Homotopy generators for the total fibre of the cube of configuration spaces

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MPIM

Topics Course

with Rob

Recall top of link map tower for manifolds M, N

$M = \coprod_{i=1}^{m} M_i$ with M_i, N connected:

$$\begin{tikzcd}
 & \mathcal{L}_m \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(M^{\underline{m}}, \mathrm{Tot}(C_{\leq m} N)) \arrow[d] \\
LM(M,N) \arrow[r, "ev_m"] \arrow[dr, "ev_{m-1}"] \arrow[ddr, bend right, "ev_1"'] & T_m \arrow[r, "f_m"] \arrow[d] & \mathrm{Map}(M^{\underline{m}}, C_m N) \\
 & T_{m-1} \arrow[d, dotted] & \\
 & T_1 = \mathrm{Map}(M,N) &
\end{tikzcd}$$

Here $f_m \circ ev_m(L) = C_m(L)$ sends $(x_1, \dots, x_m) \longmapsto (L_1(x_1), \dots, L_m(x_m))$, generalizing Gauß map $T^2 \to S^2$, $m=2$, $M_i = S^1$, $N = \mathbb{R}^3$.

Thm.: [Kosanović - S-T]: Assume $\dim M = 1$, $\dim N = 3$:

1) L is almost trivial $\Longleftrightarrow$ $ev_{m-1}(L) \simeq ev_{m-1}(U)$

where $U: M \twoheadrightarrow \{p_1, \ldots, p_m\} \subseteq N$ is unlink map

2) In this case, $\exists$ isomorphisms of S_m-modules

$$\mathbb{Z}[S_{m-2} \times \pi_1 N^{m-1}] \cong Lie_m(\pi_1 N) \cong [T^m, Tot_\bullet(N)]$$

such that $\mu_L(\text{length } m) \leftrightarrow [ev_m(L)]$.

Moreover, $\mu_L \equiv 0 \Leftrightarrow ev_m(L) \simeq ev_m(U) \Leftrightarrow [L] = [U]$.

In particular, the following three filtrations agree:

- Goodwillie-Weiss link map tower: $ev_n(L) \simeq ev_n(U)$,
- Whitney towers of order $n-1$ on L in $N \times I$,
- Milnor invariant tower $\mu_L(\text{length} \le n) = 0$.

They all detect whether $[L] = [U]$. $[L] = [L']$?

We'll show what's going on for $n=2,3$, $N=\mathbb{R}^3$:

$n=2$:

$$
\begin{tikzcd}
 & \mathcal{L}_2 \arrow[r, "\simeq"] \arrow[d, "\simeq\ T"] & \mathrm{Map}(T^2, \mathrm{Tot}(C_{\leq 2})) \arrow[d, "\simeq"'] \\
LM(S^1 \sqcup S^1, \mathbb{R}^3) \arrow[r, "ev_2"] & T_2 \arrow[r, "f_2", "\simeq"'] \arrow[d] & \mathrm{Map}(T^2, C_{12}) \\
 & T_1 \simeq *
\end{tikzcd}
$$

$$\Rightarrow \quad \pi_0(T_2) \cong \pi_0\big(\mathrm{Map}(T^2, C_{12})\big) \cong [T^2, S^2] \cong \mathbb{Z}.$$

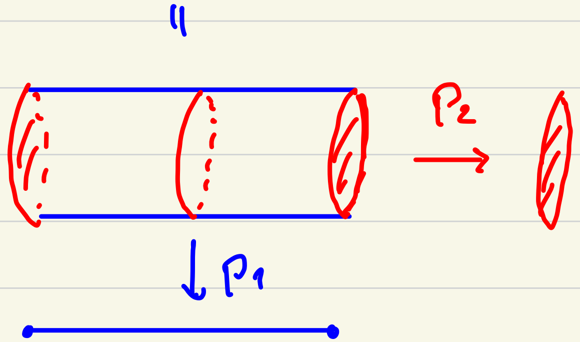
Our theorem says here that $\mathrm{degree}(ev_2(L)) = \lambda_0(W) = D_1 \pitchfork D_2$

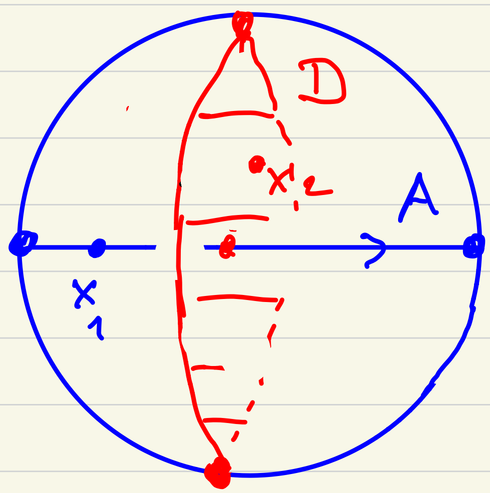
$$\underset{lk_{Gauß}(L)}{\overset{\parallel}{\mathrm{degree}(ev_2(L))}} = \underset{lk_W(L)}{\overset{\parallel}{\lambda_0(W)}}$$

where $W = D_1 \sqcup D_2 \to D^4$ are discs bounding L, an order 0 Whitney tower.

Key Lemma: $\pi_2 \mathrm{Tot}\, C_{\underset{\sim}{2}} \cong \pi_2 C_{12} \cong \mathbb{Z}$ is generated by

$S^2 = \partial\left(D^1 \times D^2\right) \xrightarrow{A \times D} C_{12} \subseteq \mathbb{R}^3 \times \mathbb{R}^3$, $\quad A : D^1 \longrightarrow \mathbb{R}^3$, $\quad D : D^2 \longrightarrow \mathbb{R}^3$





Proof: Compute degree of $S^2 \xrightarrow{A \times D} C_{12} \underset{\simeq}{\xrightarrow{g_{12}}} S^2$,

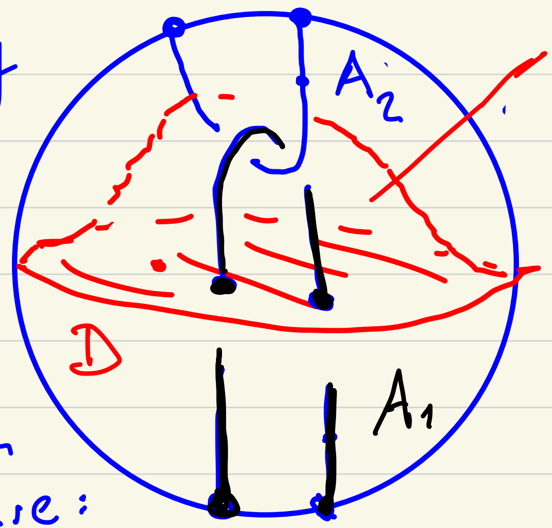
i.e. count inverse images of $\uparrow$: $(x_1, x_2) \longmapsto \dfrac{x_2 - x_1}{|x_1 - x_2|}$

$\Longrightarrow \quad x_1 \in a_1 \cap d_2$ and so $x_2 \in \partial D \Rightarrow x_2 = \uparrow \underset{\text{reg. value}}{\Longrightarrow} \deg = \pm 1$ $\boxtimes$

m = 3:

$$\begin{tikzcd}
 & \mathcal{L}_3 \arrow[r, "\simeq"] \arrow[d] \arrow[dr, dashed, "\tilde{f}_3"] & \mathrm{Map}(T^3, \mathrm{Tot}(C_{\leq 3})) \arrow[d] \\
LM(S^1 \sqcup S^1 \sqcup S^1, \mathbb{R}^3) \arrow[r, "ev_3"] \arrow[dr, "ev_2"'] & T_3 \arrow[r, "f_3"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(T^3, C_{123}) \arrow[d] \\
 & T_2 \arrow[r, "f_2"] & \mathrm{Map}(T^3, C_{12} \times C_{23} \times C_{13})
\end{tikzcd}$$

Key lemma: $\pi_3(\mathrm{Tot}\, C_3) \cong \mathbb{Z}$ is gen. by



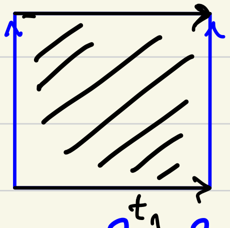
$\mathbb{D}(0)$ $\big\{$ W.move $\downarrow$ $\mathbb{D}(|t_2|)$

$\mathbb{D}(1) = \mathbb{D}(-1)$

$$\begin{array}{ccc}
 & & \mathrm{Tot}(C_{\le 3}) \\
 & \overset{W}{\nearrow} & \downarrow \\
S^3 = \partial(\mathbb{D}^1 \times \mathbb{D}^1 \times \mathbb{D}^2) & \longrightarrow & C_{123} \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \\
\cap & & \downarrow \\
\mathbb{D}^4 = \mathbb{D}^1 \times \mathbb{D}^1 \times \mathbb{D}^2 & \xrightarrow{A_1 \times A_2 \times \mathbb{D}} & C_{12} \times C_{23} \times C_{31}
\end{array}$$

$\left.\begin{array}{l} A_1 = A_1(t_1) \\ A_2 = A_2(t_2) \end{array}\right\}$ as in figure:

$\mathbb{D} := \mathbb{D}(|t_2|)$:



$\xrightarrow{p_2}$ $[-1, 1] \ni t_2$ $\xrightarrow{|t_2|}$ $[0, 1]$ $\xrightarrow{\text{W.-move}}$ $\left\{ \begin{array}{l} \text{space of discs} \\ \text{with boundary} \\ \text{equal to equator} \end{array} \right\}$

$\mathbb{R}^3 \smallsetminus \{x_1, x_2\} \simeq S^2 \vee S^2 \subseteq S^2 \times S^2 \implies \mathrm{Tot}(C_{\le 3}) \simeq \mathrm{hof}(S^2 \vee S^2 \to S^2 \times S^2)$

$$\begin{array}{ccc}
\mathbb{R}^3 \smallsetminus \{x_1, x_2\} & & \\
\cap & \swarrow \simeq & \\
C_{123} & \longrightarrow & C_{13} \times C_{23} \\
\downarrow & & \\
C_{12} & &
\end{array}$$

Hilton-Milnor: This is 2-connected with $\pi_3 \cong \mathbb{Z}$, generated by Whitehead product $[x_{13}, x_{23}]$, detected by $\mathrm{lk}(F_{12}^{-1}(p), F_{23}^{-1}(p))$.