Final Proofs: G group $\Rightarrow L(G) := \bigoplus_{k\geq 0} G_k / G_{k+1}$ is a $\mathbb{N}$-graded **Lie algebra** under $(+, [\,,\,]) = (\cdot, [\,,\,])$

$$ab-ba \qquad ghg^{-1}h^{-1}$$

F free group on generators $x_1, \dots, x_m \Rightarrow L(F)$ free Lie alg. on x_i

Magnus expansion is the monom. $F \longrightarrow R[[X_1, \dots, X_m]]$ non-comm. power series

$$x_i \longmapsto 1+X_i$$

$$\begin{tikzcd} F \arrow[r] \arrow[d] & R[[X_1,\dots,X_m]] \arrow[d] \\ M(m) \arrow[r, "e"] & NR(m) \end{tikzcd}$$

finite rank!

We showed that in Lemma below, $M(m)$ acts on $(NR(m), +)$ via left mult. by e

$$\forall g \in M(m), Y \in NR(m): \quad g\, j(Y)\, g^{-1} = j(e(g) \cdot Y) \qquad (*)$$

Lemma: There is a split extension, $M(F) =: M(m)$

$$(NR(m), +) \xrightarrow{j} M(m+1) \underset{\longleftarrow}{\xrightarrow{p}} M(m), \quad p(x_i) = \begin{cases} x_i & i \geq 1 \\ 1 & i=0 \end{cases}$$

where $NR(m) :=$ free ring on non-repeating $y_1, \dots, y_m$

$= \mathbb{Z}\langle y_1, \dots, y_m \rangle /$ ideal gen. by repeating monomials.

So $(NR(m), +)$ is free abelian of rank $\sum_{k=0}^{m} \binom{m}{k} \cdot k!$

Proof: Recall $j(y_{i_1}\cdots y_{i_s}) := [x_{i_1},[x_{i_2},\dots,[x_{i_s},x_0]]\dots]$,

$j(1) := x_0$, is well-defined and lies in Ker p.

(i) $\operatorname{Ker} p \subseteq \operatorname{im}(j)$ follows from $\boxed{x_0^{g} = j(e(g))}$:

Ind. on word length of g: length $0 \Leftrightarrow g = 1$ ✓

$$x_0^{x_i g} = x_i(x_0^{g})x_i^{-1} \underset{\text{Ind.}}{=} x_i\, je(g)\, x_i^{-1} \underset{(*)}{=} j((1+y_i)\cdot e(g))$$
$$= j(e(x_i)e(g)) = je(x_i g).$$

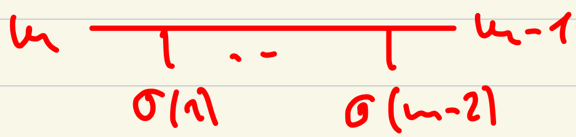
(ii) j is injective because $NR(n) \xrightarrow{j} M(n+1) \xrightarrow{e} NR(n+1)^{\times}$
$y_I \longmapsto x_{[I,0]} \longmapsto 1 + y_{[I,0]}$

is clearly injective by

noticing that the first term in $y_{[I,0]}$ is $y_I \cdot y_0$ ∎

The crucial diagram: 1) First it's algebraic part:

$\sigma \longmapsto$



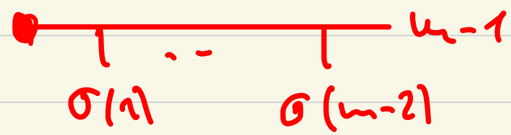
$$\begin{tikzcd} \mathbb{Z}[S_{m-2}] \arrow[r, tail] \arrow[dr, "j"'] & \Lambda_{m-2}(m) \arrow[d] \\ & M(m-1)_{m-1} \end{tikzcd}$$

$y_\sigma = y_{\sigma(1)} \cdots y_{\sigma(m-2)}$ $\cong$

$\longmapsto [x_{\sigma_1}, [x_{\sigma_2}, \ldots [x_{\sigma_{m-2}}, x_{m-1}]] \ldots]$

is onto by lengthening geodesics from $m-1$ to m inside tree!

$\Longrightarrow$ all 3 maps are isom.



read rooted tree toward the root. means that we start with $[x_{\sigma(m-2)}, x_{m-1}]$ and add $x_{\sigma(i)}$ to left ending with $\sigma(1)$

2) Geometric part: $L = (\ell_1, \ldots, \ell_m)$

$$\left\{ \begin{matrix} \text{order } (m-2) \text{ non-repeating} \\ \text{Whitney towers in } D^4 \end{matrix} \right\} \xrightarrow{\;\partial\;} \left\{ \begin{matrix} \text{almost homotopically} \\ \text{trivial links in } S^3 \end{matrix} \right\}$$

$$\Big\downarrow \lambda_m \qquad\qquad\qquad \cong \Big\downarrow [\ell_m]$$

$$\Lambda_{m-2}(m) \cong M(m-1)_{m-1} \xrightarrow{\cong} M(L \setminus \ell_m)_{m-1}$$

Important note: $M(L\setminus \ell_m)_{n-1}$ is central in $M(L\setminus \ell_m)$ and hence changing ℓ_m (or x_i!) by conjugation does not change our invariant $[\ell_m]$.

This is wrong in general, e.g. take $L=(\ell_1,\ell_2,\ell_3)$ with $lk(\ell_1,\ell_2)=0$, so up to link hom. $(\ell_1,\ell_2)=$ unlink.

$\Rightarrow$ $\ell_3 \in M(\ell_1,\ell_2) \cong M(2)$ is free and fits into

$$\mathbb{Z}\cdot 1 \oplus \mathbb{Z} y_1 = NR(y_1) \xrightarrow{j} M(2) \xrightarrow{x_2 \equiv 1} M(x_1) = \mathbb{Z}.$$

$lk(\ell_1,\ell_3)=0 \Rightarrow$

$$\ell_3 = j(a\cdot 1 + b\cdot y_1)$$

$a = lk(\ell_2,\ell_3)$

$b = \mu_L(123)$

$j(1) = x_2$, $j(y_1) = [x_1,x_2]$

Recall: $x_1^{k}\, j(Y)\, x_1^{-k} = j((1+k y_1)\cdot Y)$

$\ell_3 = j(a\cdot 1 + b\cdot y_1) \Rightarrow x_1^{k}\, \ell_3\, x_1^{-k} = j(a\cdot 1 + (b+ka)\cdot y_1)$

$[L] = [\ell_1,\ell_2, g\ell_3 g^{-1}] \Longrightarrow \mu_L(123)$ not well-defined

e.g. $a=3$, $b=1$: