# $L^2$-Betti numbers

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Bonn, 24. & 26. April 2018

# Outline

- We introduce $L^2$-Betti numbers.
- We present their basic properties and tools for their computation.
- We compute the $L^2$-Betti numbers of all 3-manifolds.
- We discuss the Atiyah Conjecture and the Singer Conjecture.

# Basic motivation

- Given an invariant for finite $CW$-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- Examples:

| Classical notion | generalized version |
|---|---|
| Homology with coefficients in $\mathbb{Z}$ | Homology with coefficients in representations |
| Euler characteristic $\in \mathbb{Z}$ | Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$ |
| Lefschetz numbers $\in \mathbb{Z}$ | Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$ |
| Signature $\in \mathbb{Z}$ | Surgery invariants in $L_*(\mathbb{Z}G)$ |
| — | torsion invariants |

- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Here are two naive attempts which fail:
  - $\dim_{\mathbb{C}}(H_n(\widetilde{X};\mathbb{C}))$
  - $\dim_{\mathbb{C}\pi}(H_n(\widetilde{X};\mathbb{C}))$,
    where $\dim_{\mathbb{C}\pi}(M)$ for a $\mathbb{C}[\pi]$-module could be chosen for instance as $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M)$.
- The problem is that $\mathbb{C}\pi$ is in general not Noetherian and $\dim_{\mathbb{C}\pi}(M)$ is in general not additive under exact sequences.
- We will use the following successful approach which is essentially due to Atiyah [1].

# Group von Neumann algebras

- Throughout these lectures let $G$ be a discrete group.
- Given a ring $R$ and a group $G$, denote by $RG$ or $R[G]$ the group ring.
- Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero.
- Addition is given by adding the coefficients.
- Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).
- In general $RG$ is a very complicated ring.

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

## Definition

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded $G$-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

## Example (Finite $G$)

If $G$ is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\operatorname{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient $\lambda_e$.

## Example ($G = \mathbb{Z}^n$)

Let $G$ be $\mathbb{Z}^n$. Let $L^2(T^n)$ be the Hilbert space of $L^2$-integrable functions $T^n \to \mathbb{C}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

Let $L^\infty(T^n)$ be the Banach space of essentially bounded measurable functions $f\colon T^n \to \mathbb{C}$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f$$

where $M_f\colon L^2(T^n) \to L^2(T^n)$ is the bounded $\mathbb{Z}^n$-operator $g \mapsto g \cdot f$.

Under this identification the trace becomes

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^n)}\colon L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.$$

# von Neumann dimension

## Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$.
A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f \colon V \to W$ is a bounded $G$-equivariant operator.

## Definition (von Neumann dimension)

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p \colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$.
Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in \mathbb{R}^{\geq 0}.$$

## Example (Finite $G$)

For finite $G$ a finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is the same as a unitary finite dimensional $G$-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

## Example ($G = \mathbb{Z}^n$)

Let $G$ be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

In particular each $r \in \mathbb{R}^{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

## Definition (Weakly exact)

A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is weakly exact at $V$ if the kernel $\ker(p)$ of $p$ and the closure $\overline{\mathrm{im}(i)}$ of the image $\mathrm{im}(i)$ of $i$ agree.

A map of Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a weak isomorphism if it is injective and has dense image.

## Example

The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1}\colon L^2(\mathbb{Z}) = L^2(S^1) \to L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z-1)\cdot u(z)$$

is a weak isomorphism, but not an isomorphism.

## Theorem (Main properties of the von Neumann dimension)

1. *Faithfulness*

   *We have for a finitely generated Hilbert $\mathcal{N}(G)$-module $V$*

   $$V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0;$$

2. *Additivity*

   *If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then*

   $$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V);$$

3. *Cofinality*

   *Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of $V$, directed by inclusion. Then*

   $$\dim_{\mathcal{N}(G)}\left(\overline{\bigcup_{i \in I} V_i}\right) = \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\}.$$

# $L^2$-homology and $L^2$-Betti numbers

## Definition ($L^2$-homology and $L^2$-Betti numbers)

Let $X$ be a connected $CW$-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular $L^2$-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its $n$-th $L^2$-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its $n$-th $L^2$-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

## Theorem (Main properties of $L^2$-Betti numbers)

*Let $X$ and $Y$ be connected CW-complexes of finite type.*

- *Homotopy invariance*

  *If $X$ and $Y$ are homotopy equivalent, then*

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{Y});$$

- *Euler-Poincaré formula*

  *We have*

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X});$$

- *Poincaré duality*

  *Let $M$ be a closed manifold of dimension $d$. Then*

$$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

## Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th $L^2$-Betti number*

  *We have*

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  *If $X \to Y$ is a finite covering with $d$ sheets, then*

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

## Example (Finite $\pi$)

If $\pi$ is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$

## Example ($S^1$)

Consider the $\mathbb{Z}$-$CW$-complex $\widetilde{S^1}$. We get for $C_*^{(2)}(\widetilde{S^1})$

$$\ldots \to 0 \to L^2(\mathbb{Z}) \xrightarrow{M_{z-1}} L^2(\mathbb{Z}) \to 0 \to \ldots$$

and hence $H_n^{(2)}(\widetilde{S^1}) = 0$ and $b_n^{(2)}(\widetilde{S^1}) = 0$ for all $\geq 0$.

## Example ($\pi = \mathbb{Z}^d$)

Let $X$ be a connected $CW$-complex of finite type with fundamental group $\mathbb{Z}^d$. Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$. Then

$$b_n^{(2)}(\widetilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\widetilde{X})\right)$$

Obviously this implies

$$b_n^{(2)}(\widetilde{X}) \in \mathbb{Z}.$$

- For a discrete group $G$ we can consider more generally any free finite $G$-$CW$-complex $\overline{X}$ which is the same as a $G$-covering $\overline{X} \to X$ over a finite $CW$-complex $X$. (Actually proper finite $G$-$CW$-complex suffices.)

- The universal covering $p\colon \widetilde{X} \to X$ over a connected finite $CW$-complex is a special case for $G = \pi_1(X)$.

- Then one can apply the same construction to the finite free $\mathbb{Z}G$-chain complex $C_*(\overline{X})$. Thus we obtain the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\overline{X};\mathcal{N}(G)) := H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})),$$

and define

$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\big(H_n^{(2)}(\overline{X};\mathcal{N}(G))\big) \in \mathbb{R}^{\geq 0}.$$

- Let $i\colon H \to G$ be an injective group homomorphism and $C_*$ be a finite free $\mathbb{Z}H$-chain complex.
- Then $i_*C_* := \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*$ is a finite free $\mathbb{Z}G$-chain complex.
- We have the following formula

$$\begin{aligned} &\dim_{\mathcal{N}(G)}\bigl(H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} i_*C_*)\bigr) \\ &\qquad\qquad = \dim_{\mathcal{N}(H)}\bigl(H_n^{(2)}(L^2(H) \otimes_{\mathbb{Z}H} C_*)\bigr). \end{aligned}$$

## Lemma

*If $\overline{X}$ is a finite free $H$-CW-complex, then we get*

$$b_n^{(2)}(i_*\overline{X};\mathcal{N}(G)) = b_n^{(2)}(\overline{X};\mathcal{N}(H)).$$

- The corresponding statement is wrong if we drop the condition that $i$ is injective.

- An example comes from $p\colon \mathbb{Z} \to \{1\}$ and $\widetilde{X} = \widetilde{S^1}$ since then $p_*\widetilde{S^1} = S^1$ and we have for $n = 0, 1$

$$b_n^{(2)}(\widetilde{S^1}; \mathcal{N}(\mathbb{Z})) = b_n^{(2)}(\widetilde{S^1}) = 0,$$

and

$$b_n^{(2)}(p_*\widetilde{S^1}; \mathcal{N}(\{1\})) = b_n(S^1) = 1.$$

# The $L^2$-Mayer Vietoris sequence

## Lemma

*Let* $0 \to C_*^{(2)} \xrightarrow{i_*^{(2)}} D_*^{(2)} \xrightarrow{p_*^{(2)}} E_*^{(2)} \to 0$ *be a weakly exact sequence of finite Hilbert* $\mathcal{N}(G)$*-chain complexes.*

*Then there is a long weakly exact sequence of finitely generated Hilbert* $\mathcal{N}(G)$*-modules*

$$\cdots \xrightarrow{\delta_{n+1}^{(2)}} H_n^{(2)}(C_*^{(2)}) \xrightarrow{H_n^{(2)}(i_*^{(2)})} H_n^{(2)}(D_*^{(2)}) \xrightarrow{H_n^{(2)}(p_*^{(2)})} H_n^{(2)}(E_*^{(2)})$$
$$\xrightarrow{\delta_n^{(2)}} H_{n-1}^{(2)}(C_*^{(2)}) \xrightarrow{H_{n-1}^{(2)}(i_*^{(2)})} H_{n-1}^{(2)}(D_*^{(2)})$$
$$\xrightarrow{H_{n-1}^{(2)}(p_*^{(2)})} H_{n-1}^{(2)}(E_*^{(2)}) \xrightarrow{\delta_{n-1}^{(2)}} \cdots .$$

## Lemma

*Let*

$$\begin{array}{ccc} \overline{X_0} & \longrightarrow & \overline{X_1} \\ \downarrow & & \downarrow \\ \overline{X_2} & \longrightarrow & \overline{X} \end{array}$$

*be a cellular G-pushout of finite free G-CW-complexes, i.e., a G-pushout, where the upper arrow is an inclusion of a pair of free finite G-CW-complexes and the left vertical arrow is cellular.*

*Then we obtain a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules*

$$\begin{aligned} \cdots \to H_n^{(2)}(\overline{X_0};\mathcal{N}(G)) \to H_n^{(2)}(\overline{X_1};\mathcal{N}(G)) \oplus H_n^{(2)}(\overline{X_2};\mathcal{N}(G)) \\ \to H_n^{(2)}(\overline{X};\mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X_0};\mathcal{N}(G)) \\ \to H_{n-1}^{(2)}(\overline{X_1};\mathcal{N}(G)) \oplus H_{n-1}^{(2)}(\overline{X_2};\mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X};\mathcal{N}(G)) \to \cdots . \end{aligned}$$

## Proof.

- From the cellular $G$-pushout we obtain an exact sequence of $\mathbb{Z}G$-chain complexes

$$0 \to C_*(\overline{X}_0) \to C_*(\overline{X}_1) \oplus C_*(\overline{X}_2) \to C_*(\overline{X}) \to 0.$$

- It induces an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_0) \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_1) \oplus L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_2) \\ \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}) \to 0.$$

- Now apply the previous result.

□

## Definition ($L^2$-acyclic)

A finite (not necessarily connected) $CW$-complex $X$ is called $L^2$-acyclic, if $b_n^{(2)}(\widetilde{C}) = 0$ holds for every $C \in \pi_0(X)$ and $n \in \mathbb{Z}$.

- If $X$ is a finite (not necessarily connected) $CW$-complex, we define

$$b_n^{(2)}(\widetilde{X}) := \sum_{C \in \pi_0(X)} b_n^{(2)}(\widetilde{C}) \in \mathbb{R}^{\geq 0}.$$

**Definition ($\pi_1$-injective)**

A map $X \to Y$ is called $\pi_1$-injective, if for every choice of base point in $X$ the induced map on the fundamental groups is injective.

- Consider a cellular pushout of finite $CW$-complexes

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

such that each of the maps $X_i \to X$ is $\pi_1$-injective.

## Lemma

*We get under the assumptions above for any $n \in \mathbb{Z}$*

- *If $X_0$ is $L^2$-acyclic, then*

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{X}_1) + b_n^{(2)}(\widetilde{X_2}).$$

- *If $X_0$, $X_1$ and $X_2$ are $L^2$-cyclic, then $X$ is $L^2$-acyclic.*

## Proof.

- Without loss of generality we can assume that $X$ is connected.
- By pulling back the universal covering $\widetilde{X} \to X$ to $X_i$, we obtain a cellular $\pi = \pi_1(X)$-pushout

$$\begin{array}{ccc} \overline{X}_0 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ \overline{X}_2 & \longrightarrow & \widetilde{X} \end{array}$$

- Notice that $\overline{X}_i$ is in general not the universal covering of $X_i$.

□

## Proof continued.

- Because of the associated long exact $L^2$-sequence and the weak exactness of the von Neumann dimension, it suffices to show for $n \in \mathbb{Z}$ and $i = 1, 2$

$$\begin{aligned} H_n^{(2)}(\overline{X_0}; \mathcal{N}(\pi)) &= 0; \\ b_n^{(2)}(\overline{X_i}; \mathcal{N}(\pi)) &= b_n^{(2)}(\widetilde{X}_i). \end{aligned}$$

- This follows from $\pi_1$-injectivity, the lemma above about $L^2$-Betti numbers and induction, the assumption that $X_0$ is $L^2$-acyclic, and the faithfulness of the von Neumann dimension.

□

**Example (Finite self coverings)**

We get for a connected $CW$-complex $X$ of finite type, for which there is a selfcovering $X \to X$ with $d$-sheets for some integer $d \geq 2$,

$$b_n^{(2)}(\widetilde{X}) = 0 \quad \text{for } n \geq 0.$$

This implies for each connected $CW$-complex $Y$ of finite type that $S^1 \times Y$ is $L^2$-acyclic.

## Example ($L^2$-Betti number of surfaces)

- Let $F_g$ be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- Since $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\ b_n^{(2)}(\widetilde{F_0}) &= 0 \quad \text{for} \quad n \neq 1. \end{aligned}$$

## Theorem ($S^1$-actions, Lück)

*Let $M$ be a connected compact manifold with $S^1$-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to M, \quad z \mapsto zx$ is $\pi_1$-injective.*

*Then $M$ is $L^2$-acyclic.*

## Proof.

Each of the $S^1$-orbits $S^1/H$ in $M$ satisfies $S^1/H \cong S^1$. Now use induction over the number of cells $S^1/H_i \times D^n$ and a previous result using $\pi_1$-injectivity and the vanishing of the $L^2$-Betti numbers of spaces of the shape $S^1 \times X$. □

## Theorem ($S^1$-actions on aspherical manifolds, Lück)

*Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then*

1. *The action has no fixed points;*
2. *The map $S^1 \to M, \ z \mapsto zx$ is $\pi_1$-injective for $x \in M$;*
3. $b_n^{(2)}(\widetilde{M}) = 0$ *for* $n \geq 0$ *and* $\chi(M) = 0$.

## Proof.

The hard part is to show that the second assertion holds, since $M$ is aspherical. Then the first assertion is obvious and the third assertion follows from the previous theorem. □

## Theorem ($L^2$-Hodge - de Rham Theorem, Dodziuk [2])

*Let $M$ be a closed Riemannian manifold. Put*

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0,\ ||\widetilde{\omega}||_{L^2} < \infty\}$$

*Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules*

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

## Corollary ($L^2$-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, dvol\,.$$

*where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the $\pi$-action.*

## Theorem (hyperbolic manifolds, Dodziuk [3])

*Let $M$ be a hyperbolic closed Riemannian manifold of dimension $d$. Then:*

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

## Proof.

A direct computation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that $M$ is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. □

## Corollary

*Let $M$ be a hyperbolic closed manifold of dimension $d$. Then*

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *$M$ carries no non-trivial $S^1$-action.*

## Proof.

(1) We get from the Euler-Poincaré formula and the last result

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathbb{H}^d$ is contractible, $M$ is aspherical. Now apply a previous result about $S^1$-actions. □

## Theorem (3-manifolds, Lott-Lück [7])

*Let the 3-manifold $M$ be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds $M_j$. Assume that $\pi_1(M)$ is infinite. Then*

$$\begin{aligned} b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\,\pi_1(M_j)\,|} - \chi(M) \\ &\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\ b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\,\pi_1(M_j)\,|} \\ &\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\ b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2. \end{aligned}$$

## Proof.

- We have already explained why a closed hyperbolic 3-manifold is $L^2$-acyclic.
- One of the hard parts of the proof is to show that this is also true for any hyperbolic 3-manifold with incompressible toral boundary.
- Recall that these have finite volume.
- One has to introduce appropriate boundary conditions and Sobolev theory to write down the relevant analytic $L^2$-deRham complexes and $L^2$-Laplace operators.
- A key ingredient is the decomposition of such a manifold into its core and a finite number of cusps.

□

## Proof continued.

- This can be used to write the $L^2$-Betti number as an integral over a fundamental domain $\mathcal{F}$ of finite volume, where the integrand is given by data depending on $\mathbb{H}^3$ only:

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, d\mathrm{vol}\,.$$

- Since $\mathbb{H}^3$ has a lot of symmetries, the integrand does not depend on $\tilde{x}$ and is a constant $C_n$ depending only on $\mathbb{H}^3$.
- Hence we get

$$b_n^{(2)}(\widetilde{M}) = C_n \cdot \mathrm{vol}(M).$$

- From the closed case we deduce $C_n = 0$.

□

## Proof continued.

- Next we show that any Seifert manifold with infinite fundamental group is $L^2$-acyclic.

- This follows from the fact that such a manifold is finitely covered by the total space of an $S^1$-bundle $S^1 \to E \to F$ over a surface with injective $\pi_1(S^1) \to \pi_1(E)$ using previous results.

- In the next step one shows that any irreducible 3-manifold $M$ with incompressible or empty boundary and infinite fundamental group is $L^2$-acyclic.

- Recall that by the Thurston Geometrization Conjecture we can find a family of incompressible tori which decompose $M$ into hyperbolic and Seifert pieces. The tori and all these pieces are $L^2$-acyclic.

- Now the claim follows from the $L^2$-Mayer Vietoris sequence.

□

## Proof continued.

- In the next step one shows that any irreducible 3-manifold $M$ with incompressible boundary and infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.
- This follows by considering $N = M \cup_{\partial M} M$ using the $L^2$-Mayer-Vietoris sequence, the already proved fact that $N$ is $L^2$-acyclic and the previous computation of the $L^2$-Betti numbers for surfaces.
- In the next step one shows that any irreducible 3-manifold $M$ with infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.

□

## Proof continued.

- This is reduced by an iterated application of the Loop Theorem to the case where the boundary is incompressible. Namely, using the Loop Theorem one gets an embedded disk $D^2 \subseteq M$ along which one can decompose $M$ as $M_1 \cup_{D^2} M_2$ or as $M_1 \cup_{S^0 \times D^2} D^1 \times D^2$ depending on whether $D^2$ is separating or not.
- Since the only prime 3-manifold that is not irreducible is $S^1 \times S^2$, and every manifold $M$ with finite fundamental group satisfies the result by a direct inspection of the Betti numbers of its universal covering, the claim is proved for all prime 3-manifolds.
- Finally one uses the $L^2$-Mayer Vietoris sequence to prove the claim in general using the prime decomposition.

□

## Corollary

*Let $M$ be a 3-manifold. Then $M$ is $L^2$-acyclic if and only if one of the following cases occur:*

- *$M$ is an irreducible 3-manifold with infinite fundamental group whose boundary is empty or toral.*
- *$M$ is $S^1 \times S^2$ or $\mathbb{RP}^3 \sharp \mathbb{RP}^3$.*

## Corollary

*Let $M$ be a compact $n$-manifold such that $n \leq 3$ and its fundamental group is torsionfree.*

*Then all its $L^2$-Betti numbers are integers.*

## Theorem (mapping tori, Lück [9])

*Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex $X$ of finite type. Let $T_f$ be the mapping torus. Then*

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \text{for } n \geq 0.$$

## Proof.

- As $T_{f^d} \to T_f$ is up to homotopy a $d$-sheeted covering, we get

$$b_n^{(2)}(\widetilde{T_f}) = \frac{b_n^{(2)}(\widetilde{T_{f^d}})}{d}.$$

□

# Proof continued.

- If $\beta_n(X)$ is the number of $n$-cells, then there is up to homotopy equivalence a $CW$-structure on $T_{f^d}$ with $\beta_n(T_{f^d}) = \beta_n(X) + \beta_{n-1}(X)$. We have

$$b_n^{(2)}(\widetilde{T_{f^d}}) = \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(C_n^{(2)}(\widetilde{T_{f^d}}))\right) \leq \dim_{\mathcal{N}(G)}\left(C_n^{(2)}(\widetilde{T_{f^d}})\right) = \beta_n(T_{f^d}).$$

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T}_f) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.

□

- Let $M$ be an irreducible manifold $M$ with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.
- Agol proved the Virtually Fibering Conjecture for such $M$.
- This implies by the result above that $M$ is $L^2$-acyclic.

# The fundamental square and the Atiyah Conjecture

**Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)**

*Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$*

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If $G$ is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \text{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

## Conjecture (Atiyah Conjecture for torsionfree groups)

*Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.*

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

  $$\dim_{\mathcal{N}(G)}\big(\ker\big(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n\big)\big)$$

  is an integer. In this case this dimension agrees with

  $$\dim_{\mathcal{D}(G)}\big(\ker\big(r_A \colon \mathcal{D}(G)^m \to \mathcal{D}(G)^n\big)\big).$$

- The general version above is equivalent to the one stated before if $G$ is finitely presented.

- The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.
- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.
- However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski [4].

**Theorem (Linnell [6], Schick [11])**

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

## Strategy to prove the Atiyah Conjecture

1. Show that $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective
   (This is implied by the Farrell-Jones Conjecture)
2. Show that $K_0(\mathbb{C}G) \to K_0(\mathcal{D}(G))$ is surjective.
3. Show that $\mathcal{D}(G)$ is semisimple.

# Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the $L^2$-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected $CW$-complex $X$ of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

- Given an integer $l \geq 1$ and a sequence $r_1, r_2, \ldots, r_l$ of non-negative rational numbers, we can construct a group $G$ such that $BG$ is of finite type and

$$\begin{array}{lcll} b_n^{(2)}(BG) & = & r_n & \text{for } 1 \leq n \leq l; \\ b_n^{(2)}(BG) & = & 0 & \text{for } l+1 \leq n; \\ b_n(BG) & = & 0 & \text{for } n \geq 1. \end{array}$$

- For any sequence $s_1, s_2, \ldots$ of non-negative integers there is a $CW$-complex $X$ of finite type such that for $n \geq 1$

$$\begin{array}{rcl} b_n(X) & = & s_n; \\ b_n^{(2)}(\widetilde{X}) & = & 0. \end{array}$$

**Theorem (Approximation Theorem, Lück [8])**

*Let $X$ be a connected CW-complex of finite type. Suppose that $\pi$ is residually finite, i.e., there is a nested sequence*

$$\pi = G_0 \supset G_1 \supset G_2 \supset \dots$$

*of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let $X_i$ be the finite $[\pi : G_i]$-sheeted covering of $X$ associated to $G_i$.*

*Then for any such sequence $(G_i)_{i \geq 1}$*

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

- Ordinary Betti numbers are not multiplicative under finite coverings, whereas the $L^2$-Betti numbers are. With the expression

$$\lim_{i\to\infty} \frac{b_n(X_i)}{[G : G_i]},$$

we try to force the Betti numbers to be multiplicative by a limit process.

- The theorem above says that $L^2$-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

# Applications to deficiency and signature

## Definition (Deficiency)

Let $G$ be a finitely presented group. Define its deficiency

$$\operatorname{defi}(G) := \max\{g(P) - r(P)\}$$

where $P$ runs over all presentations $P$ of $G$ and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation $P$.

## Example

- The free group $F_g$ has the obvious presentation $\langle s_1, s_2, \dots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\operatorname{defi}(F_g) = g$.
- If $G$ is a finite group, $\operatorname{defi}(G) \leq 0$.
- The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n \rangle$ realizes the deficiency.
- The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is $-1$, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ realizes the deficiency.

## Example (deficiency and free products)

The deficiency is not additive under free products by the following example due to Hog-Lustig-Metzler. The group

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$$

has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

One may think that its deficiency is $-2$. However, it turns out that its deficiency is $-1$ realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

## Lemma

*Let G be a finitely presented group. Then*

$$\operatorname{defi}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

## Proof.

We have to show for any presentation $P$ that

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let $X$ be a $CW$-complex realizing $P$. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{aligned} b_n^{(2)}(\widetilde{X}) &= b_n^{(2)}(G) \qquad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) &\geq b_2^{(2)}(G). \end{aligned}$$

## Theorem (Deficiency and extensions, Lück)

*Let* $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\operatorname{defi}(G) \leq 1$;
3. *Let M be a closed oriented* 4*-manifold with G as fundamental group. Then*
$$|\operatorname{sign}(M)| \leq \chi(M).$$

# The Singer Conjecture

## Conjecture (Singer Conjecture)

*If $M$ is an aspherical closed manifold, then*

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \textit{if } 2n \neq \dim(M).$$

*If $M$ is a closed Riemannian manifold with negative sectional curvature, then*

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2n \neq \dim(M); \\ > 0 & \textit{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
- The Singer Conjecture gives also evidence for the Atiyah Conjecture.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies the following conjecture provided that $M$ has non-positive sectional curvature.

## Conjecture (Hopf Conjecture)

*If $M$ is a closed Riemannian manifold of even dimension with sectional curvature* $\operatorname{sec}(M)$*, then*

$$\begin{array}{rcclrcl}
(-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & < & 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \leq & 0; \\
\chi(M) & = & 0 & \textit{if} & \operatorname{sec}(M) & = & 0; \\
\chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \geq & 0; \\
\chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & > & 0.
\end{array}$$

## Definition (Kähler hyperbolic manifold)

A Kähler hyperbolic manifold is a closed connected Kähler manifold $M$ whose fundamental form $\omega$ is $\widetilde{d}$(bounded), i.e. its lift $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded 1-form $\eta \in \Omega^1(\widetilde{M})$.

## Theorem (Gromov [5])

*Let $M$ be a closed Kähler hyperbolic manifold of complex dimension $c$. Then*

$$\begin{array}{rcll} b_n^{(2)}(\widetilde{M}) & = & 0 & \text{if } n \neq c; \\ b_n^{(2)}(\widetilde{M}) & > & 0; & \\ (-1)^m \cdot \chi(M) & > & 0; & \end{array}$$

- Let $M$ be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, generally $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial.

- A consequence of the theorem above is that any Kähler hyperbolic manifold is a projective algebraic variety.

M. F. Atiyah.
Elliptic operators, discrete groups and von Neumann algebras.
*Astérisque*, 32-33:43–72, 1976.

J. Dodziuk.
de Rham-Hodge theory for $L^2$-cohomology of infinite coverings.
*Topology*, 16(2):157–165, 1977.

J. Dodziuk.
$L^2$ harmonic forms on rotationally symmetric Riemannian manifolds.
*Proc. Amer. Math. Soc.*, 77(3):395–400, 1979.

Ł. Grabowski.
On Turing dynamical systems and the Atiyah problem.
*Invent. Math.*, 198(1):27–69, 2014.

M. Gromov.
Kähler hyperbolicity and $L_2$-Hodge theory.
*J. Differential Geom.*, 33(1):263–292, 1991.

P. A. Linnell.
Division rings and group von Neumann algebras.
*Forum Math.*, 5(6):561–576, 1993.

J. Lott and W. Lück.
$L^2$-topological invariants of 3-manifolds.
*Invent. Math.*, 120(1):15–60, 1995.

W. Lück.
Approximating $L^2$-invariants by their finite-dimensional analogues.
*Geom. Funct. Anal.*, 4(4):455–481, 1994.

W. Lück.
$L^2$-Betti numbers of mapping tori and groups.
*Topology*, 33(2):203–214, 1994.

W. Lück.
*$L^2$-Invariants: Theory and Applications to Geometry and $K$-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in*

*Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*.
Springer-Verlag, Berlin, 2002.

T. Schick.
Integrality of $L^2$-Betti numbers.
*Math. Ann.*, 317(4):727–750, 2000.