Topic 2b: Link homotopy classification for almost trivial links in 3-space.

Nov. 6, 2022

MPIM

Topics Course

with Rob

Recall: $\{\text{almost trivial links}\} \subseteq LM\left[\coprod^m S^1, \mathbb{R}^3\right]$

$$\cong \downarrow \mu$$

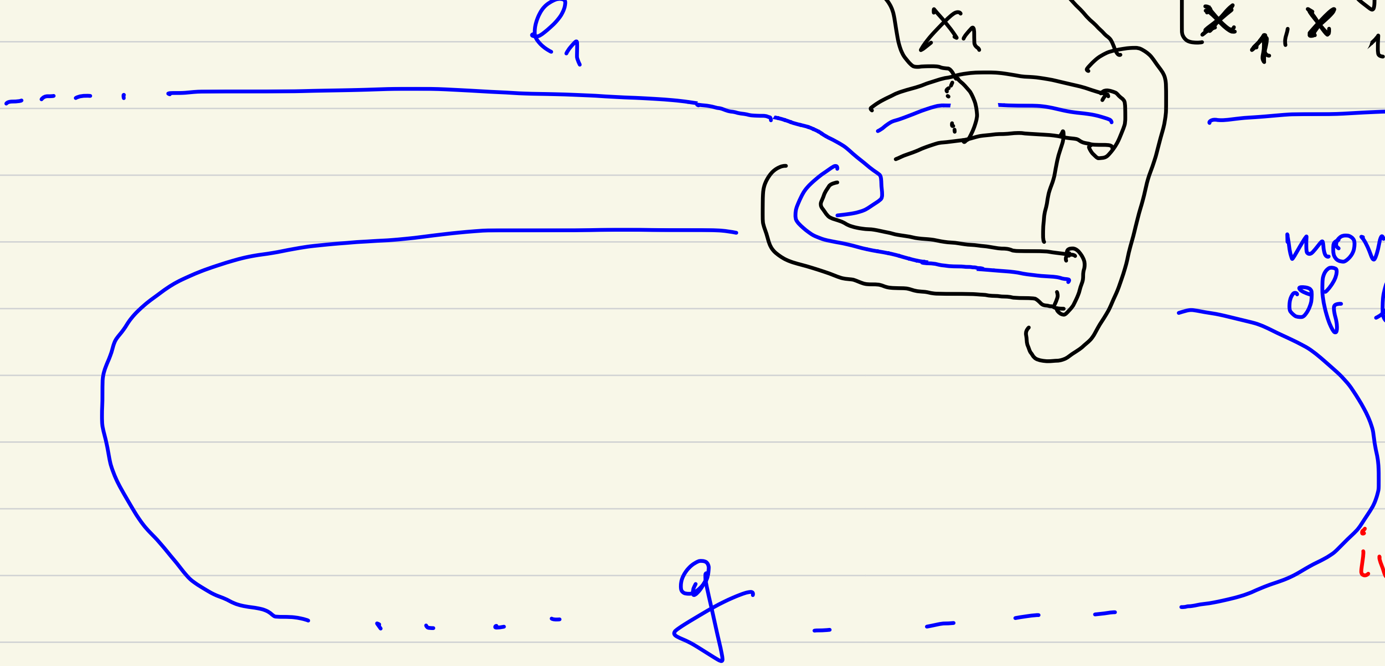
Today: proof of Milnor's result

$$\mathbb{Z}[S_{m-2}]$$

free abelian group on the symmetric group.

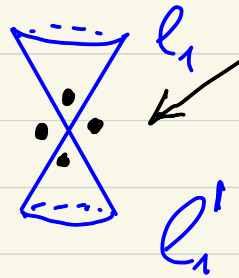
$$\mu = \sum_{\sigma \in S_{m-2}} \sigma \cdot \mu(\sigma(1), \dots, \sigma(m-2), m-1, m) = \sum_{\sigma} \mu_L(\sigma) \cdot \sigma$$

(non-repeating) **Milnor invariants**



$[x_1, x_1^g] =$ top-cell of Clifford torus

movie of h_t



$M^3 \times I \setminus h \simeq (M \setminus \ell_1) \cup e^2_{[x_1, x_1^g]}$

invariant of link homotopy

Def.: The **Milnor group of L** is the quotient

$$M(L) := \pi_1(\mathbb{R}^3 \setminus L) \Big/ [x_i, x_i^g] = 1 \quad \forall i = 1, \dots, m \quad \forall g \in \pi_1$$

In Rob's talk today: $L = (\ell_1, \dots, \ell_m)$

1) L is almost trivial $\iff$ L bounds a non-repeating Whitney tower W of order $m-2$.

2) The intersection invariant $\tau(W) \in \left\langle \begin{array}{c}\text{non-repeating trees}\\ \text{of order } m-2\end{array} \right\rangle \Big/ \text{AS, Jacobi}$

translates to μ_L via the following isomorphism:

$$\begin{array}{ccc} \overset{m}{\bullet\!-\!\!\!-\!\!\!\!\underset{\sigma(1)}{\bullet}\!\!-\!\!\underset{\sigma(2)}{\bullet}\cdots\underset{\sigma(m-2)}{\bullet}\!-\!\!\!-}\overset{m-1}{} & \longleftarrow & \\ \uparrow & & \uparrow \cong \\ \sigma & \in & \mathbb{Z}[S_{m-2}] \\ & & \wr\!\wr \end{array}$$

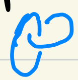
Corollary: Can figure out all $[\ell_i] \in M(L \setminus \ell_i) \cong M(m-1)$ in terms of ℓ_m by "shaking the tree": $\operatorname{Ker}\left(M(m-1) \to \prod_{i=1}^{m-1} M(m-2)\right)$

$$[\ell_m] = \sum_{\sigma \in S_{m-2}} \mu_L(\sigma) \cdot \bullet\!\!-\!\!\underset{\sigma(1)}{|}\cdots\underset{\sigma(m-2)}{|}\!\!-\,{}^{m-1} = j\left(\sum_{\sigma} \mu_L(\sigma) \cdot y_{\sigma(1)} \cdots y_{\sigma(m-2)}\right) \overset{j\ \cup\!\wr}{} NR(m-2)_{m-2}$$

$$\Rightarrow \quad [\ell_i] \mathrel{\hat{=}} \sum_{\sigma} \mu_L(\sigma)\ {}^{m}\!\!-\!\!\underset{\sigma(1)}{|}\cdots\underset{\sigma(m-2)}{|}\!\!-\,{}^{m-1} = \sum_{\sigma} \mu_L(\sigma) \cdot \left(\sum_{t} a_{\sigma,t} \cdot {}^{i}\!\!-\!\!\underset{t(\cdot)}{|}\cdots\underset{t(\cdots)}{|}\!\!-\!\!\bullet\,{}^{i-1}\right)$$

Some algebra of Milnor groups for $L = (l_1, \dots, l_m)$

1) $M(L)$ is nilpotent of class $\le m$, i.e. $M(L)_{m+1} = \{1\}$

class $M($  $)$ is $= 1 \le 2$.

2) $M(L)$ is generated by $x_1, \dots, x_m$, i.e. one meridian per component

3) If L, L' are almost trivial then $[L] = [L']$ iff

$$[l_m] = [l_m'] \in M(L \setminus l_m) \underset{\text{mer}}{\overset{\cong}{\longleftarrow}} M(\text{unlink}) \underset{\text{mer}}{\overset{\cong}{\longrightarrow}} M(L' \setminus l_m')$$

Rem.: This is independent of choices by (1) + (2)

$\Downarrow$ $\Downarrow$

whiskers not i-port $\Leftarrow$ l_m central   iso's unique.

Proof: 1) $\bigcap_{i=1}^{m} \operatorname{Ker} p_i \rightarrowtail M(L) \xrightarrow{\prod p_i} \prod_{i=1}^{m} M(L \setminus l_i)$ is central ext.

2) Induction using $m_1^{a m_1^{l} b} = m_1^{ab}$

3) follows from remark and finger move relations the 4D-computation of π_1. ∎

Def.: Magnus expansion is the homom. $M(n) \xrightarrow{e} NR(n)^\times$

where $NR(n) = \mathbb{Z}\langle y_1, \dots, y_n\rangle$ is the free ring on non-repeating y_i, i.e. on y_i modulo ideal spanned by repeating monomials.

$M(\text{unlink}) = MF(x_1, \dots, x_n)$

$e(x_i) = 1 + y_i$ commutes with $e(x_i^g) = 1 + e(g) y_i e(g)^{-1}$.

This is well-defined! Note $e(x_i^{-1}) = 1 - y_i$ and $(y_1 + y_2)^2 = y_1 y_2 + y_2 y_1$

e.g. $(NR(2), +) = \langle 1, y_1, y_2, y_1 y_2, y_2 y_1 \rangle \cong \mathbb{Z}^5$ is non-comm.

$\text{rk}\, NR(n) = \sum_{k=1}^{n} k! \cdot \binom{n}{k} = \sum_{k=1}^{n} \frac{n!}{(n-k)!} = \text{rk}\, M(n+1) - \text{rk}\, M(n)$

Lemma: There is a split extension $(NR(n), +) \overset{j}{\rightarrowtail} M(n+1) \overset{p_0}{\twoheadrightarrow} M(n)$, $x_0 \longmapsto 1$

with $j(y_{i_1} \cdots y_{i_s}) := [x_{i_1}, [x_{i_2}, \dots [x_{i_s}, x_0]] \dots]$

The conjugation action is $g\, j(Y)\, g^{-1} = j(e(g) \cdot Y)$ (left mult. in $NR(n)$)

Cor.: $\langle y_{\sigma(1)} \cdots y_{\sigma(n)} \rangle_{\sigma \in S_n} \overset{j}{\rightarrowtail} M(n+1) \xrightarrow{\prod p_i} \prod_{i=0}^{n} M(n)$ is exact

Def.: $\mu_L \longmapsto \ell_0 \longmapsto 1$ iff $L = (\ell_0, \dots, \ell_{n+1})$ is almost trivial

$\Longrightarrow$ Milnor's Thm. via 3) above.

$\prod p_i$ not onto for $n=2$: $\text{rk}\, MF(3) = 5 + 3 < 3 \cdot \text{rk}\, MF(2) = 3 \cdot 3$

Proof of Lemma: 1) $\mathrm{Ker}\, p$ = normal closure of x_0 is abelian
$\Rightarrow$ j is well-defined since it lies in $\mathrm{Ker}\, p$

2) j is injective because $NR(m)_+ \xrightarrow{j} MF(m+1) \xrightarrow{e} NR(m+1)^\times$ is inj.

3) $\mathrm{Ker}(p) \leq \mathrm{im}(j) \Leftarrow$ $\underset{(*)}{x_0^g = je(g)}$ $\forall g \in MF(m)$

$y_I \longmapsto [x_I, x_0] \longmapsto 1 + [y_I, y_0]$

$$x_0^{x_i} = [x_i, x_0] \cdot x_0 = j(y_i) \cdot j(1) = j(1+y_i) = je(x_i)$$

By ind. $x_0^{x_i g} = x_i(je(g))x_i^{-1} = j((1+y_i) \cdot e(g)) = j(e(x_i)e(g)) = je(x_i g)$

conj. = left mult.

$x_i\, j(y_I)\, x_i^{-1}$

$$= x_i [x_I] x_i^{-1} = [x_i, x_I] \cdot [x_I] = j(y_i \cdot y_I) \cdot j(y_I) = j((y_i + 1) \cdot y_I)$$

Next goal: Identify this μ with non-repeating μ_L by pushing off parallel copies.