Topics in 4-dim. topology.

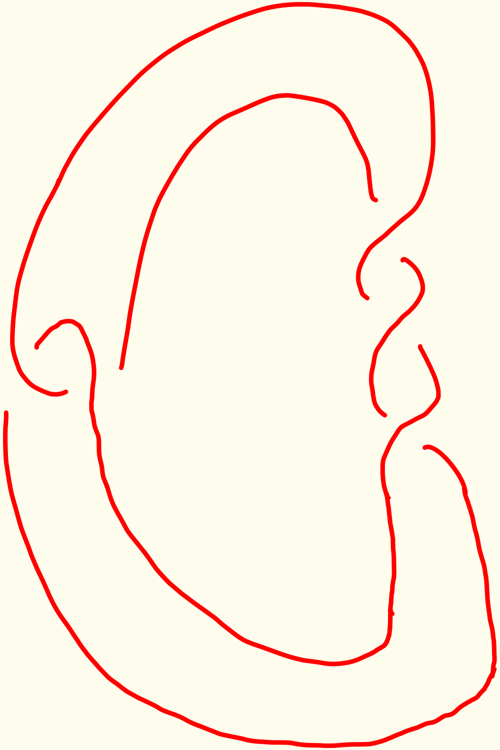
Lectures in winter 2022/23

joint with Rob Schneiderman

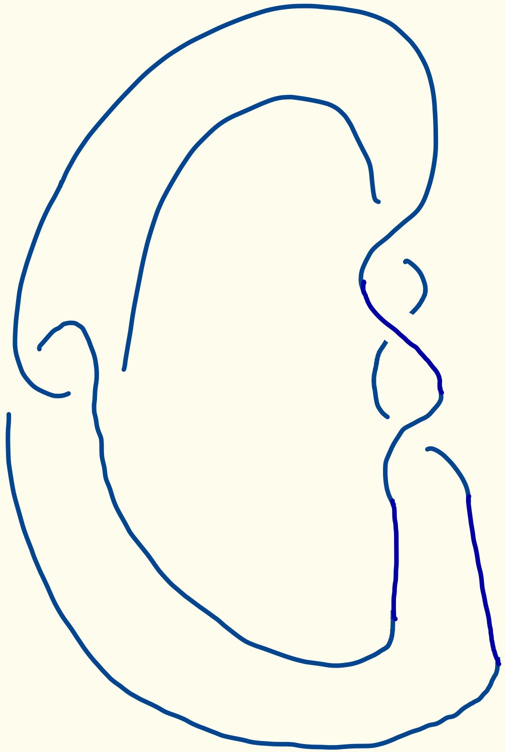

**Topic #1:** Knots, links and the group of disjoint spheres in $\mathbb{R}^4$.

Oct. 10, 22

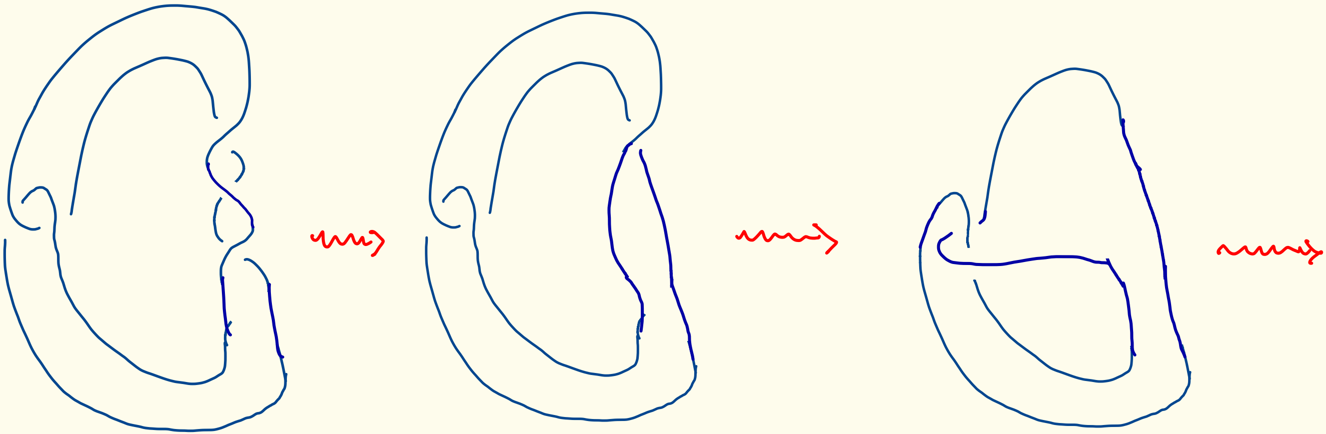
A **knot** $S^1 \overset{C^\infty}{\hookrightarrow} \mathbb{R}^3$:

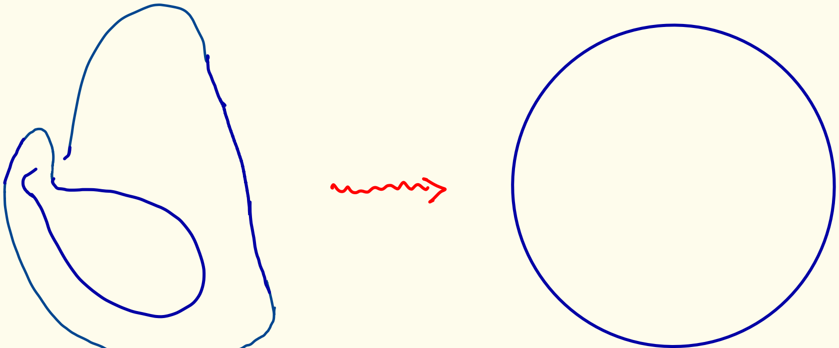


The **unknot**:



# Proof: An isotopy of circles in $\mathbb{R}^3$



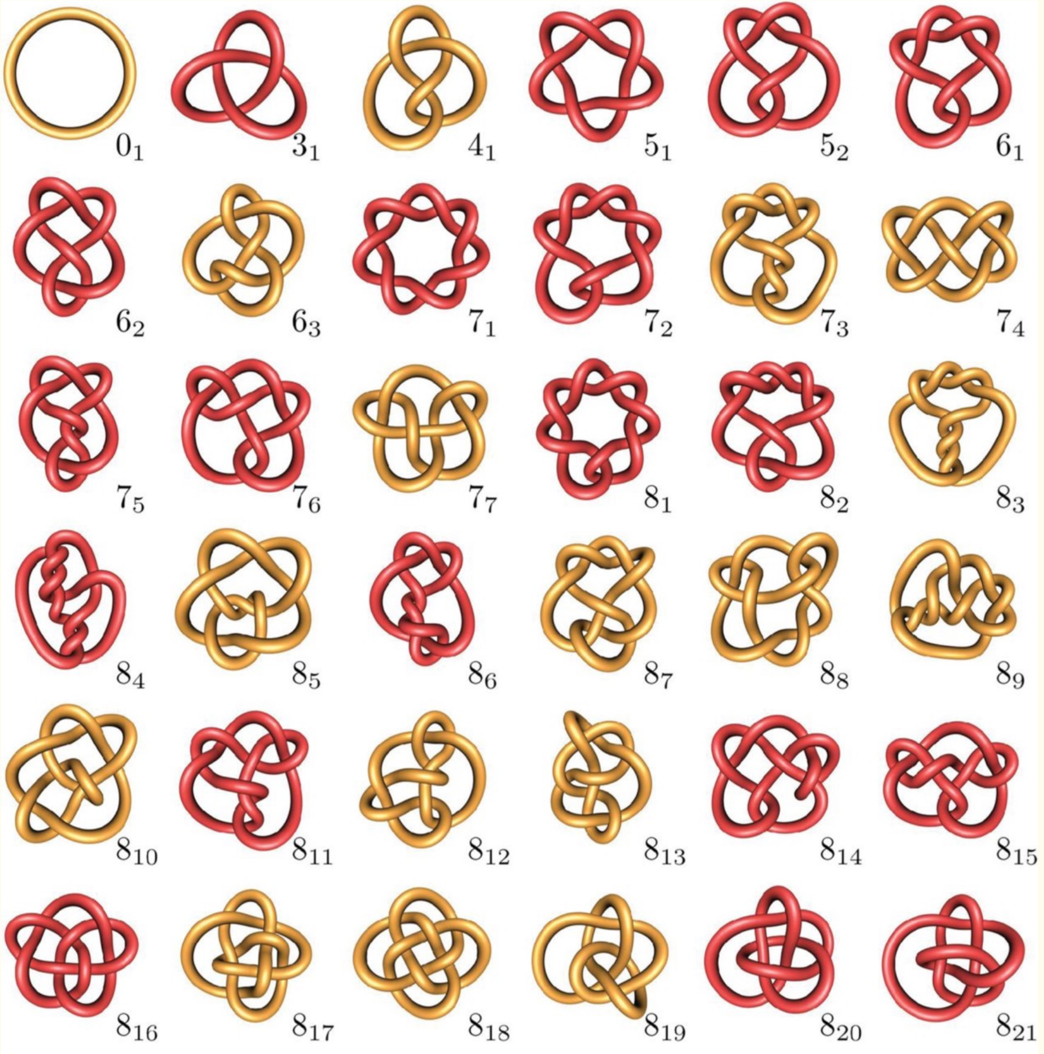


$=$ standard embedding $S^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$.

Our basic objects:

**Knots** := isotopy classes of embeddings $S^1 \hookrightarrow \mathbb{R}^3$.

picture better than label, some names ....



Knots have a direction (orientation of $S^1$)

and also a mirror image (reflect in $\mathbb{R}^3$)

$-K :=$ reverse mirror image $= (K^m)^r$

will be the inverse of $K$ in the

knot concordance group $\mathcal{C} :=$

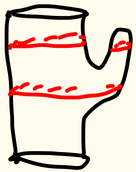
$$\text{knots} / \underset{4D}{\sim} \quad \text{where} \quad K_0 \underset{4D}{\sim} K_1 \quad \text{if}$$

$$\exists \quad S^1 \times I \overset{C}{\hookrightarrow} \mathbb{R}^3 \times I \quad \text{s.t.} \quad C_i = K_i$$

$I := [0,1]$. $\qquad$ for $i = 0, 1$

**Alternative def.** of a knot :=

1-dim. submanifold (connected, oriented) in $\mathbb{R}^3$. Get same set of isotopy classes of embeddings $S^1 \hookrightarrow \mathbb{R}^3$ by calling two submanifolds $K_0, K_1 \subseteq \mathbb{R}^3$ isotopic if $\exists$ codim. 2 submanifold $J \subseteq \mathbb{R}^3 \times I$ s.t. $J_i = K_i$ $i = 0,1$ and $\forall t \in I$ $J_t$ is connected.



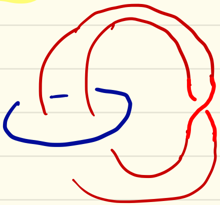
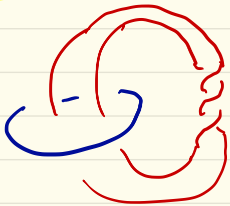
**Prime decomposition** w.r.t. connected sum:

$K_1 \# K_2 :=$ 


**Thm.:** The **knot group** $\pi_1(\mathbb{R}^3 \setminus K)$ ← Wirtinger presentation

classifies prime knots.

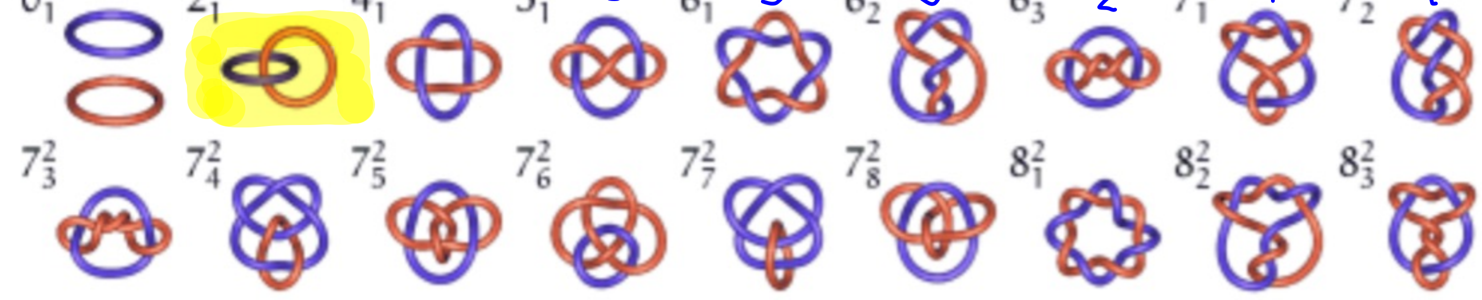
This is not true for links, e.g. the following **links $L_n$** have diffeomorphic **exteriors** $\forall n$:

$L_1 =$  odd number $n$ of twists, $L_3 =$  .

# Links with two components:

unlink Hopf link Whitehead link linking number

0 1 2 0 3 3 2 1 1




Note: The two knots are trivial in most cases!

**Links modulo knots** [John Milnor, 1950's]:

$$LM^3 := \frac{\left\{\begin{array}{c}\text{2-component} \\ \text{links in } \mathbb{R}^3\end{array}\right\}}{\text{link homotopy}} \cong \mathbb{Z}$$

linking number

allow only self-intersections of each component.

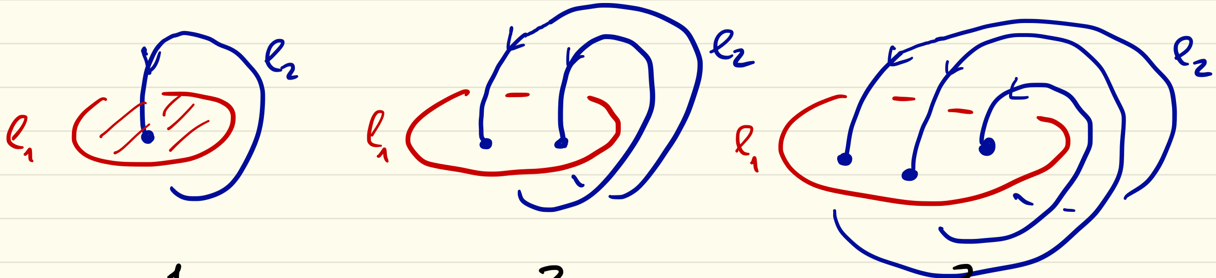
$$\# \longleftrightarrow +$$

This is the

**group of two disjoint** knots **in $\mathbb{R}^3$**

up to self-intersections.

Back to **Linking numbers**:




$=$ 1 2 3

$L = (\ell_1, \ell_2) : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$ 2-comp. link

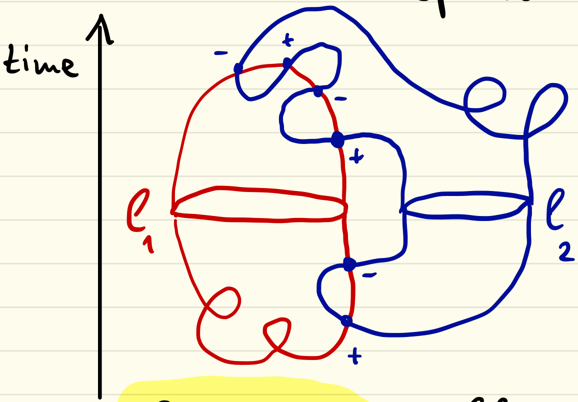
$\ell k(L) := \#$ transverse intersections of $\ell_1$ and $\ell_2$ during null homotopy

$$\ell k(L) = [\ell_1] \in H_1(\mathbb{R}^3 \setminus \ell_2) \cong \mathbb{Z}$$

# Idea of a proof for $LM^3 \cong \mathbb{Z}$

$$L \longmapsto lk(L)$$

**Step 1:** $lk$ is well-defined, i.e. does not depend on choice of null homotopy:




$\ell \subseteq \mathbb{R}^3 \times \mathbb{R}$

**Reason:** Two 2-spheres in $\mathbb{R}^4$ intersect alg. trivially because $\pi_2(\mathbb{R}^4) = 0$. ■

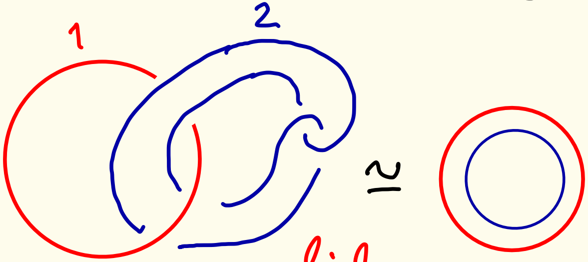
**Step 2:** $lk$ is injective (surjectivity is clear!)

Unknot $\ell_2$ in $\mathbb{R}^3 \setminus \ell_1$. Then $\pi_1(\mathbb{R}^3 \setminus \ell_2) \cong \mathbb{Z}$

$$\ell_1 \longmapsto \underset{\overset{\shortparallel}{0}}{lk(L)}$$

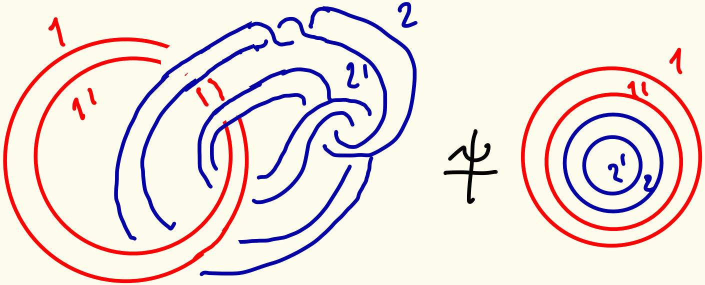
Hence $\ell_1$ shrinks in $\mathbb{R}^3 \setminus \ell_2$ ■

Example: The Whitehead link is link homotopic to the unlink:




link homotopy

but

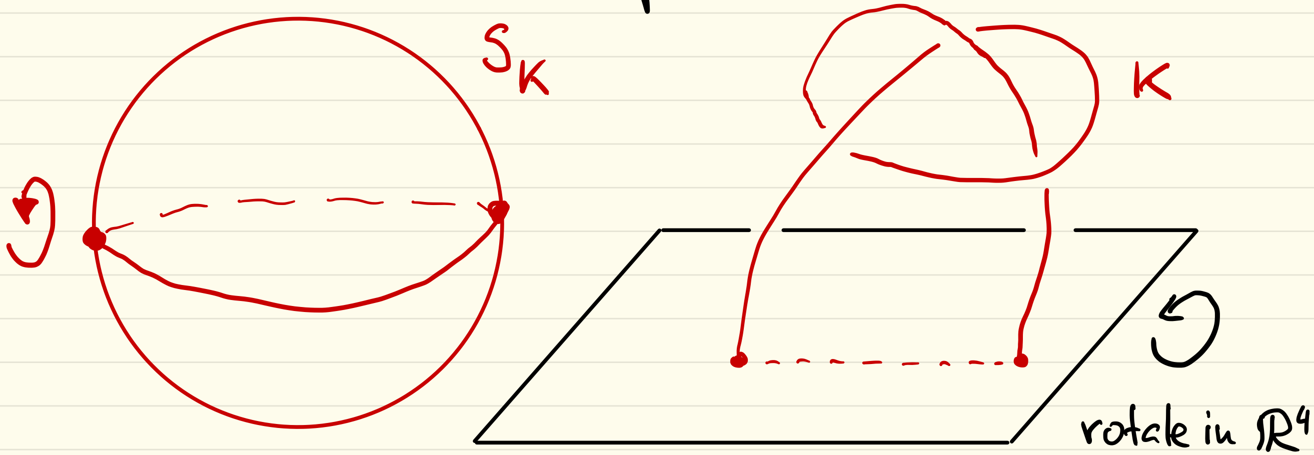



Its actual non-triviality is detected by higher order Milnor invariants $\mu(1 1' 2 2') \neq 0$.

$\mu(12) = lk = 0 \implies$ all $\mu(11\cdots122) \in \mathbb{Z}$ are link invariants by Tim Cochran's work.

# Brief history of 2-knots $S^2 \hookrightarrow \mathbb{R}^4$:

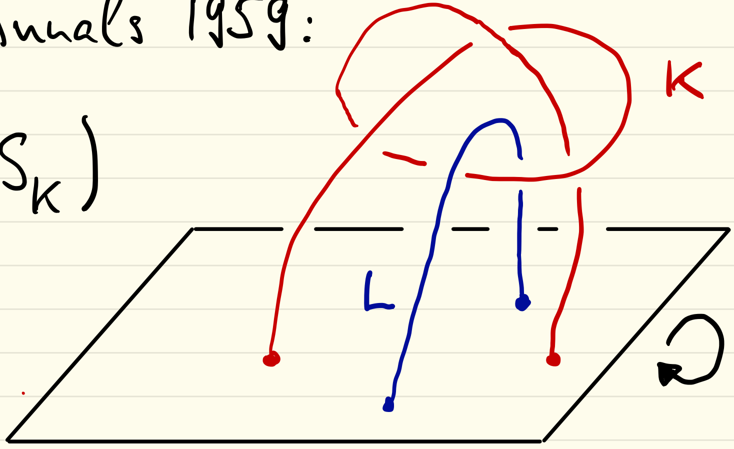
E. Artin, 1925 : Spun 2-knots




$\pi_1(\mathbb{R}^4 \setminus S^2_K) \cong \pi_1(\mathbb{R}^3 \setminus K)$ so 2-knots are more complicated than 1-knots!

**Andrews-Curtis**, Annals 1959:

$$0 \neq S_L \in \pi_2(\mathbb{R}^4 \setminus S_K)$$




unlike asphericity of 1-knot complement $S^3 \setminus K$.

**Note:**

**$S_L$ is a 2-unknot**, so $\pi_2(\mathbb{R}^4 \setminus S_L) = 0$, in particular, $S_K$ shrinks in the complement of $S_L$ and the 2-link **$(S_L, S_K)$ is link homotopic** to 2-unlink.

allow self-intersections again!

**Def.:** A **link map** is a continuous map $f: X \longrightarrow Y$ s.t. $\pi_0(X) \hookrightarrow \pi_0(f(X))$,

i.e. distinct components stay disjoint under $f$.

$$\boxed{LM^n_{p,q} :=} \left\{ \frac{\text{link maps } f: S^p \sqcup S^q \to S^n}{\text{link homotopy} := \text{homotopy through link maps}} \right\}$$

e.g. $LM^n_{1,n-2} \underset{lk}{\cong} \mathbb{Z} \quad \forall\, n \geq 3.$

First unknown case was $\boxed{LM^4 := LM^4_{2,2}}$.

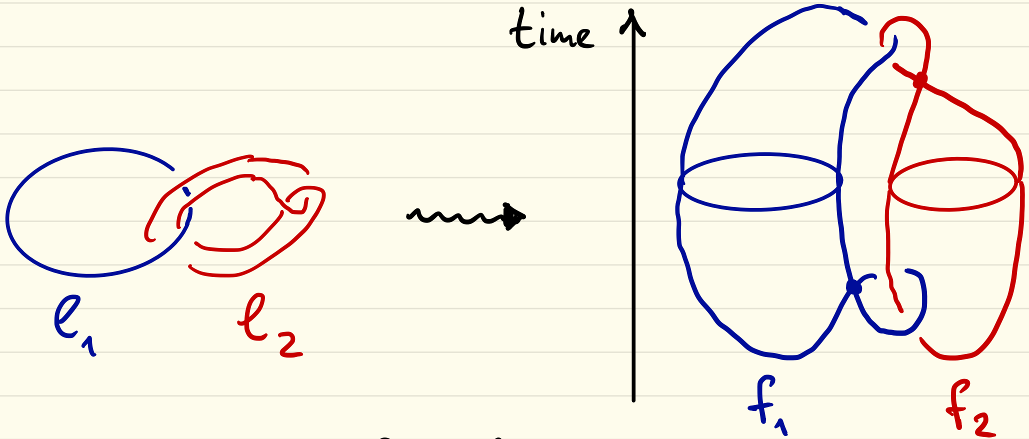
# Fenn–Rolfsen, 1986 :

There is a homotopically essential link map

$$FR := (f_1, f_2) : S^2 \sqcup S^2 \longrightarrow \mathbb{R}^4 ,$$

a "suspension" of the Whitehead link:

$\mathbb{R}^3 \supseteq$




$\subseteq \mathbb{R}^3 \times \mathbb{R}$

Note that both $f_i$ have a self-intersection!

$\sigma(f_1) \in \mathbb{Z}[t]$ measures these using $\left|\operatorname{lk}_{\mathbb{R}^4}(\mathrm{dp}\ell., f_2)\right|$.

Jin - Kirk, 1988, similarly gave 3D to 4D map:

$$\left\{ \begin{array}{l} L = (\ell_1, \ell_2) : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3 \\ \text{with } \ell_i \text{ unknots } lk(L) = 0 \end{array} \right\} \xrightarrow{JK} (LM^4, \#)$$

$$\mu : \{L\} \to \bigoplus^{\infty} \mathbb{Z} \leftarrow (LM^4, \#)$$

Kirk invariant $(\sigma(\ell_1), \sigma(\ell_2))$

Theorem [R. Schneiderman & P.T. 2019, Annals]

$JK(L) = JK(L') \iff L \ \& \ L'$ have the same Cochran - Milnor invariants $\mu_L(\underbrace{11 \cdots 1}_{n} 22), \mu_L(\underbrace{22 \cdots 2}_{n} 11) \ \forall n \geq 2.$

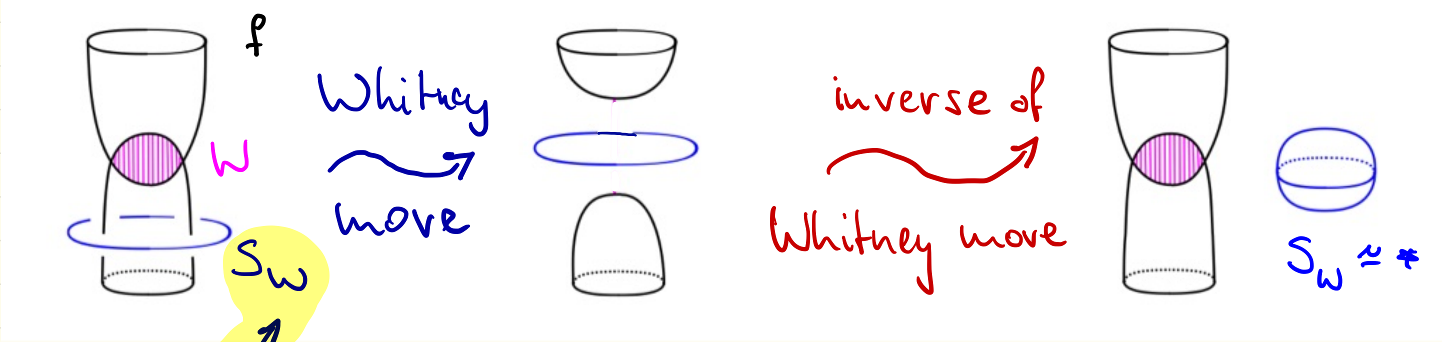
Moreover, $LM^4$ is a free $\dfrac{\mathbb{Z}[z_1, z_2]}{(z_1 \cdot z_2)}$ - module on one generator FR!

## Corollaries:

(i) $LM^4 \cong \bigoplus \mathbb{Z}$ as abelian group under $\#$.

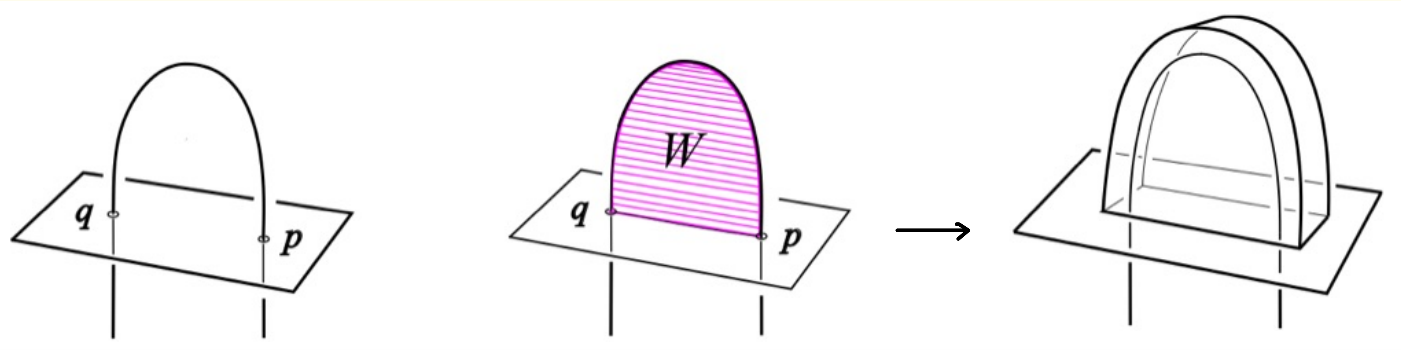
(ii) A link map $(f_1, f_2): S^2 \sqcup S^2 \to \mathbb{R}^4$ is link hom. trivial if $f_1$ is a (topological) embedding.

Main idea in our proof: Dimensionally reduced figure:

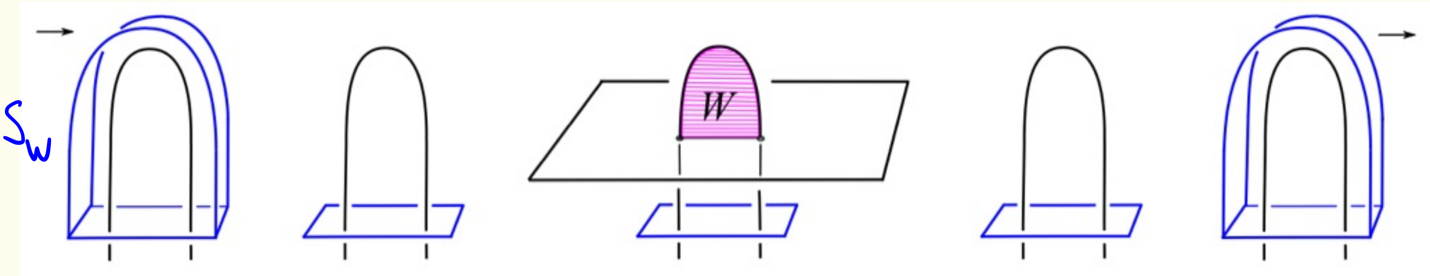


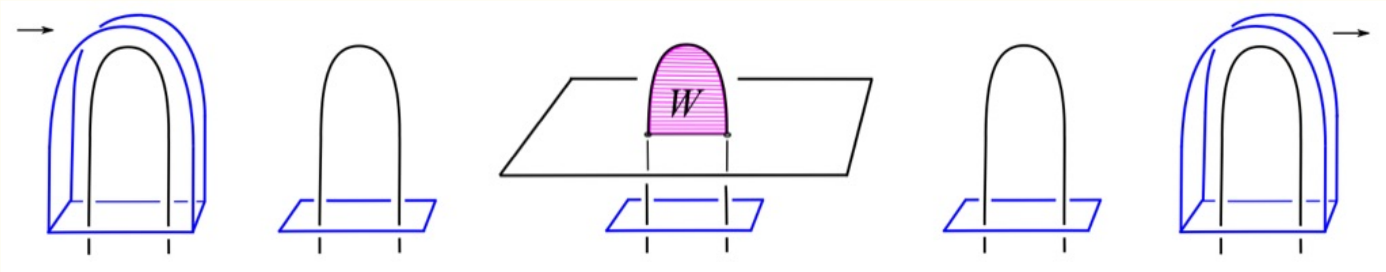

Whitney sphere shrinks in between Whitney moves.

Whitney move, using time as 4-th dimension:



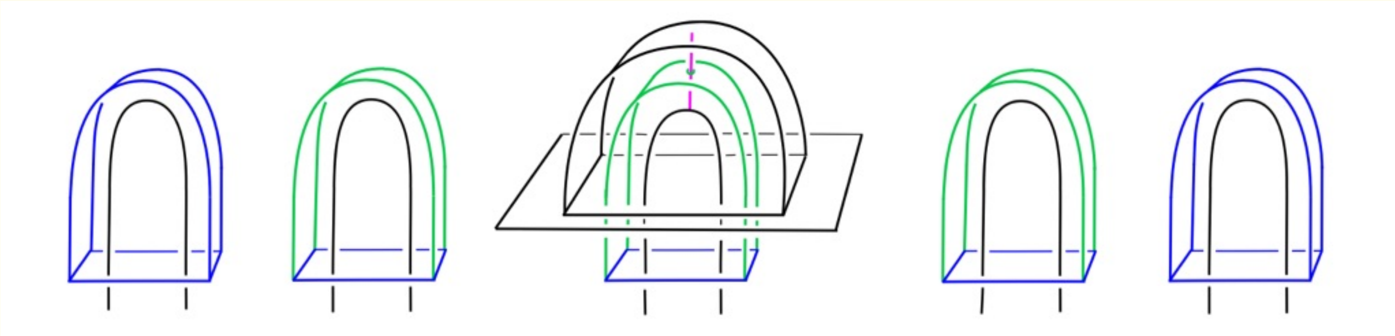

Precise picture of Whitney sphere $S_W$:






After the Whitney move we see that

Whitney sphere bounds 3-ball:



$S_W \simeq *$ is called a Whitney homotopy.

Wish list for notions in this Topic #1 to be discussed in more detail:

1) linking numbers

2) (self-) intersection invariants