Topic 2c: Link homotopy and non-repeating Whitney towers

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MPIM
Topics Course
with Rob

Def.: $L = (\ell_1, \dots, \ell_m)$ **almost trivial**, consider

$$[\ell_m] \in M(L \setminus \ell_m)_{m-1} \cong M(m-1)_{m-1} \overset{\cong}{\underset{j}{\longleftarrow}} \mathbb{Z}[S_{m-2}]$$

Then $j^{-1}[\ell_m] = \sum_\sigma \sigma \cdot \mu_L(\sigma 1, \dots, \sigma m-2, m-1, m)$, $X_\sigma := [x_{\sigma(1)}, [x_{\sigma(2)}, \dots [x_{\sigma(m-2)}, x_{m-1}]]] \longleftarrow \sigma$

Open problem: $\mathcal{K}(L) := [S^1 \times \dots_m \times S^1, C_m \mathbb{R}^3]$

$(t_1, \dots, t_m) \longmapsto (L(t_1), L(t_2), \dots, L(t_m))$

Is the map $LM[\coprod^m S^1, \mathbb{R}^3] \nearrow$ injective?

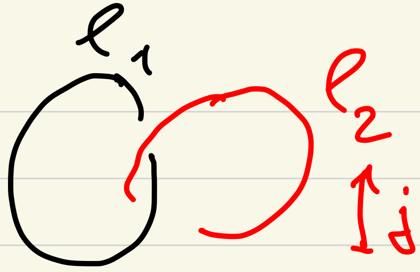
Rem.: It detects Milnor invariants, i.e. almost trivial links

For links L in 3-mfds M, the almost trivial case works as before, giving a bijection with

$\mathbb{Z}[\pi_1 M^{m-1} \times S_{m-2}]$, where each leave has a $g_i \in \pi_1 M$.
we have a root at ℓ_m &

Ex : L =

(0)

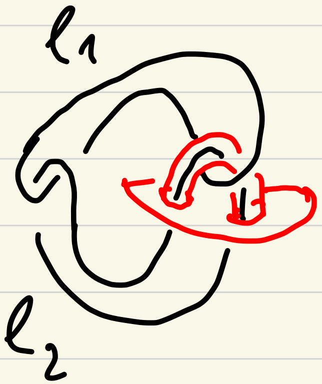


$\ell_2 \overset{j}{\uparrow} 1$

$\ell_2 = x_1 \iff lk(\ell_1, \ell_2) = 1$

$lk(\ell_1, \ell_2) = \mu_L(12)$

(1)

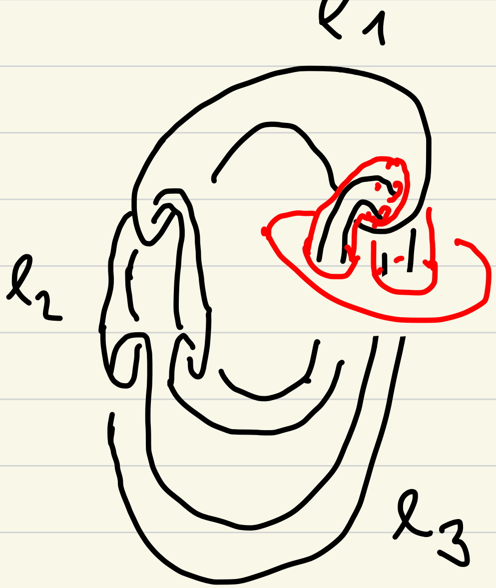


$\ell_3 = [x_1, x_2] \iff \mu_L(1\,2\,3) = 1$

$\uparrow j$

y_1

(2)



$\ell_4 = [x_1, [x_2, x_3]] \iff \mu_L(1234) = 1$

$\uparrow j$

$y_1 \cdot y_2$

etc.

λ_{n-2}

{ non-repeating W. towers of order $n-2$ }

$\downarrow$ $S_n \circlearrowleft$

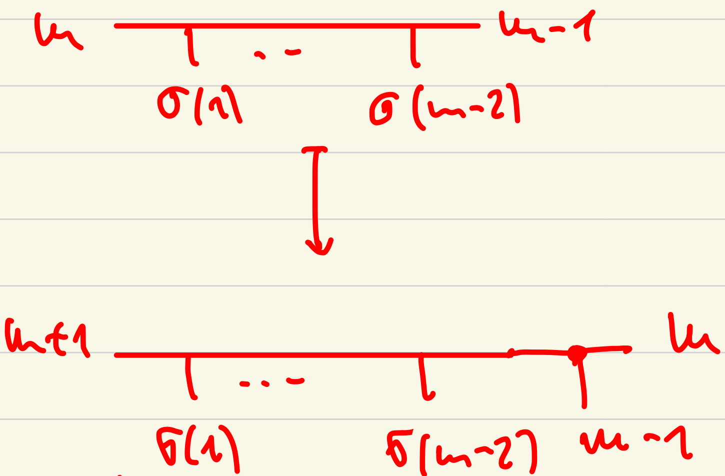
$$S_n \curvearrowright \Lambda_{n-2}(n) \cong \mathbb{Z}[S_{n-2}] \underset{\ell_n}{\cong} \{\text{almost triv. } n\text{-links}\}$$

$\downarrow$ incl. $\downarrow$ $Y \mapsto Y \cdot y_{n-1}$ $\downarrow$ Bing double ℓ_{n-1} !

$$\Lambda_{n-1}(n+1) \cong \mathbb{Z}[S_{n-1}] \underset{\ell_{n+1}}{\cong} \{\text{almost triv. } (n+1)\text{-links}\}$$

(to the left)

read commutator towards n.



$x_{n-1} \mapsto [x_{n-1}, x_n]$

— ıı — $(n+1)$

$$\mathbb{Z}[S_{n-2}] \rightarrowtail \Lambda_{n-2}(n)$$

onto sy shortest geodesic

add root to n

$$\searrow \quad M(n-1)_{n-1}$$

$$y_I \mapsto [x_{I,n-1}] \quad \cong$$

Recall: If G is normally generated by $x_1, \dots, x_m$ then
$$M(G) := G \Big/ [x_i, x_i^{g_i}] = 1 \quad \forall i = 1, \dots, m, \; g_i \in G$$
is nilpotent of class m and it's generated by $x_1, \dots, x_m$. Ex.: F = free group on $x_1, \dots, x_m$

Lemma: There is a split extension, $M(F) =: M(m)$
$$(NR(m), +) \xrightarrow{j} M(m+1) \underset{\longleftarrow}{\overset{p}{\twoheadrightarrow}} M(m), \quad p(x_i) = \begin{cases} x_i & i \geq 1 \\ 1 & i = 0 \end{cases}$$
where $NR(m) :=$ free ring on non-repeating $y_1, \dots, y_m$
$= \mathbb{Z}(y_1, \dots, y_m) \,/\,$ ideal gen. by repeating monomials.

So $(NR(m), +)$ is free abelian of rank $\sum_{k=0}^{m} \binom{m}{k} \cdot k!$
$$\Longrightarrow \quad \operatorname{rk} M(2) = 3, \quad \operatorname{rank} M(3) = 3 + (1 + 2 + 2) = 8 < 9 = 3 \cdot \operatorname{rk} M(2).$$

Here $j(y_{i_1} \cdots y_{i_s}) := [x_{i_1}, [x_{i_2}, \ldots, [x_{i_s}, x_0]] \cdots]$

clearly lies in $Ker(p)$ = normal closure of x_0

$\Rightarrow$ j is group homom. (an abelian group!)

The conjugation action of $M(m)$ on $NR(m)$

is determined by $x_i \, j(Y) \, x_i^{-1} \overset{(*)}{=} j((1+y_i) \cdot Y)$

i.e. it uses ring multiplication and in fact

gives a homom. $M(m) \xrightarrow{e} NR(m)^\times$ $e(x_i) = 1+y_i$

s.t. $\forall g \in M(m), Y \in NR(m)$ $g \, j(Y) g^{-1} = j(e(g) \cdot Y)$

Proof: (i) e is well-defined because the foll. commute

$e(x_i) = 1+y_i$ and $e(x_i^g) = 1 + e(g) \, y \, e(g)^{-1}$ ∎

(ii) j is injective because $NR(m) \xrightarrow{j} M(m+1) \xrightarrow{e} NR(m+1)^\times$

$y_I \longmapsto x_{[I,0]} \longmapsto 1 + y_{[I,0]}$ ∎

(*) $x_i\, j(Y_I)\, x_i^{-1} = x_i (x_{[I,0]}) x_i^{-1} = [x_i, x_{[I,0]}] \cdot x_{[I,0]}$

$= j(y_i \cdot Y_I) \cdot j(Y_I) = j((1+y_i)\cdot Y_I)$ ∎

(iii) $\ker p \subseteq \operatorname{im}(j)$ follows from $x_0^g = j(e(g))$:

Ind. on word length of g: length $0 \iff g=1$ ✓

$x_0^{x_i g} = x_i (x_0^g) x_i^{-1} \underset{\text{Ind.}}{=} x_i\, je(g)\, x_i^{-1} \overset{(*)}{=} j((1+y_i)\cdot e(g))$

$= j(e(x_i)e(g)) = je(x_i g)$ ∎

Def.: L s.t. $[L \setminus \ell_m]$ trivial triv.

$$M(L\setminus \ell_m) \xleftarrow[\text{mer.}]{\cong} M(m-1) \xrightarrow{p_{m-1}} M(m-2)$$

$p_{m-1}(\ell_m) = 1$

$\iff [L \setminus \ell_{m-1}]$ triv. Ind. on m

$M(m-1) \uparrow j$

$[\ell_m] \mapsto \sum_{|I|\leq m-2} a_I(L)\cdot y_I \in NR(m-2)$

$\mu_L(I, m-1, m) := a_I(L)$

L almost trivial $\overset{\text{Ind.}}{\iff} a_I(L) = 0 \;\forall\, |I| < m-2$.

Remark: Choice of meridians implies that $a_I(L)$ is only well-defined for shortest $|I|$, i.p. for almost trivial L: $\forall\, g_{i_k} \in M(m-1)$

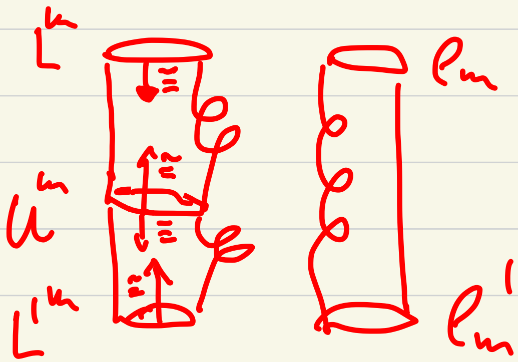
$$[x_{i_1},[x_{i_2},\dots x_{i_{m-1}}]\dots] = [x_{i_1}^{g_{i_1}},[x_{i_2}^{g_{i_2}},\dots x_{i_{m-1}}^{g_{i_{m-1}}}]\dots]$$

because $M(m-1)$ is nilpotent of class $m-1$

we can work modulo m-fold commutators where bilinearity of $[\,,]$ holds!

Final argument in proof of Milnor's theorem:

$$\begin{array}{ccccc} M(L^m)_{m-1} & \xleftarrow{\cong} & M(m-1)_{m-1} & \xrightarrow{\cong} & M(L'^m)_{m-1} \\ \cong\ \ \ell_m & \longleftarrow & \| & \longmapsto & \ell'_m \\ \mathbb{Z}[S_{m-2}] & & M(\text{link homotopy}) & \mathbb{R}^3\times I \setminus & \end{array}$$



$[L]=[L'] \iff$ ℓ_m freely hom. to ℓ'_m (in some $\mathbb{R}^3\times I \setminus$ l.h.) $\iff [\ell_m]=[\ell'_m]$ ∎