. He thanks warmly the staff of the Center for their hospitality. We will say a few words about the first proof, since the starting point for it was a general and very pretty formula due to B. C. Berndt [1] that can undoubtedly be applied to many other situations of this type, namely the following "character analogue of the Poisson summation formula":

$$\sum_{a\le l\le b}{}^{*}\chi(l)\,G(l)=\frac{1}{\tau(\overline{\chi})}\sum_{n=-\infty}^{\infty}\overline{\chi}(n)\int\limits_a^b G(x)\,e^{2\pi inx/M}\,dx.$$

Here $G$ is a continuous function on the interval $[a,b]$, $\chi$ is a primitive Dirichlet character modulo $M$, and the star means that the term $\chi(l)\,G(l)$ is to be divided by 2 if $l=a$ or $l=b$. (To prove this identity, one can write $\chi(l)$ as $\tau(\overline{\chi})^{-1}\sum_{k=1}^{M}\overline{\chi}(k)e^{2\pi ikl/M}$ and apply the usual Poisson summation formula to the functions $G(x)\,e^{2\pi ikx/M}$.) Taking $G(x)=x^{m-1}$, after some calculations one obtains an expression for the sum on the left-hand side of (6) as a linear combination of sums of the form $\sum_{n\ne0}\overline{\chi}(n)\,\zeta^n\,n^{-m}$ with $\zeta$ an $r$th root of unity, and these can be written in turn as finite linear combinations of generalized Bernoulli numbers and polynomials, giving (6). The rest of the proof is the same.

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