Classical torsion and L^2-torsion

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Bonn, 29. May & 12. June

Outline

- The first algebraic K-group and the Whitehead group.
- Torsion invariants for chain complex.
- Whitehead torsion.
- The Alexander polynomial.
- L^2-torsion and its applications.

The first algebraic K-group and the Whitehead group

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R to be the abelian group $K_1(R)$, whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get

$$[f_1] = [f_0] + [f_2];$$

- Given two automorphisms f, g of the same finitely generated projective R-module, we get

$$[g \circ f] = [f] + [g].$$

- $K_1(R)$ is isomorphic to $GL(R)/[GL(R), GL(R)]$.

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the reduced K_1-group

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

  which in general is not bijective. It is bijective, if R is a field.

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

Proof.

- The ring $\mathbb{Z}$ possesses an Euclidean algorithm.
- Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1).
- This implies that any element in $K_1(\mathbb{Z})$ is represented by ± 1.

□

- Let G be a finite group. Let F be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
- Define $r_F(G)$ to be the number of irreducible F-representations of G.
- The Whitehead group $\mathrm{Wh}(G)$ is a finitely generated abelian group of rank $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- The torsion subgroup of $\mathrm{Wh}(G)$ is the kernel of the map $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.
- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable.

Example (Non-vanishing of $\mathrm{Wh}(\mathbb{Z}/5)$)

- The ring homomorphism $f\colon \mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$, which sends the generator of $\mathbb{Z}/5$ to $\exp(2\pi i/5)$, and the map $\mathbb{C} \to \mathbb{R}^{>0}$ taking the norm of a complex number, yield a homomorphism of abelian groups

$$\phi\colon \mathrm{Wh}(\mathbb{Z}/5) \to \mathbb{R}^{>0}.$$

- Since $(1 - t - t^{-1}) \cdot (1 - t^2 - t^3) = 1$ in $\mathbb{Z}[\mathbb{Z}/5]$, we get the unit $1 - t - t^{-1} \in \mathbb{Z}[\mathbb{Z}/5]^\times$. Its class in the Whitehead group is sent to $(1 - 2\cos(2pi/5)) \neq 1$ and hence is an element of infinite order.

- Indeed, this element is a generator of the infinite cyclic group $\mathrm{Wh}(\mathbb{Z}/5)$.

Conjecture

If G is torsionfree, then $\mathrm{Wh}(G)$ *is trivial.*

Torsion invariants for chain complex

- Let R be an associative ring with unit, not necessarily commutative.
- R-modules are by default left R-modules.
- An R-chain complex C_* is called finite if there exists a natural number N such that $C_n = 0$ for every $n \in \mathbb{Z}$ with $|n| > N$ and C_n is finitely generated for every $n \in \mathbb{Z}$.
- An R-chain complex C_* is called free or projective if C_n is free or projective for all $n \in \mathbb{Z}$.
- A free R-chain complex C_* is called based free if C_n comes with a (unordered) basis B_n for $n \in \mathbb{Z}$.
- An R-chain complex C_* is called acyclic if $H_n(C_*)$ vanishes for every $n \in \mathbb{Z}$.

- An chain contraction γ_* for a R-chain complex $C_* = (C_*, c_*)$ is a sequence of R-maps $\gamma_n \colon C_n \to C_{n+1}$ satisfying $c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$.

- An R-chain complex C_* is called contractible if it possesses a chain contraction.

- A contractible R-chain complex is acyclic.

- The converse is not true in general, e.g., $R = \mathbb{Z}$ and C_* is concentrated in dimensions 0,1,2 and given there by the exact sequence $\mathbb{Z} \xrightarrow{2 \cdot \mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\mathrm{pr}} \mathbb{Z}/2$.

- However, a projective R-chain complex C_* is acyclic if and only if it is contractible.

- Let C_* be an acyclic finite projective R-chain complex.
- Put $C_{\mathrm{odd}} = \bigoplus_{n \in \mathbb{Z}} C_{2n+1}$ and $C_{\mathrm{ev}} = \bigoplus_{n \in \mathbb{Z}} C_{2n}$.
- Let γ_* and δ_* be two chain contractions. Put $\Theta_* = \gamma_{*+1} \circ \delta_*$. Then the composite

$$C_{\mathrm{odd}} \xrightarrow{(c+\gamma)_{\mathrm{odd}}} C_{\mathrm{ev}} \xrightarrow{(\mathrm{id} + \Theta)_{\mathrm{ev}}} C_{\mathrm{ev}} \xrightarrow{(c+\delta)_{\mathrm{ev}}} C_{\mathrm{odd}}$$

  is given by an upper triangular matrix whose entries on the diagonal are identity morphisms. Also $(\mathrm{id} + \Theta)_{\mathrm{ev}}$ is given by an upper triangular matrix whose entries on the diagonal are identity morphisms. The analogous statement holds if we interchange odd and ev, and γ_* and δ_*.
- In particular we see that $(c + \gamma)_{\mathrm{odd}} \colon C_{\mathrm{odd}} \to C_{\mathrm{ev}}$ is an isomorphism.

- Let $f\colon M \xrightarrow{\cong} N$ be an isomorphism of finitely generated based free R-modules. Let A be the matrix describing f with respect to the given bases. Then we define

$$\tau(f) \in \widetilde{K}_1(R)$$

by the class of A coming from the identification of $K_1(R)$ with $GL(R)/[GL(R), GL(R)]$, where

$$\widetilde{K}_1(R) := \operatorname{cok}(K_1(\mathbb{Z}) \to K_1(R)) = K_1(R)/\{(\pm 1)\}.$$

- Equivalently, choose an isomorphism $b\colon N \to M$ which respects the given bases. Then $\tau(f) = [b \circ f]$.

- The fact that the bases are only unordered does not affect the definition of $\tau(f)$ since we are working in $\widetilde{K}_1(R)$.

- If we have a commutative diagram of finitely generated based free R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow g & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

  such that the vertical arrows are bijective and the rows are based exact, then we get in $\widetilde{K}_1(R)$.

$$\tau(g) = \tau(f) + \tau(h).$$

- If $f\colon M \to N$ and $g\colon N \to P$ are isomorphisms of finitely generated based free R-modules, then we get in $\widetilde{K}_1(R)$

$$\tau(g \circ f) = \tau(g) + \tau(f).$$

- Let C_* be an acyclic finite based free R-chain complex. Choose a chain contraction γ_*. Define

$$\tau(C_*) \in \widetilde{K}_1(R)$$

by $\tau\big((c+\gamma)_{\mathrm{odd}} \colon C_{\mathrm{odd}} \to C_{\mathrm{ev}}\big)$.

- This is independent of the choice of γ_*, since we get for any other chain contraction δ_* from the facts above and the observation that an upper triangular matrix with identities on the diagonal represents zero in $\widetilde{K}_1(R)$ the equality in $\widetilde{K}_1(R)$

$$\tau\big((c+\gamma)_{\mathrm{odd}}\big) = -\tau\big((c+\delta)_{\mathrm{ev}}\big).$$

- Let $f_*\colon C_* \to D_*$ be an R-chain homotopy equivalence of finite based free R-chain complexes. Let $\operatorname{cone}(f_*)$ be its mapping cone

$$\cdots \to C_n \oplus D_{n+1} \xrightarrow{\begin{pmatrix} -c_n & 0 \\ f_n & d_{n+1} \end{pmatrix}} C_{n-1} \oplus D_n \to \cdots$$

This is a contractible finite based free R-chain complex.

Definition (Whitehead torsion)

Define the Whitehead torsion of f_* by

$$\tau(f_*) := \tau(\operatorname{cone}(f_*)) \in \widetilde{K}_1(R).$$

- If we have a commutative diagram of finitely generated based free R-chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \\ & & \downarrow f_* & & \downarrow g_* & & \downarrow h_* & & \\ 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \end{array}$$

such that the vertical arrows are R-chain homotopy equivalences and the rows are based exact, then we get in $\widetilde{K}_1(R)$

$$\tau(g_*) = \tau(f_*) + \tau(h_*).$$

- If $f_*\colon C_* \to D_*$ and $g_*\colon D_* \to E_*$ are R-chain homotopy equivalences of finite based free R-chain complexes, then we get in $\widetilde{K}_1(R)$
$$\tau(g_* \circ f_*) = \tau(g_*) + \tau(f_*).$$

- Let $f_*, g_*\colon C_* \to D_*$ be R-chain homotopy equivalences of finite based free R-chain complexes. If they are R-chain homotopic, then we get in $\widetilde{K}_1(R)$
$$\tau(f_*) = \tau(g_*).$$

- Let $f_*\colon C_* \to D_*$ be an isomorphism of finite based free R-chain complexes, not necessarily preserving the bases. Then
$$\tau(f_*) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \tau(f_n).$$

Example (1-dimensional case)

If the acyclic finite based free C_* is concentrated in two consecutive dimensions n and $n-1$, then the n-th differential $c_n\colon C_n \to C_{n-1}$ is an isomorphism of finitely generated based free R-modules and

$$\tau(C_*) = (-1)^{n+1} \cdot \tau(c_n).$$

since we get a chain contraction γ_* by putting $\gamma_{n-1} = c_n^{-1}$ and $[c_n] = -[\gamma_n]$ holds in $\widetilde{K}_1(R)$.

Example (2-dimensional case)

- Suppose that the acyclic based free R-chain complex C_* is concentrated in two dimensions. Then it is the same as a short exact sequence of finitely generated based free R-modules

$$0 \to C_2 \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \to 0$$

- One easily checks that there exists a R-map $\gamma_1 \colon C_1 \to C_2$ with $\gamma_1 \circ c_2 = \mathrm{id}_{C_2}$. Moreover, for any such γ_1, one can find a R-map $\gamma_0 \colon C_0 \to C_1$, such that we get a chain contraction γ_*.
- Hence $\tau(C_*)$ is represented by the isomorphism of finitely generated based free R-modules

$$C_1 \xrightarrow{\begin{pmatrix} c_1 \\ \gamma_1 \end{pmatrix}} C_0 \oplus C_2$$

$\mathbb{R}$-chain complexes

Example ($R = \mathbb{R}$)

- Taking the logarithm of the absolute value of the determinant of an invertible matrix induces an isomorphism
$$\widetilde{K}_1(\mathbb{R}) \xrightarrow{\cong} \mathbb{R}, \quad [A] \mapsto \ln(|\det(A)|).$$
Hence the Whitehead torsion is just a real number.

- Consider the finite based free 1-dimensional $\mathbb{R}$-chain complexes C_* and D_* given by $c_1 \colon \mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}} \mathbb{R}^2$ and $d_1 \colon \mathbb{R} \xrightarrow{0} \mathbb{R}$.

Example (Continued)

- Define a chain map $f_* \colon C_* \to D_*$ by the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}} & \mathbb{R}^2 \\ {\scriptstyle \begin{pmatrix} 1 & 0 \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} -8 & 4 \end{pmatrix}} \\ \mathbb{R} & \xrightarrow[\begin{pmatrix} 0 \end{pmatrix}]{} & \mathbb{R} \end{array}$$

- One easily checks that f_* induces an isomorphism on homology and hence is a $\mathbb{R}$-chain homotopy equivalence

Example (Continued)

- Its mapping cone is the 2-dimensional finite acyclic based free $\mathbb{R}$-chain complex

$$\mathbb{R}^2 \xrightarrow{\begin{pmatrix} -1 & -2 \\ -2 & -4 \\ 1 & 0 \end{pmatrix}} \mathbb{R}^2 \oplus \mathbb{R} \xrightarrow{\begin{pmatrix} -8 & 4 & 0 \end{pmatrix}} \mathbb{R}$$

- A retraction $\gamma_1 \colon \mathbb{R}^2 \oplus \mathbb{R} \to \mathbb{R}^2$ of its second differential is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ -1/2 & 0 & -1/2 \end{pmatrix}$$

Example (Continued)

- Hence $(c+\gamma)_{\text{odd}}$ is given by the isomorphism of finitely generated based free $\mathbb{R}$-modules

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 0 & 0 & 1 \\ -1/2 & 0 & -1/2 \\ -8 & 4 & 0 \end{pmatrix}} \mathbb{R}^3$$

- Since its determinant is -2, we get

$$\tau(f_*) = \ln(2).$$

- Notice that we do not need a $\mathbb{R}$-basis, it suffices to have a Hilbert space structure on each $\mathbb{R}$-chain module.

- Namely, then one can just choose an orthonormal basis and define the torsion using this basis. If we choose another orthonormal basis, then the change of bases matrix is an orthogonal matrix and its determinant is ± 1.

- Hence we can define for any $\mathbb{R}$-chain homotopy equivalence $f_*\colon C_* \to D_*$ of finite Hilbert $\mathbb{R}$-chain complexes its Whitehead torsion

$$\tau(f_*) \in \mathbb{R}.$$

Definition (Laplace operator)

Let C_* be a finite Hilbert $\mathbb{R}$-chain complex. Define the n-th Laplace operator

$$\Delta_n = c_n^* \circ c_n + c_{n+1} \circ c_{n+1}^* \colon C_n \to C_n.$$

- Δ_n is a positive $\mathbb{R}$-homomorphism, and we have the orthogonal decomposition

  $$C_n = \bigoplus_{\lambda \geq 0} E_\lambda(\Delta_n),$$

  where $E_\lambda(\Delta_n)$ is the eigenspace of Δ_n for the eigenvalue $\lambda > 0$.

- Suppose that Δ_n is invertible, or, equivalently, that 0 is no eigenvalue. Then by functional calculus we get a $\mathbb{R}$-map

  $$\ln(\Delta_n) \colon C_n \to C_n$$

  This is the operator which is $\ln(\lambda) \cdot \mathrm{id}$ on $E_\lambda(\Delta_n)$.

- Obviously we get

$$\ln(\det(\Delta_n)) = \mathrm{tr}(\ln(\Delta_n)),$$

provided that Δ_n is invertible.

- Notice that both sides of the equation above are defined without choosing any basis.

Lemma

Let C_* be a finite Hilbert $\mathbb{R}$-chain complex.

1. *C_* is acyclic if and only if Δ_n is an isomorphism for each $n \in \mathbb{Z}$;*
2. *If C_* is acyclic, then we get*

$$\tau(C_*) = -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)) \in \mathbb{R}.$$

Proof.

- We have for $x \in C_n$

$$\begin{aligned}
\langle \Delta_n(x), x \rangle &= \langle c_n^* \circ c_n(x) + c_{n+1} \circ c_{n+1}^*(x), x \rangle \\
&= \langle c_n^* \circ c_n(x), x \rangle + \langle c_{n+1} \circ c_{n+1}^*(x), x \rangle \\
&= \langle c_n(x), c_n(x) \rangle + \langle c_{n+1}^*(x), c_{n+1}^*(x) \rangle \\
&= ||c_n(x)||^2 + ||c_{n+1}^*(x)||^2.
\end{aligned}$$

- This implies

$$\begin{aligned}
\ker(\Delta_n) = \ker(c_n) \cap \ker(c_{n+1}^*) &= \ker(c_n) \cap \operatorname{im}(c_{n+1})^{\perp} \\
&\xrightarrow{\cong} \ker(c_n)/\operatorname{im}(c_{n+1}) = H_n(C_*).
\end{aligned}$$

□

Proof (Continued).

- We explain the proof of the second assertion only in the special case, where C_* is concentrated in dimensions p and $(p-1)$.
- We have
$$\det(c_p)^2 = \det(c_p \circ c_p^*) = \det(c_p^* \circ c_p).$$
- Hence we get
$$\ln(|\det(c_p)|) = \frac{1}{2} \cdot \ln(\det(\Delta_{p-1})) = \frac{1}{2} \cdot \ln(\det(\Delta_p)).$$

□

Proof (Continued).

- We compute

$$\begin{aligned}
\tau(C_*) &= (-1)^{p+1} \cdot \ln(|\det(c_p)|) \\
&= (-1)^{p+1} \cdot \frac{1}{2} \cdot \ln(\det(\Delta_p)) \\
&= -\frac{1}{2} \cdot \big((-1)^p \cdot p \cdot \ln(\det(\Delta_p)) \\
&\qquad\qquad +(-1)^{p-1} \cdot (p-1) \cdot \ln(\det(\Delta_p))\big) \\
&= -\frac{1}{2} \cdot \big((-1)^p \cdot p \cdot \ln(\det(\Delta_p)) \\
&\qquad\qquad +(-1)^{p-1} \cdot (p-1) \cdot \ln(\det(\Delta_{p-1}))\big) \\
&= -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)).
\end{aligned}$$

□

Change of rings

- Often we will also consider the following situation, where $\phi\colon R \to S$ is a fixed ring homomorphism.
- Let C_* be a finite free R-chain complex, not necessarily based. Suppose that $\phi_* C_* := S \otimes_R C_*$ is acyclic.
- Next we define an element
$$\overline{\tau}(C_*) \in \operatorname{cok}\bigl(\phi_*\colon K_1(R) \to K_1(S)\bigr).$$
- Choose an R-basis B_* for C_*. It induces an S-basis $\phi_* B$ for $\phi_* C_*$ in the obvious way. Hence $(C_*, \phi_* B)$ is an acyclic finite based free S-chain complex and $\tau(C_*, \phi_* B) \in \widetilde{K}_1(S)$ is defined.
- Let $\overline{\tau}(C_*) \in \operatorname{cok}\bigl(\phi_*\colon K_1(R) \to K_1(S)\bigr)$ be the class of $\tau(C_*; \phi_* B)$.
- One easily checks that $\overline{\tau}(C_*)$ does not depend on the choice of B_*.

Example (Milnor torsion)

- Let $\phi\colon R \to S$ be the inclusion $\mathbb{Z} \to \mathbb{Q}$.
- Let $C*$ be a finite free $\mathbb{Z}$-chain complex such that $\mathbb{Q} \otimes_{\mathbb{Z}} C_*$ is acyclic. The latter condition is equivalent to the requirement that $H_n(C_*)$ is a finite abelian group for every $n \in \mathbb{Z}$.
- The cokernel of $K_1(\mathbb{Z}) \to K_1(\mathbb{Q})$ is by definition $\widetilde{K}_1(\mathbb{Q})$. Taking the norm of the determinant of an invertible matrix yields an isomorphism

$$\widetilde{K}_1(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^{>0}.$$

  Hence $\overline{\tau}(C_*)$ is a positive rational number.
- It is not hard to check

$$\overline{\tau}(C_*) = \prod_{n \in \mathbb{Z}} |H_n(C_*)|^{(-1)^n}.$$

Whitehead torsion

- Let $f\colon X \to Y$ be a homotopy equivalence of connected finite CW-complexes.
- If $\pi = \pi_1(X) = \pi_1(Y)$ is the fundamental group, f lifts to a π-homotopy equivalence $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ of the universal coverings which are finite free π-CW-complexes.
- By passing to the cellular chain complexes, we obtain a $\mathbb{Z}\pi$-chain homotopy equivalence $C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$.
- After a choice of a lift of each open cell e in X to a open cell $\widetilde{e}$ in $\widetilde{X}$ and of an orientation on $\widetilde{e}$, we obtain a preferred $\mathbb{Z}\pi$-basis for $C_*(\widetilde{X})$, and analogously for $C_*(\widetilde{Y})$.

- Hence we obtain an element $\tau(C_*(\widetilde{f})) \in \widetilde{K}_1(\mathbb{Z}\pi)$.
- The class $[\tau(C_*(\widetilde{f}))]$ in $\mathrm{Wh}(\pi)$ is independent of the choices.

Definition (Whitehead torsion)

We define Whitehead torsion of the homotopy equivalence $f\colon X \to Y$ of connected finite CW-complexes

$$\tau(f) := [\tau(C_*(\widetilde{f}))] \in \mathrm{Wh}(\pi).$$

- The pair (D^n, S^{n-1}_+) carries an obvious relative CW-structure with one $(n-1)$-cell and one n-cell.
- Define Y as the pushout $X \cup_f D^n$ for any map $f\colon S^{n-1}_+ \to X$.
- The inclusion $j\colon X \to Y$ is a homotopy equivalence and called an elementary expansion.
- There is a map $r\colon Y \to X$ with $r \circ j = \mathrm{id}_X$. This map is unique up to homotopy relative $j(X)$ and is called an elementary collapse.

Definition (Simple homotopy equivalence)

Let $f\colon X \to Y$ be a map of finite CW-complexes. We call it a simple homotopy equivalence, if there is a sequence of maps

$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each f_i is an elementary expansion or elementary collapse and f is homotopic to the composition of the maps f_i.

Theorem (Main properties of Whitehead torsion)

- *A homotopy equivalence of connected finite CW-complexes is a simple homotopy equivalence if and only if $\tau(f) \in \mathrm{Wh}(\pi)$ vanishes;*
- *The Whitehead torsion $\tau(f)$ is a homotopy invariant;*
- *There are sum and product formulas for it;*
- *If X and Y are finite CW-complexes and $f \colon X \to Y$ is a homeomorphism, then f is a simple homotopy equivalence;*
- *The Whitehead group $\mathrm{Wh}(\pi)$ and the Whitehead torsion of an h-cobordism W over M_0 defined in a previous lecture coincide with the Whitehead group and the Whitehead torsion of the inclusion $M_0 \to W$ in the sense of this lecture.*

- The notion of Reidemeister torsion for lens spaces, which led to the classification of lens spaces $L(V) := SV/(\mathbb{Z}/m)$ up to isometric diffeomorphism, or diffeomorphism, or homeomorphism, is a special case of the constructions above for the ring homomorphism $\phi\colon \mathbb{Z}[\mathbb{Z}/m] \to \mathbb{Q}[\mathbb{Z}/m] \to \mathbb{Q}(\mathbb{Z}[\mathbb{Z}/m])/(N)$, where N is the norm element.

- The point is that for a lens space $L(V)$ the $\mathbb{Q}(\mathbb{Z}[\mathbb{Z}/m])/(N)$-chain complex $\phi_* C_*(\widetilde{LV})$ is acyclic, which is a direct consequence of the fact that $\mathbb{Z}/m$ acts trivially on $H_*(SV) = H_*(\widetilde{L(V)})$.

Use "Franz independence lemma".

The Alexander polynomial

Definition (knot)

- A knot $K \subseteq S^3$ is a connected oriented 1-dimensional smooth submanifold of S^3.
- We call two knots $K \subseteq S^3$ and $K' \subseteq S^3$ equivalent if there exists an orientation preserving diffeomorphism $f\colon S^3 \to S^3$ such that $f(K) = K'$ and the induced diffeomorphism $f|_K\colon K \to K'$ respects the orientations.

- One can define knots also as smooth embeddings $S^1 \to S^3$ and then equivalent means isotopy of embeddings.

- If $(N, \partial N)$ is a tubular neighborhood of K, then define
$$M_K = S^3 \setminus \mathrm{int}(N).$$
This is a compact 3-manifold M_K whose boundary consists of one component which is a 2-torus.

- If $(N', \partial N')$ is another tubular neighborhood, then there is a diffeomorphism of compact 3-manifolds with boundary
$$S^3 \setminus \mathrm{int}(N) \xrightarrow{\cong} S^3 \setminus \mathrm{int}(N').$$
Hence we write M_K without taking the tubular neighborhood into account.

- M_k is homotopy equivalent to the knot complement $M \setminus K$.

- The knot K is trivial if and only if M_K is homeomorphic to $S^1 \times D^2$.

- If K is non-trivial, then M_k is an irreducible compact 3-manifold with incompressible boundary.

- Consider a knot $K \subseteq S^3$.
- By Alexander-Lefschetz duality $H^1(M_K; \mathbb{Z}) \cong \mathbb{Z}$. Hence there is a preferred infinite cyclic covering $p\colon \overline{M_K} \to M_k$.
- Consider the inclusion $\phi\colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Q}[\mathbb{Z}] \to \mathbb{Q}[\mathbb{Z}]_{(0)}$, where $\mathbb{Q}[\mathbb{Z}]_{(0)}$ is the quotient field of the integral principal ideal domain $\mathbb{Q}[\mathbb{Z}]$. Then it turns out that $\phi_* C_*(\overline{M_K})$ is $\mathbb{Q}[\mathbb{Z}]_{(0)}$-acyclic.
- We have defined above

$$\overline{\tau}(C_*(\overline{M_K})) \in \operatorname{cok}\bigl(K_1(\mathbb{Z}[\mathbb{Z}]) \to K_1(\mathbb{Q}[\mathbb{Z}]_{(0)})\bigr).$$

- The determinant induces an isomorphism

$$K_1(\mathbb{Q}[\mathbb{Z}]_{(0)}) \xrightarrow{\cong} (\mathbb{Q}[\mathbb{Z}]_{(0)})^{\times}.$$

  Elements in $(\mathbb{Q}[\mathbb{Z}]_{(0)})^{\times}$ are quotients $p(t)/q(t)$ for $p(t), q(t) \in \mathbb{Q}[t]$ with $q \neq 0$. Hence we get an identification

$$\operatorname{cok}\bigl(K_1(\mathbb{Z}[\mathbb{Z}]) \to K_1(\mathbb{Q}[\mathbb{Z}]_{(0)})\bigr) \xrightarrow{\cong} (\mathbb{Q}[\mathbb{Z}]_{(0)})^{\times}/\{\pm t^n\}.$$

- So we get a knot invariant

$$\overline{\tau}(C_*(\overline{M_K})) \in (\mathbb{Q}[\mathbb{Z}]_{(0)})^\times/\{\pm t^n\}.$$

- One can assign to a knot K its Alexander polynomial Δ_K which is a symmetric finite Laurent series in $\mathbb{Z}[\mathbb{Z}]$ such that its evaluation at $t = 1$ is 1.

- We have the following values of Δ_K

K	Δ_K
unknot	t^0
trefoil	$t^2 - t^0 + t^{-2}$
figure eight knot	$-t + 3t^0 - t^{-1}$

Theorem (Alexander polynomial and torsion invariants, (Milnor))

If $K \subseteq S^3$ is a knot, then we get in $(\mathbb{Q}[\mathbb{Z}]_{(0)})^\times/\{\pm t^n\}$ the equality

$$[(t-1)\cdot\Delta_K(t)] = \overline{\tau}(C_*(\overline{M_K})).$$

In particular Δ_K and $\overline{\tau}(C_*(\overline{M_K}))$ determine one another.

Question

Can we define an interesting torsion invariant for the universal covering of M_K which gives new information about K?

The definition of L^2-torsion

- We have defined for an acyclic finite Hilbert $\mathbb{R}$-chain complex C_* its torsion to be the real number

$$\tau(C_*) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)) \in \mathbb{R}.$$

- So one can try to make sense of the same expression when we consider a finite $\mathcal{N}(G)$-chain complex $C_*^{(2)}$, and declare this to be the L^2-torsion of $C_*^{(2)}$.
- The condition acyclic should become the condition weakly acyclic, i.e., $b_n^{(2)}(C_*^{(2)}) = 0$, or, equivalently, $H_n^{(2)}(C_*^{(2)}) = 0$ for all $n \in \mathbb{Z}$.
- The Laplace operator can be defined as before

$$\Delta_n^{(2)} := (c_n^{(2)})^* \circ c_n^{(2)} + c_{n+1}^{(2)} \circ (c_{n+1}^{(2)})^* \colon C_n^{(2)} \to C_n^{(2)}.$$

- The Laplace operator $\Delta_n^{(2)}$ is a weak isomorphism for all $n \in \mathbb{Z}$ if and only if $C_*^{(2)}$ is acyclic.
- The main problem is to make sense of $\ln(\det(\Delta_n^{(2)}))$. If this has been solved, we can define

Definition (L^2-torsion (Lück-Rothenberg))

Let X be a connected finite CW-complex. Then we define the L^2-torsion

$$\rho^{(2)}(\widetilde{X}) := -\frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln(\det(\Delta_n^{(2)})) \quad \in \mathbb{R},$$

where $\Delta_n^{(2)} \colon C_n^{(2)}(\widetilde{X}) \to C_n^{(2)}(\widetilde{X})$ is the Laplace operator for the finite Hilbert $\mathcal{N}(\pi)$-chain complex $C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$.

- The definition above extends to finite CW-complexes by taking the sum of the L^2-torsion for each path component.

- There is also an analytic definition in terms of heat kernels of the universal covering of a closed Riemannian manifold due to Matthey and Lott. Both approaches have been identified by Burghelea-Friedlander-Kappeler-Mc Donald.

- Explicit computations and the proof of some general properties are based on both approaches.

The Fuglede-Kadison determinant

- Here is more information about the term $\ln(\det(\Delta_n^{(2)}))$.
- Consider a bijective positive operator $f\colon V \to V$ of finite-dimensional Hilbert spaces with trivial kernel. Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be its eigenvalues and μ_i be the multiplicity of λ_i. Then

$$\ln(\det(f)) = \sum_{i \geq 1} \mu_i \cdot \ln(\lambda_i) = \mathrm{tr}(\ln(f)).$$

- Define the spectral density function $F\colon [0,\infty) \to [0,\infty)$ to be the right-continuous step function, which has a jump at each of the eigenvalues of height its multiplicity, and which is zero for $\lambda < 0$.

- We can write

$$\ln(\det(f)) = \sum_{i \geq 1} \mu_i \cdot \ln(\lambda_i) = \int_{0+}^{\infty} \ln(\lambda) dF$$

where dF is the measure on $\mathbb{R}$ associated to the monotone increasing right-continuous function F which is given by $dF\big((a,b]\big) := F(b) - F(a)$.

- If $f\colon L^2(G)^k \to L^2(G)^k$ is a positive bounded G-equivariant operator, we define its spectral density function

$$F(f)(\lambda) := \dim_{\mathcal{N}(G)}\big(\operatorname{im}(E_\lambda^f)\big) = \operatorname{tr}_{\mathcal{N}(G)}\big(E_\lambda^f\big)$$

where $E_\lambda^f\colon L^2(G)^k \to L^2(G)^k$ is its spectral projection for $\lambda \geq 0$.

- Now the following expression makes sense:

$$\int_{0+}^{\infty} \ln(\lambda) dF \in \mathbb{R} \amalg \{-\infty\}.$$

- We define the logarithm of the Fuglede-Kadison determinant

$$\ln(\det(f)) := \int_{0+}^{\infty} \ln(\lambda) dF \in \mathbb{R},$$

provided that $\int_{0+}^{\infty} \ln(\lambda) dF > -\infty$ holds.

- We have $\int_{0+}^{\infty} \ln(\lambda) dF > -\infty$ if f is bijective, but there are weak isomorphisms f with $\int_{0+}^{\infty} \ln(\lambda) dF = -\infty$.

- Now the observation comes into play that the Laplace operator coming from a cellular structure lives already over the integers and the following

Conjecture (Determinant Conjecture)

Let $A \in M_{a,b}(\mathbb{Z}G)$ be a matrix. It defines a bounded G-equivariant operator $r_A^{(2)} \colon L^2(G)^m \to L^2(G)^n$. We have

$$\ln\left(\det\left((r_A^{(2)})^* \circ r_A^{(2)}\right)\right) := \int_{0+}^{\infty} \ln(\lambda)\, dF \geq 0.$$

- This conjecture is known for a very large class of groups, for instance for all sofic groups.

- Therefore we will tacitly assume that this conjecture holds and $\ln(\det(\Delta_n^{(2)}))$ is defined for the n-th Laplace operator $\Delta_n^{(2)}$.

Lehmer's problem

- Before we return to L^2-torsion, here is a very interesting aside concerning Fuglede-Kadison determinants and Mahler measures.

Definition (Mahler measure)

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-trivial element. Write it as $p(z) = c \cdot z^k \cdot \prod_{i=1}^{r}(z - a_i)$ for an integer $r \geq 0$, non-zero complex numbers c, a_1, ..., a_r and an integer k. Define its Mahler measure

$$M(p) = |c| \cdot \prod_{\substack{i=1,2,\ldots,r\\ |a_i|>1}} |a_i|.$$

- The following famous and open problem goes back to a question of Lehmer [6].

Problem (Lehmer's Problem)

Does there exist a constant $\Lambda > 1$ *such that for all non-trivial elements* $p(z) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ *with* $M(p) \neq 1$ *we have*

$$M(p) \geq \Lambda.$$

- There is even a candidate for which the minimal Mahler measure is attained, namely, Lehmer's polynomial

$$L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

- It is actual $-z^5 \cdot \Delta(z)$ for the Alexander polynomial $\Delta(z)$ of a bretzel knot given by $(2, 3, 7)$.
- It is conceivable that for any non-trivial element $p \in \mathbb{Z}[\mathbb{Z}]$ with $M(p) > 1$

$$M(p) \geq M(L) = 1.17628\ldots$$

holds.
- For a survey on Lehmer's problem we refer for instance to [1, 2, 4, 8].

Lemma

The Mahler measure $m(p)$ is the square root of the Fuglede-Kadison determinant of the operator $L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ given by multiplication with $p(z) \cdot \overline{p(z)}$.

Definition (Lehmer's constant of a group)

Define Lehmer's constant of a group G

$$\Lambda(G) \in [1, \infty)$$

to be the infimum of the set of Fuglede-Kadison determinants

$$\det{}^{(2)}_{\mathcal{N}(G)}\big(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r\big),$$

where A runs through all (r, r)-matrices with coefficients in $\mathbb{Z}G$ for all $r \geq 1$, for which $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r$ is a weak isomorphism and the Fuglede-Kadison determinant satisfies $\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)}) > 1$.

- We can show

$$\Lambda(\mathbb{Z}^n) \geq M(L)$$

for all $n \geq 1$, provided that Lehmer's problem has a positive answer.

- We know $1 \leq \Lambda(G) \leq M(L)$ for torsionfree G.

Problem (Generalized Lehmer's Problem)

For which torsionfree groups G does

$$\Lambda(G) = M(L)$$

hold?

Basic properties of L^2-torsion

- Next we record the basic properties of L^2-torsion. It behaves similar to the Euler characteristic.

Theorem (Simple homotopy invariance)

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Suppose that $\widetilde{X}$ and hence also $\widetilde{Y}$ are L^2-acyclic.

Then there is a homomorphism depending only on π

$$\Phi_\pi \colon \operatorname{Wh}(\pi) \to \mathbb{R}$$

sending $\tau(f)$ to $\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X})$.

- If $\operatorname{Wh}(\pi)$ vanishes, the L^2-torsion is a homotopy invariant.

Theorem (Sum formula)

Let X be a finite CW-complex with subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are L^2-acyclic and the inclusions $X_i \to X$ are π-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

Theorem (Fibration formula)

Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that $\widetilde{F}$ is L^2-acyclic and the inclusion $F \to E$ is π-injective.

Then $\widetilde{E}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- By Poincaré duality we have $\rho^{(2)}(\widetilde{M}) = 0$ for every even dimensional closed manifold M, provided that $\widetilde{M}$ is L^2-acyclic.

- The L^2-torsion is multiplicative under finite coverings, i.e., if $X \to Y$ is a d-sheeted covering of connected finite CW-complexes and $\widetilde{X}$ is L^2-acyclic, then $\widetilde{Y}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y}).$$

- In particular $\widetilde{S^1}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{S^1}) = 0.$$

Theorem (S^1-actions on aspherical manifolds (Lück))

Let M be an aspherical closed manifold with non-trivial S^1-action.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

Theorem (L^2-torsion and aspherical CW-complexes, Wegner)

Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup.

Then $\widetilde{X}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = 0.$$

Theorem (Hyperbolic manifolds, (Hess-Schick, Olbrich))

There are (computable) rational numbers $r_n > 0$ such that for every hyperbolic closed manifold M of odd dimension $2n+1$ the universal covering $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- Since for every hyperbolic manifold M we have $\mathrm{Wh}(\pi_1(M)) = 0$, we rediscover the fact that the volume of an odd-dimensional hyperbolic closed manifold depends only on $\pi_1(M)$.
- We also rediscover the theorem that any S^1-action on a closed hyperbolic manifold is trivial.

- The proof is based on the fact that the analytic version of L^2-torsion is of the shape

$$\rho^{(2)}(\widetilde{M}) = \int_{\mathcal{F}} f(x) \, d\mathrm{vol}_{\mathbb{H}^{2n+1}}$$

where $\mathcal{F}$ is a fundamental domain of the π-action on the hyperbolic space $\mathbb{H}^{2n+1}$ and $f(x)$ is an expression in terms of the heat kernel $k(x,x)(t)$.

- By the symmetry of $\mathbb{H}^{2n+1}$ this function $k(x,x)(t)$ is independent of x and hence $f(x)$ is independent of x.

- If we take $r_n = (-1)^n \cdot \pi^n \cdot f(x)$ for any $x \in \mathbb{H}^{2n+1}$, we get

$$\int_{\mathcal{F}} f(x) \, d\mathrm{vol}_{\mathbb{H}^{2n+1}} = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(\mathcal{F}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- We have $r_1 = \frac{1}{6}$, $r_2 = \frac{31}{45}$, $r_7 = \frac{221}{70}$.

Theorem (Lott-Lück-,Lück-Schick)

Let M be an irreducible closed 3-manifold with infinite fundamental group. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

- The proof of the result above is based on the meanwhile approved Thurston Geometrization Conjecture. It reduces the claim to Seifert manifolds with incompressible torus boundary and to hyperbolic manifolds with incompressible torus boundary using the sum formula. The Seifert pieces are treated analogously to aspherical closed manifolds with S^1-action. The hyperbolic pieces require a careful analysis of the cusps.

Combinatorial approach in low dimensions

- Here is a recipe to read of the L^2-torsion for an irreducible 3-manifold with incompressible toroidal boundary from its fundamental group π, provided that π is infinite.

- Let
$$\pi = \langle s_1, s_2, \ldots s_g \mid R_1, R_2, \ldots R_r \rangle$$
be a presentation of π.

- Let the (r, g)-matrix
$$F = \begin{pmatrix} \frac{\partial R_1}{\partial s_1} & \ldots & \frac{\partial R_1}{\partial s_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial R_r}{\partial s_1} & \ldots & \frac{\partial R_r}{\partial s_g} \end{pmatrix}$$
be the Fox matrix of the presentation (see [3, 9B on page 123], [5], [7, page 84]).

- Now there are two cases:
  1. Suppose ∂M is non-empty and $g = r + 1$. Define A to be the $(g-1, g-1)$-matrix with entries in $\mathbb{Z}\pi$ obtained from the Fox matrix F by deleting one of the columns.
  2. Suppose ∂M is empty and $g = r$. Define A to be the $(g-1, g-1)$-matrix with entries in $\mathbb{Z}\pi$ obtained from the Fox matrix F by deleting one of the columns and one of the rows.

- Let K be any positive real number satisfying $K \geq \|R_A^{(2)}\|$. A possible choice for K is the product of $(g-1)^2$ and the maximum over the word length of those relations R_i whose Fox derivatives appear in A.

- Define for $x = \sum_{w \in \pi} n_w \cdot w \in \mathbb{Z}\pi$

$$\mathrm{tr}_{\mathbb{Z}\pi}(x) = \lambda_e \in \mathbb{Z}.$$

- Then the sum of non-negative rational numbers

$$\sum_{p=1}^{L} \frac{1}{2p} \cdot \mathrm{tr}_{\mathbb{Z}\pi}\big((1 - K^{-2} \cdot AA^*)^p\big)$$

converges for $L \to \infty$ to $\rho^{(2)}(\widetilde{M}) + (g-1) \cdot \ln(K)$.

- More precisely, there is a constant $C > 0$ and a number $\alpha > 0$ such that we get for all $L \geq 1$

$$0 \leq \rho^{(2)}(\widetilde{M}) + (g-1) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{2p} \cdot \mathrm{tr}_{\mathbb{Z}\pi}\big((1 - K^{-2} \cdot AA^*)^p\big) \leq \frac{C}{L^{\alpha}}.$$

Example (Figure eight knot)

- Let $K \subseteq S^3$ be the figure eight knot;
- Its complement is a hyperbolic 3-manifold.
- It fibers over S^1 and the fiber is a surface whose fundamental group is the free group F_2 in two generators s_1 and s_2. The automorphism of F_2 is given by $s_1 \mapsto s_2$ and $s_2 \mapsto s_2^3 s_1^{-1}$.

  We get the presentation for $\pi = \pi_1(M_K) \cong F_2 \rtimes \mathbb{Z}$

$$\pi \; = \; \langle s_1, s_2, t \mid ts_1 t^{-1} s_2^{-1} = ts_2 t^{-1} s_1 s_2^{-3} = 1 \rangle.$$

Example (continued)

- If we delete from the Fox matrix the column belonging to s_2, we obtain the matrix

$$A = \begin{pmatrix} t & 1 - s_2 \\ s_2^3 s_1^{-1} & 1 - s_2^3 s_1^{-1} \end{pmatrix}$$

- The number $K = 4$ is greater or equal to the operator norm of the bounded π-equivariant operator induced by A.

Example (continued)

- Define the $(2,2)$-matrix $B = (b_{i,j})$ over $\mathbb{Z}\pi$ by

$$\begin{aligned} b_{1,1} &= 13 + s_2 + s_2^{-1}; \\ b_{1,2} &= -1 + s_2 + s_1 s_2^3 - s_2 s_1 s_2^{-3} - t s_1 s_2^{-3}; \\ b_{2,1} &= -1 + s_2^{-1} + s_2^3 s_1^{-1} - s_2^3 s_1^{-1} s_2^{-1} - s_2^3 s_1^{-1} t^{-1}; \\ b_{2,2} &= 13 + s_2^3 s_1^{-1} + s_1 s_2^{-3}. \end{aligned}$$

- Since $B = 16 - AA^*$, we get:

$$\frac{1}{6\pi} \operatorname{vol}(M_K) = -\ln(\rho(K_8)) = 8 \cdot \ln(2) - \sum_{p=1}^{\infty} \frac{1}{p \cdot 16^p} \cdot \operatorname{tr}_{\mathbb{Z}\pi}(B^p).$$

- The volume of M_K is about 2.02988.

Group automorphisms

- Let $f\colon G \to G$ be an automorphism of a group G for which there exists a finite CW-model X for BG.
- Let $\widehat{f}\colon X \to X$ be any selfhomotopy equivalence such that $\pi_1(\widehat{f})$ and f agree up to inner automorphisms of G.
- Then the mapping torus $T_{\widehat{f}}$ is a connected finite CW-complex, which is L^2-acyclic. Hence its L^2-torsion $\rho^{(2)}(\widetilde{T}_f) \in \mathbb{R}$ is defined.
- It depends only on f and not on the choices of X and $\widehat{f}$ since the simple homotopy type of $T_{\widehat{f}}$ is independent of these choices.
- Hence we get a well-defined element

$$\rho^{(2)}(f) := \rho^{(2)}(T_{\widehat{f}}) \in \mathbb{R}.$$

Theorem

Suppose that all groups appearing below have finite CW-models for their classifying spaces.

- *Suppose that G is the amalgamated product $G_1 *_{G_0} G_2$ for subgroups $G_i \subset G$ and the automorphism $f\colon G \to G$ is the amalgamated product $f_1 *_{f_0} f_2$ for automorphisms $f_i\colon G_i \to G_i$. Then*

$$\rho^{(2)}(f) = \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0);$$

- *Let $f\colon G \to H$ and $g\colon H \to G$ be isomorphisms of groups. Then*

$$\rho^{(2)}(f \circ g) = \rho^{(2)}(g \circ f).$$

  In particular $\rho^{(2)}(f)$ is invariant under conjugation with automorphisms;

Theorem (continued)

- *Suppose that the following diagram of groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{p} & G_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{p} & G_3 & \longrightarrow & 1 \end{array}$$

  commutes, has exact rows and its vertical arrows are automorphisms. Then

$$\rho^{(2)}(f_2) = \chi(BG_3) \cdot \rho^{(2)}(f_1);$$

- *Let $f\colon G \to G$ be a group automorphism. Then for all integers $n \geq 1$*

$$\rho^{(2)}(f^n) = n \cdot \rho^{(2)}(f);$$

Theorem (continued)

- *Suppose that G contains a subgroup G_0 of finite index $[G : G_0]$. Let $f\colon G \to G$ be an automorphism with $f(G_0) = G_0$. Then*

$$\rho^{(2)}(f) = \frac{1}{[G : G_0]} \cdot \rho^{(2)}(f|_{G_0});$$

- *We have $\rho^{(2)}(f) = 0$ if G satisfies one of the following conditions:*
    - *All L^2-Betti numbers of the universal covering of BG vanish;*
    - *G contains an amenable infinite normal subgroup.*
- *If $h\colon S \to S$ is a pseudo-Anosov selfhomeomorphism of a connected orientable surface, and $f\colon \pi_1(S) \xrightarrow{\cong} \pi_1(S)$ is the induced automorphism, then its mapping torus T_h is a hyperbolic 3-manifold and*

$$\rho^{(2)}(f) = -\frac{1}{6\pi} \cdot \mathrm{vol}(T_h).$$

Approximation

- The following conjecture combines and generalizes Conjectures by Bergeron-Venkatesh, Hopf, Singer, and Lück.
- If G is a finitely generated group, we denote by $d(G)$ the minimal number of generators.
- A chain for a group G is a sequence of in G normal subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that $[G : G_i] < \infty$ and $\bigcap_{i \geq 0} G_i = \{1\}$.

- We denote by RG the rank gradient introduced by Lackenby.

$$RG(G, \{G_i\}) = \lim_{i \to \infty} \frac{d(G_i)}{[G : G_i]}.$$

Conjecture (Homological growth and L^2-invariants for aspherical closed manifolds)

Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then

- *For any natural number n with $2n \neq d$ we get*

  $$b_n^{(2)}(\widetilde{M}) = 0.$$

  If $d = 2n$, we have

  $$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \geq 0.$$

  If $d = 2n$ and M carries a Riemannian metric of negative sectional curvature, then

  $$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) > 0;$$

Conjecture (Continued)

- *Let* $(G_i)_{i \geq 0}$ *be any chain. Put* $M[i] = G_i \backslash \widetilde{M}$*.*
  Then we get for any natural number n and any field F

$$b_n^{(2)}(\widetilde{M}) = \lim_{i \to \infty} \frac{b_n(M[i];F)}{[G:G_i]} = \lim_{i \to \infty} \frac{d\big(H_n(M[i];\mathbb{Z})\big)}{[G:G_i]};$$

and for $n = 1$

$$b_1^{(2)}(\widetilde{M}) = \lim_{i \to \infty} \frac{b_1(M[i];F)}{[G:G_i]} = \lim_{i \to \infty} \frac{d\big(G_i/[G_i,G_i]\big)}{[G:G_i]}$$

$$= RG(G,(G_i)_{i \geq 0}) = \begin{cases} 0 & d \neq 2; \\ -\chi(M) & d = 2; \end{cases}$$

Conjecture (Continued)

- *If $d = 2n+1$ is odd, we have*

$$(-1)^n \cdot \rho^{(2)}(\widetilde{M}) \geq 0;$$

If $d = 2n+1$ is odd and M carries a Riemannian metric with negative sectional curvature, we have

$$(-1)^n \cdot \rho^{(2)}(\widetilde{M}) > 0;$$

Conjecture (Continued)

- *Let $(G_i)_{i \geq 0}$ be a chain. Put $M[i] = G_i \backslash \widetilde{M}$.*
  Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = 0,$$

  and we get in the case $d = 2n+1$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = (-1)^n \cdot \rho^{(2)}\left(\widetilde{M}\right) \geq 0.$$

- The conjecture above is very optimistic, but we do not know a counterexample.
- It is related to the Approximation Conjecture for the Fuglede-Kadison determinant.
- The main issue here are uniform estimates about the spectrum of the n-th Laplace operators on $M[i]$ which are independent of i.
- Abert-Nikolov have settled the rank gradient part if G contains an infinite normal amenable subgroup.
- Kar-Kropholler-Nikolov have settled the part about the growth of the torsion in the homology if G is infinite amenable.
- Abert-Gelander-Nikolov deal with the rank gradient and the growth of the torsion in the homology for right angled lattices.
- Li-Thom deal with the vanishing of L^2-torsion for amenable G.
- Bridson-Kochloukova deal with limit groups, where the limits are not necessarily zero.

Theorem (Lück)

Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$ and fields F and any chain $(G_i)_{i \geq 0}$

$$
\begin{aligned}
\lim_{i\to\infty} \frac{b_n(M[i];F)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{d\big(H_n(M[i];\mathbb{Z})\big)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\ln\big(\big|\operatorname{tors}\big(H_n(M[i])\big)\big|\big)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M}) &= 0;\\
\rho^{(2)}(\widetilde{M}) &= 0.
\end{aligned}
$$

- Let M be a closed hyperbolic 3-manifold. Then the conjecture above predicts for any chain $(G_i)_{i \geq 0}$

$$\lim_{i \to \infty} \frac{\ln\bigl(\bigl|\operatorname{tors}\bigl(H_1(G_i)\bigr)\bigr|\bigr)}{[G : G_i]} = \frac{1}{6\pi} \cdot \operatorname{vol}(M).$$

  Since the volume is always positive, the equation above implies that $|\operatorname{tors}\bigl(H_1(G_i)\bigr)|$ growth exponentially in $[G : G_i]$.

- In particular this would allow to read of the volume from the profinite completion of $\pi_1(M)$.

D. Boyd, D. Lind, F. R. Villegas, and C. Deninger.
The many aspects of Mahler's measure.
Report about the workshop the many aspects of Mahler measure, April 2003, Birs, Banff, 2003.

D. W. Boyd.
Speculations concerning the range of Mahler's measure.
Canad. Math. Bull., 24(4):453–469, 1981.

G. Burde and H. Zieschang.
Knots.
Walter de Gruyter & Co., Berlin, 1985.

M. Carrizosa.
Survey on Lehmer problems.
São Paulo J. Math. Sci., 3(2):317–327, 2009.

R. H. Fox.
Free differential calculus. I. Derivation in the free group ring.
Ann. of Math. (2), 57:547–560, 1953.

D. H. Lehmer.
Factorization of certain cyclotomic functions.
Ann. of Math. (2), 34(3):461–479, 1933.

W. Lück.
L^2-torsion and 3-manifolds.
In *Low-dimensional topology (Knoxville, TN, 1992)*, pages 75–107. Internat. Press, Cambridge, MA, 1994.

C. Smyth.
The Mahler measure of algebraic numbers: a survey.
In *Number theory and polynomials*, volume 352 of *London Math. Soc. Lecture Note Ser.*, pages 322–349. Cambridge Univ. Press, Cambridge, 2008.