Configuration spaces & Whitney towers

Dec. 5, 2022
MPIM
Topics Course
with Rob

Recall that for any two spaces M, N have

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted] & \\
 & T_n \arrow[dd, dotted] \arrow[r, phantom, "="] & \mathrm{HoNat}_{I_n}(C_{\leq n}M, C_{\leq n}N) \\
\mathrm{Emb}(M,N) \arrow[ur, "ev_n"] \arrow[dr, "ev_1"'] & & \\
 & T_1 \arrow[r, phantom, "="] & \mathrm{Map}(M,N)
\end{tikzcd}$$

$I_n =$ finite sets of order $\leq n$, injections

uncompactified Goodwillie-Weiss tower (no doubling, no little discs)

[Arone-Krushkal, 2022] There is a comm. diagram

$$\begin{tikzcd}
\text{"layer"} & \mathcal{L}_n \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}_{\Sigma_n}(C_n M, \mathrm{Tof}(C_{\leq n} N)) \arrow[d] \\
{} \arrow[r, "ev_n"] & T_n \arrow[r, "f_n"] \arrow[d, "p_n"] & \mathrm{Map}_{\Sigma_n}(C_n M, C_n N) \\
 & T_{n-1} &
\end{tikzcd}$$

$f_n \circ ev_n(j) = C_n(j)$

$\shortparallel$

total fibre of the n-cube of configurations

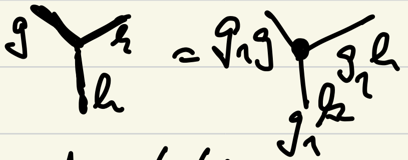
If N^d is a manifold then $\mathrm{Tof}(C_{\leq n}N)$ is $(n-1)(d-2)$-connected with $\pi_{(n-1)(d-2)+1}(-\shortparallel-) \cong \Lambda_{n-2}(\pi_1 N)$

where $\Lambda^{n}_{n-2}(G) = \dfrac{\text{unitrivalent trees with non-repeating indices from } \underline{n} = \{1, \dots, n\} \text{ and labels from } G \text{ on edges}}{\text{AS, IHX, Hol}}$

$\mathbin{/\!/\!\!/}\!\!\wr$

$\cong$

$\mathbb{Z}[\Sigma_{n-2} \times G^{n-1}] \wr \Sigma_n$

Hol :


They use this to find obstructions for embedding simplicial complexes into $\mathbb{R}^d$.

We want a different tower for link maps $LM(M,N)$.
To get maps ev_n, need to restrict $C_\bullet M$ to

non-repeating configurations $C^{n.r.}_{n} M_m := \{ c \in C_n M \mid$ each component of M contains at most one point of $c \}$

If $M = \coprod_{i=1}^{m} M_i$, M_i & N connected, then

$C^{n.r.}_{n} M = \emptyset$ if $n > m$ and $= \prod_{\underline{n} \underset{j}{\hookrightarrow} \underset{\substack{\cong \\ \pi_0 M}}{\underline{m}}} \left(\prod_{i=1}^{n} M_{j(i)} \right)$

where Σ_n-action precomposes j.

$\Longrightarrow$ Σ_n acts freely on j with orbits $S = \mathrm{im}(j) \subseteq \underline{m}$

$$\Longrightarrow \quad \mathrm{Map}_{\Sigma_n}\left(\overset{u.r.}{C_n} M, C_n N \right) \cong \prod_{\substack{S \subseteq \underline{m} \\ |S| = n}} \mathrm{Map}\left(\underset{\underset{\prod_{i \in S} M_i}{\parallel}}{M^S}, C_S N \right)$$

In particular, the top. of tower looks as follows:

$$\begin{tikzcd}
 & \mathcal{L}_m \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}(M^{\underline{m}}, \mathrm{Tot}(C_{\leq m} N)) \arrow[d] \\
LM(M,N) \arrow[r, "ev_m"] \arrow[rd, "ev_{m-1}"] \arrow[rdd, bend right, "ev_1"'] & T_m \arrow[r, "f_m"] \arrow[d] & \mathrm{Map}(M^{\underline{m}}, C_m N) \\
 & T_{m-1} \arrow[d, dotted] & \\
 & T_1 = \mathrm{Map}(M,N) &
\end{tikzcd}$$

$f_n \circ ev_n(j) = C_m(j)$, e.g. Gauß-map for $M = S^1 \sqcup S^1$
$N = \mathbb{R}^3$.

[Kosanović - S - T]: Assume $\dim M = 1$, $\dim N = 3$:

same in $\pi_0 T_{m-1}$

1) L is almost trivial $\Longleftrightarrow$ $ev_{m-1}(L) \simeq ev_{m-1}(U)$

where $U : M \twoheadrightarrow \{p_1, \dots, p_m\} \subseteq N$ is "unlink map",

$U[M_i] = p_i \ \forall i \in \underline{m}$

2) In this case,

$$\begin{array}{ccccc} & & \mu_L(\text{top length}) & \hat{=} & [ev_m(L)] \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathbb{Z}[\Sigma_{m-2} \times \pi_1 N^{m-1}] & \subseteq & Lie_m(\pi_1 N) & \cong & [T^m, Tot_\bullet(N)] \end{array}$$

In particular, the two filtrations defined by

- Goodwillie-Weiss link map tower and
- Whitney towers in $N \times I$

agree and both determine whether $[L] = [U]$. For string-links we even get $[L] = [L']$ via group-like composition.

We'll show what's going on for $n=2,3$, $N=\mathbb{R}^3$:

$n=2$:

$$\begin{tikzcd}
 & \mathcal{L}_2 \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{Map}(T^2, \mathrm{Tot}(C_{\leq 2})) \arrow[d, "\simeq"'] \\
LM(S^1 \sqcup S^1, \mathbb{R}^3) \arrow[r, "ev_2"] \arrow[dd, "ev_2"'] \arrow[ddr, "\text{Gauß}"] & T_2 \arrow[r, "f_2"] \arrow[d] & \mathrm{Map}(T^2, C_{12}) \\
 & * \simeq T_1 & \\
\Longrightarrow\ T_2 \arrow[r, phantom, "\simeq"] & \mathrm{Map}(T^2, S^2) &
\end{tikzcd}$$

2-cube of configurations is very simple

$$\begin{tikzcd}
 & * \simeq C_2 \arrow[r] & C_\emptyset = pt \\
 & C_{12} \arrow[r] \arrow[u] & C_1 \simeq * \arrow[u] \\
\mathrm{Tot}(C_{\leq 2}) \arrow[r, "\simeq"] & F_{12} \arrow[r] \arrow[u, "\simeq"'] & F_1 \simeq * \arrow[u]
\end{tikzcd}$$

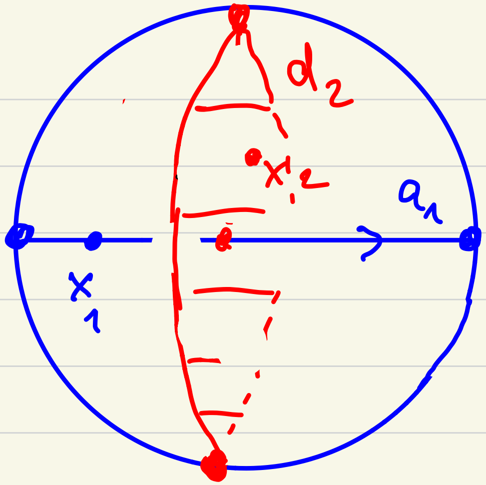
Our theorem says here that $\mathrm{lk}_{\text{Gauß}}(L) = \mathrm{lk}_W(L)$ where $W = D_1 \sqcup D_2 \to D^4$ are discs bounding L.

$\Longleftrightarrow$ $\mathrm{degree}(ev_2) = D_1 \pitchfork D_2 \in \mathbb{Z}$.

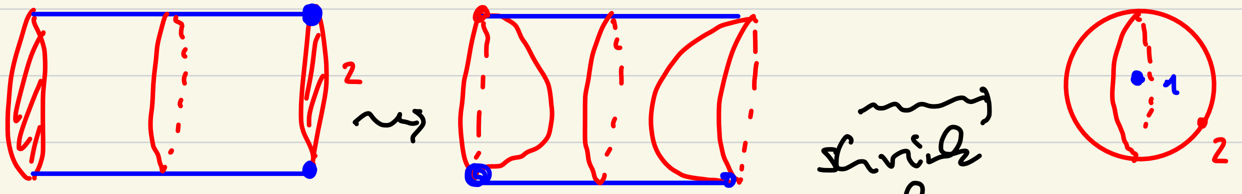
Key Lemma: $\pi_2 \mathrm{Tot}\, C_{\underline{2}} \cong \pi_2 C_{12} \cong \mathbb{Z}$

is generated by

$$\begin{tikzcd}
 & & \mathrm{Tot}\, C_{\underline{2}} = \mathrm{hof}(p_1\times p_2) \arrow[d] \\
S^2 = \partial(D^1 \times D^2) \arrow[rru, dashed, "x_{12}"] \arrow[rr, "a_1\times d_2"] \arrow[d, phantom, "\cap"] & & C_{12} = \mathbb{R}^3\times\mathbb{R}^3 \setminus \Delta \arrow[d, "p_1\times p_2"] \\
D^1\times D^2 \arrow[rr, "a_1\times d_2"] & & C_1\times C_2 = \mathbb{R}^3\times\mathbb{R}^3
\end{tikzcd}$$



Proof:



move flat disk away from x_i

$\rightsquigarrow$ shrink blue arc to midpoint.

is a generator from fibr.

$$\begin{tikzcd}
 & C_{12} \arrow[r] & C_1 \\
\mathbb{R}^3\setminus x_1 \arrow[ru] & &
\end{tikzcd}$$

∎

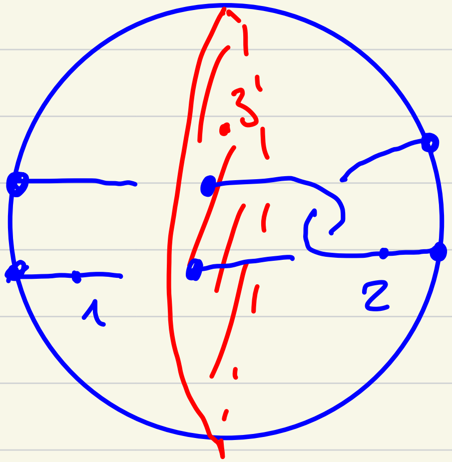
$n=3$: Conf. 3-cube is more interesting:

$$\mathrm{Tot}_3\left(\begin{tikzcd}[row sep=small, column sep=small]
 & C_1\simeq * \arrow[rr] & & C_\emptyset = pt \\
C_{13} \arrow[ru] \arrow[rr] & & C_3 \arrow[ru] & \\
 & C_{12} \arrow[uu] \arrow[rr] & & C_2\simeq * \arrow[uu] \\
C_{123} \arrow[uu] \arrow[ru] \arrow[rr] & & C_{23} \arrow[uu] \arrow[ru] &
\end{tikzcd}\right) = \mathrm{Tot}_2\left(\begin{tikzcd}
C_{13} & C_{12} \\
C_{123} \arrow[u] \arrow[ru] \arrow[r] & C_{23}
\end{tikzcd}\right) = \mathrm{hof}\left(\begin{tikzcd}
C_{123} \arrow[d] \\
C_{12}\times C_{23}\times C_{13}
\end{tikzcd}\right)$$

Key lemma: $\pi_3 \operatorname{Tot} C_3 \cong \mathbb{Z}$ is gen. by

$$\begin{tikzcd}
 & \operatorname{Tot}(C_{\leq 3}) \arrow[d] \\
S^3 = \partial(D^1\times D^1\times D^2) \arrow[ur, "W"] \arrow[r, "L"'] \arrow[d, phantom, "\cap"] & C_{123} \arrow[d] \\
D^1\times D^1\times D^2 \arrow[r, "W"'] & C_{12}\times C_{23}\times C_{31}
\end{tikzcd}$$

✓ ✓ ⇗



$$\begin{tikzcd}
\mathbb{R}^3\setminus\{x_1,x_2\} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & S^2\vee S^2 \arrow[r, phantom, "\subseteq"] & S^2\times S^2 \arrow[dl, "\simeq"] \\
C_{123} \arrow[r] \arrow[d] & C_{13}\times C_{23} & \\
C_{12} & &
\end{tikzcd}$$

W.move $\hat{=}$ move clasp to the left! = homotopy on S^3 that then extends to D^1 into C_{31}.

$\Rightarrow$ $\operatorname{Tot}(C_{\leq 3}) \simeq \operatorname{hof}(S^2\vee S^2 \to S^2\times S^2)$

$$\begin{tikzcd}
 & \mathcal{L}_3 \arrow[r, "\simeq"] \arrow[d] & \operatorname{Map}(T^3, \operatorname{Tot}(C_{\leq 3})) \arrow[d] \\
LM(S^1\sqcup S^1\sqcup S^1, \mathbb{R}^3) \arrow[r, "ev_3"] \arrow[dr, "ev_2"'] & T_3 \arrow[r, "f_3", "\Gamma"'] \arrow[d] \arrow[ur, dashed, "\tilde{f}_3"] & \operatorname{Map}(T^3, C_{123}) \arrow[d] \\
 & T_2 \arrow[r, "f_2"] & \operatorname{Map}(T^3, C_{12}\times C_{23}\times C_{13})
\end{tikzcd}$$

Hilton-Milnor: this is 2-connected with $\pi_3 \cong \mathbb{Z}$, generated by $[x_{13}, x_{23}]$

Need to show $\mu_L(123) = \tilde{f}_3 \circ ev_3(L) \in [T^3, \mathrm{Tor}(C_{\leq 3})]$

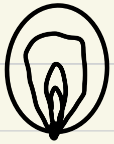
for L almost trivial $\overset{\shortparallel}{\lambda_1(W)} \in \mathbb{Z} \overset{\wr\wr}{\cong} \pi_3(\mathrm{Tor}\, C_{\leq 3})$

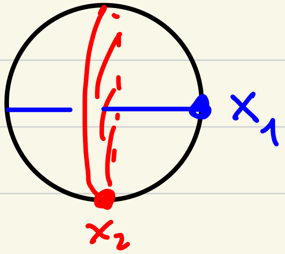
The lift $\tilde{f}_3$ actually is given by W, tower W of order 1, similarly for all higher degrees!

For inductive proof use Samelson product on the graded Lie algebra $\pi_*(\Omega X, x_0)$ over $\mathbb{Z}[\pi_1 X]$

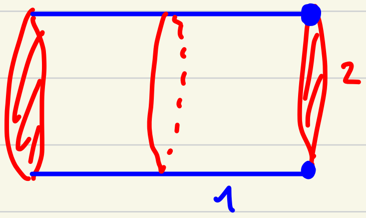
$$\Omega X \times \Omega X \xrightarrow{[\,,\,]} \Omega X$$

$$(\gamma_1, \gamma_2) \longmapsto \gamma_2 * \gamma_1^{-1} * \gamma_2 * \gamma_1$$

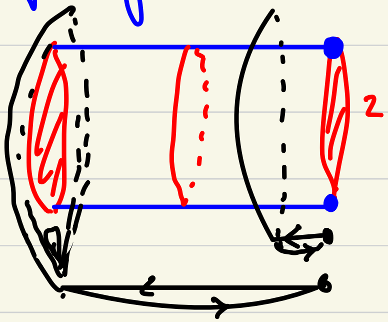




Need to turn this



into a based version like



$\in \pi_1(\Omega C_{12}, (x_1, x_2))$