Configurations, Whitney towers and the space of link maps

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MPIM
Topics Course
with Rob

The $\underline{\text{space}}$ of link maps. Why?

Warm up $LM(\underline{m}, \mathbb{R}^2) =$ configurations of m points in the plane. $\underline{m} := \{1,\dots,k\}$

This is connected and

$$\pi_1(LM(\underline{m}, \mathbb{R}^2)) = \text{pure braid group } P_m$$

e.g. $\mathbb{R}^2 \setminus \{p_1,p_2\} \longrightarrow LM(3,\mathbb{R}^2) \xrightarrow{p_{12}} LM(2,\mathbb{R}^2) \simeq S^1$

gives $F(2) \rightarrowtail P_3 \twoheadrightarrow \mathbb{Z}$ and two more p_{13}, p_{23} gives

$\cup\!\mid$

ambient handle cancellation $F(2)_2 \rightarrowtail P_3 \to \mathbb{Z}^3$

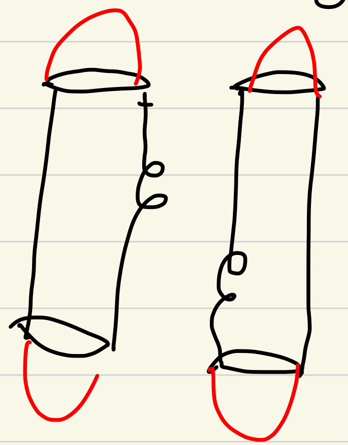
$\pi_0(LM(\cancel{S^2}\, I \times \underline{m}, \mathbb{R}^3))$ is not injective since fibre over $\mathbb{Z}^3$ (at $0,0,0$) is $\mathbb{Z} = F(2)_2 / F(2)_3$

is a group $\underset{m=3}{\cong} P_3 / F(2)_3$ $\qquad \ell_3 = [x_1, x_2]^{M_L}$

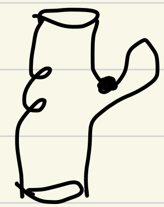
so fibre is free on ∞-many generators!

Similarly, $\pi_1 LM(\underline{m}\times S^1, \mathbb{R}^3) \twoheadrightarrow \pi_0 LM(\underline{m}\times S^2, \mathbb{R}^4)$

is not injective (!?)



$\mathbb{R}^3 \times I$, i.e. no saddles:



To understand spaces like $LM(\underline{m})$, use a simplified version of Goodwillie-Weiss embedding tower:

Fix spaces M, N and consider the space $\mathrm{Emb}(M,N)$ in various categories. Key idea:

$$\begin{array}{c}\mathrm{Emb}(M,N)\\ \downarrow \\ \mathrm{Map}_{\Sigma_k}(C_k M, C_k N)\end{array} \qquad C_k M := \mathrm{Emb}(\underline{k}, M) \xrightarrow{\mathrm{res}} C_\ell M \quad \underline{\ell} \hookrightarrow \underline{k}$$

or better

$$\begin{tikzcd} \mathrm{Emb}(M,N) \arrow[r] \arrow[d] & \mathrm{Nat}\left((\mathbb{N}, \text{injections})^{op} \arrow[r, bend left, "C_\bullet M"] \arrow[r, bend right, "C_\bullet N"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Top} \right) \arrow[d, "\mathrm{res}"'] \\ T_n \mathrm{Emb}(M,N) := \arrow[d] & \mathrm{Nat}_{\leq n} \ \text{—"—} \arrow[d, "\mathrm{res}"'] \\ T_{n-1} := & \mathrm{Nat}_{<n} \end{tikzcd}$$

$T_1 = \mathrm{Map}(M,N)$

$\Rightarrow$ Obstructions to ex. of emb. $M \hookrightarrow N$: $f \in T_1$ must lift to $T_2, T_3, \ldots$

Obstruction to isotopy of emb. : homotopy $h: I \to T_1$ must lift to T_2, T_3. Similarly for $\pi_i \operatorname{Emb}(M,N)$

Thm.: M, N smooth manifolds, $\dim M < \dim N - 2$ then $\forall i \; \exists n_i : \pi_i \operatorname{Emb} \cong \pi_i T_{n_i}$

Fine print: (1) Need $\mathbb{R}^m \hookrightarrow M^m$ rather than points, or at
(2) least need to compactify & frame $C_k M$.
(3) Need **HoNat**, not Nat

(1.) + (2.) are not needed for link maps, there our model even simplifies more: We have to use

$$C_k^{n.r.} M := \left\{ \underline{k} \overset{i}{\hookrightarrow} M \;\middle/\; \begin{array}{l} \text{each point lies in} \\ \text{a different component} \end{array} \right\} \xrightarrow[L]{} C_k N$$

$\Leftrightarrow \pi_0(i)$ is injective

$$\cong \coprod_{\substack{S \subseteq \underline{m} \\ |S| = k}} M^S \times \Sigma_k \quad , \text{ i.e. } \operatorname{Map}_{\Sigma_k}(C_k^{n.r.} M, C_k N)$$

$$M = \coprod_{i=0}^{m} M_i \quad , M_i \text{ conn.} \qquad \cong \prod_{S \subseteq \underline{m}} \operatorname{Map}(M^S, C_S N)$$

Two very interesting cod. 2 cases:

- Knots tower $\mathcal{K} \xrightarrow{ev_n} T_n\mathcal{K}$ are Vassiliev invariants of type n, known to be rationally universal. Budney Conant Koytcheff Sinha — Boavida-Horel
- Disks tower (possibly rel. to a fixed link) should contain at least repeated Milnor invariants, hopefully also higher Arf invariants.

Theorem: [Danica Kosanović - Yuqing Shi - P.T.]

$$\begin{tikzcd} \left\{\begin{matrix}\text{simple capped} \\ \text{grope cobordisms} \\ \text{of degree } n\end{matrix}\right\} \arrow[r, "\exists"] \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] & \text{Path-space of } T_n \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] \\ \mathcal{K} \arrow[r, "ev_n"] & T_n \end{tikzcd}$$

Modern view on some of Koschorke's work:

Fix $m \geq 2$, the number of components, $\underline{m} = \{1, 2, .., m\}$

Define for $S \subseteq \underline{m}$ the following spaces

- the S-tori $T^S := \prod_{s \in S} S^1_s \xleftarrow{\text{base pt. } 1} T^{S'}$, $S' \subseteq S$
  a covariant m-cube.
- $C(S) := \mathrm{Emb}(S, \mathbb{R}^3)$, a contravariant m-cube.
- $LM(S) :=$ link maps $S \times S^1 \longrightarrow \mathbb{R}^3$, also a —"— .

Koschorke's map $\kappa : LM(S) \longrightarrow \mathrm{Map}(T^S, C(S))$

is a map of m-cubes! $\quad L \longmapsto \left((\theta_s)_{s \in S} \longmapsto (s \mapsto L_s(\theta_s)) \right)$

S-cube of configuration spaces $C^S_R(N) := \mathrm{Emb}(R, N)$ $\forall R \subseteq S$

e.g.
$S = \{1, 2\} \supseteq R$

$$\begin{array}{ccccccc}
\text{fibre} = C_2 & \longrightarrow & C_2 & \longrightarrow & C_\emptyset & = & pt \\
\uparrow & & \uparrow & & \uparrow & & \\
\text{fibre} & \longrightarrow & C_{\{12\}} & \longrightarrow & C_1 & = & N \\
\uparrow & & & & & &
\end{array}$$

$\Rightarrow \mathrm{Tot}\, C^{12}_\bullet = \mathrm{fibre}\left(\underset{N \times N \setminus \Delta}{\underset{\|}{C_{12}}} \longrightarrow \underset{N \times N}{\underset{\|}{C_1 \times C_2}} \right)$, $\quad N \times N \setminus \Delta \subseteq N \times N$

$U :=$ unlink

Fix two spaces M, N and study the space

$$LM(M,N) := \{ f \in C^0(M,N) \mid M \xrightarrow{f} f(M) \text{ is injective on } \pi_0 \}$$

of link maps. $M = \coprod_{i=1}^{m} M_i$, M_i & N connected

Here $\underline{m} := \{1, \dots, m\} = \pi_0 M$, $M^S := \prod_{i \in S} M_i$, $C_\bullet^S(N) = \mathrm{Emb}(S,N)$, $S \subseteq \underline{m}$.

Thm.: [Koschorke-S-T] There is a tower of spaces, maps and layers

$$\rho_k^{-1}(u) =: \mathcal{L}_k(M,N) \simeq \prod_{\substack{S \subseteq \underline{m} \\ |S| = k}} \mathrm{Map}(M^S, \mathrm{Tot}\, C_\bullet^S(N))$$

$$\begin{tikzcd}
LM(M,N) \arrow[r, "ev_{\underline{m}}"] \arrow[dr, bend right, "ev_k"] \arrow[ddr, bend right, "ev_{k-1}"'] \arrow[dddr, bend right=40, "\text{inclusion}"'] & T_{\underline{m}}(M,N) \arrow[d, dotted] & \\
& T_k(M,N) \arrow[r] \arrow[d, "\rho_k"'] \arrow[dr, phantom, "\Gamma."] & \prod_{\substack{S \subseteq \underline{m} \\ |S|=k}} \mathrm{Map}(M^S, C_S N) \arrow[d] \\
& T_{k-1}(M,N) \arrow[r] \arrow[d, dotted] & \prod_{\substack{S \subseteq \underline{m} \\ |S|=k}} \mathrm{Map}(M^S, \operatorname{holim}_{R \in \Delta^S} C_R N) \arrow[d, dotted] \\
& T_1(M,N) \arrow[r, phantom, "="] & \mathrm{Map}(M, N)
\end{tikzcd}$$

total fiber of S-cube

such that for $\dim M = 1$, $\dim N = 3$

1) L is almost trivial $\iff$ $ev_{m-1}(L) \overset{\text{same in } \pi_0}{\simeq} ev_{m-1}(U)$

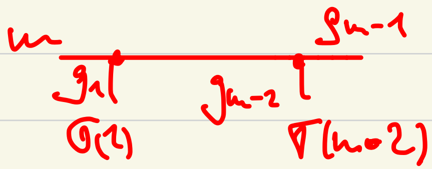
where $U: M \twoheadrightarrow \{p_1, \dots, p_m\} \subseteq N$ is "unlink map", $U(M_i) = p_i \ \forall i \in \underline{m}$

2) In this case, $\mu_L(\text{top length}) \triangleq [ev_m(L)]$

$$\qquad\qquad\qquad\cap \qquad\qquad\qquad\qquad \cap$$

$$\mathbb{Z}[S_{m-2} \times \pi_1 N^{m-1}] \subseteq \mathrm{Lie}_m(\pi_1 N) \cong [T^m, \mathrm{Tot}_\bullet(N)]$$

3) More generally, $\forall n \leq m$,



In particular, $[L] = [U] \iff ev_m(L) \simeq ev_m(U)$

if $W \subseteq N \times I$ is a non-repeating Whitney-tower of order $n-2$ then $\exists$ path $\gamma_W : I \to T_{n-1}$ from ∂W to U

and $[\gamma_W(1)] \triangleq \lambda_{n-2}(W)$

$$\cap \qquad\qquad \cap$$

in $\prod_{\substack{S \subseteq \underline{m} \\ |S| = n}} [T_1^S, \mathrm{Tot}^S C_\bullet N] \cong \mathrm{Lie}_{n-1}(\pi_1 N)$

$\qquad\quad \| \\ \qquad \pi_0 \mathcal{L}_n$

$$\begin{tikzcd} I \arrow[r] & T_{n-1} \\ \{0\} \arrow[u, phantom, "\cup"] \arrow[r, "e(\partial W)"] & T_n \arrow[u, "p_n"] \\ & \gamma_W(1) \in \mathcal{L}_n \arrow[u] \end{tikzcd}$$

In particular, the two filtrations defined by

- Goodwillie-Weiss link map tower and
- Whitney towers in $N \times I$

agree and both determine whether $[L] = [U]$. For string-links we even get $[L] = [L']$ via group-like composition.

Proof: Step 1 is geometric

space of almost trivial link maps

$$\begin{tikzcd}
\left\{ \substack{\text{non-repeating} \\ \text{level pres. W. towers} \\ \text{of order } n-2} \right\} \arrow[r, "\text{caps}", "\text{etc.}"'] & \mathrm{Tot}_S(LM(S)) \arrow[r, "\mathrm{Tot}_S(K)"] \arrow[dl] & \mathcal{L}_n = \mathrm{Map}(T^n, \mathrm{Tot}_S\, C(S)) \arrow[d, "ev"] \arrow[dl] \\
LM(\underline{n}) \arrow[r, "ev_n"] \arrow[d] \arrow[rr, bend left, "K(n)"] & T_n \arrow[r] \arrow[d] & \mathrm{Map}(T^n, C(n)) \arrow[d] \\
\underset{S \subsetneq \underline{n}}{\mathrm{holim}}\, LM(S) \arrow[r, "ev_{n-1}"] \arrow[rr, bend right, "\mathrm{holim}\, K"'] & T_{n-1} \arrow[r] & \mathrm{Map}(T^n, \underset{S \subsetneq \underline{n}}{\mathrm{holim}}\, C(S))
\end{tikzcd}$$

∂

Step 2: We show that the foll. definition satisfies properties:

$$T_k(M,N) := \operatorname*{holim}_{\substack{\emptyset \neq R \subseteq S \subseteq \underline{m} \\ |S| \le k}} \begin{array}{c} \mathrm{Map}(M^S, C_R N) \\ \downarrow \\ \mathrm{Map}(M^{S'}, C_{R'} N) \end{array}$$

where he uses poset of pairs so that we have a covariant functor $\mathrm{Map}(M^{\bullet}, C_{\bullet} N)$ into spaces. $=F$

This model follows from $\left|\mathcal{C}\downarrow_{R \subseteq S}\right| \cong I^{S \setminus R}$ and one can check directly our claims for T_1, $\mathcal{L}_k$.

$$\|$$

$$\left\{ \varphi_{R \subseteq S} : [0,1]^{S \setminus R} \quad \mathrm{Map}(M^{S_1}, C_{R_1} N) \ \text{ s.t. } \forall \ R' \subseteq R \subseteq S \subseteq S' \right\}$$

$$\begin{array}{ccc} \cap\!\shortmid & \dashv & \downarrow \\ \varphi_{R' \subseteq S'} : [0,1]^{S' \setminus R'} & \longrightarrow & \mathrm{Map}(M^{S_2}, C_{R_2} N) \end{array}$$

$$[0,1]^{S' \setminus R'} = \left\{ \alpha : \underline{m} \to [0,1] \;\middle|\; \begin{array}{ll} \alpha(r) = 0 & r \in R' \\ \alpha(i) = 1 & i \notin S' \end{array} \right\}$$

For the identification

$ev_{k+1}(L) = \lambda_k(W)$ we use the method from Danica's thesis.

Note $\mathrm{Tot}(LM) \to \mathcal{L}_m$, so to get the homotopy $ev_k(L) \simeq ev_k(W)$ we just need a point in $\mathrm{Tot}(LM)$ from a W. tower!

Step 3:

$$\begin{tikzcd}
 & \mathrm{Tot}(LM) = \mathrm{hofib}_u\Big(LM(\underline{m}) \to \overset{\text{unlink}}{\underset{S \subsetneq \underline{m}}{\mathrm{holim}}}\, LM(S)\Big) \arrow[r, "\mathrm{Tot}(\mathcal{K})"] \arrow[d, "\text{image on } \pi_0 \,=\, \text{almost trivial}"] & \mathrm{Map}(T^m, \mathrm{Tot}_S\, C(S)) \arrow[d] \\
\left\{\begin{matrix}\text{NR Order } (m-2) \\ \text{w. towers}\end{matrix}\right\} \arrow[ur, "\text{caps}"] \arrow[r, "\partial"'] & LM(\underline{m}) \arrow[r, "\mathcal{K}(\underline{m})"] \arrow[d] & \mathrm{Map}(T^m, C(\underline{m})) \arrow[d] \\
 & \underset{S \subsetneq \underline{m}}{\mathrm{holim}}\, LM(S) \arrow[r, "\mathrm{holim}\, \mathcal{K}(S)", "\text{previous step}"'] & \mathrm{Map}(T^m, \underset{S \subsetneq \underline{m}}{\mathrm{holim}}\, C(S))
\end{tikzcd}$$

$$\mathrm{Tot}(LM) \cong \left\{ L \in LM(\underline{m}),\ \begin{matrix} \Delta^{S'} \xrightarrow{\varphi_{S'}} LM_\partial(D^2_{\underline{m}\setminus S'}, D^4) \\ \big\updownarrow \qquad\qquad \big\downarrow \text{res} \\ \Delta^{S} \xrightarrow[\text{comp. conditions}]{\varphi_S} LM_\partial(D^2_{\underline{m}\setminus S}, D^4) \end{matrix} \right\}$$

level preserving?

such that $S \neq \emptyset, \underline{m}$ $\forall\, S' \subseteq S$ $\iff$ $\underline{m}\setminus S' \supseteq \underline{m}\setminus S$

D^4

$D^2 \times (\underline{m}\setminus S)$

$\shortparallel$

L & compatible null-homotopies of all L^i.

Open problems: Is $\mathcal{K}(\underline{m})$ injective on π_0, what about π_i?