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Geometric Topology and Connections with Quantum Field Theory

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Abstract. In recent years, the interplay between traditional geometric topology and theoretical physics, in particular quantum field theory, has played a significant role in the work of many researchers. The idea of this workshop was to bring these people together so that the fields will be able to grow together in the future. Most of the talks in the workshop were related to Elliptic Cohomology, Differential K-Theory, or Topological Quantum Field Theory.

Mathematics Subject Classification (2000): AMS-CLASSIFICATION.

Introduction by the Organisers

The workshop *Geometric Topology and Connections with Quantum Field Theory*, organised by Stephan Stolz (Notre Dame) and Peter Teichner (La Jolla) was held June 12th–June 18th, 2005. As mentioned above, this workshop was intendend to bring together people working in the fields of traditional geometric topology and theoretical physics. For that purpose, the organizers asked several well known expositors like Dror Bar-Natan, Dan Freed and Graeme Segal to give survey lectures in some of the most exciting connections between topology and QFT. In what follows we give a brief survey of the three most central topics.

Elliptic Cohomology: The elliptic cohomology of a space X is the home of the 'Witten genus of a family of 'string manifolds parametrized by X , similarly to the role of the K-theory of X as the home of the family version of the A-roof genus of a family of spin manifolds parametrized by X (i.e., a fiber bundle over X with spin manifold fibers). There is now a homotopy theoretic construction of elliptic cohomology (the 'topological modular form theory TMF of Hopkins and Miller) and one of the highlights of the workshop was Jacob Lurie's lecture about his new interpretation (and construction) of TMF via 'derived algebraic geometry'. This approach makes many aspects of TMF more transparent and allows for things like an equivariant version to be defined.

There is also a family version of the Witten genus with values in TMF due to Ando, Hopkins, Reszk and Strickland, however, a geometric/analytic construction of elliptic cohomology and the Witten genus (say analogous to the description of $K(X)$ as families of Fredholm operators parametrized by X) is still missing. This is despite a two decade old proposal of Graeme Segal to interpret elements of the elliptic cohomology of X essentially as families of conformal field theories parametrized by X. Segal gave the opening lecture in the workshop, surveying some of the progress along the lines of his proposal. During the last two years various more precise candidates for a geometric definition of elliptic cohomology were developed. These were represented by the talks of Rognes and Hu-Kriz and an informal evening session by Stolz-Teichner. Antony Wasserman presented an approach to construct outer representations of Lie groups which may well be the starting point of an equivariant version of one of these geometric theories.

Differential K-Theory: The differential K-Theory of a manifold is a crossover between differential forms and K-theory. For example, Freed and Hopkins have recently announced a version of the Atiyah-Singer Index Theorem which is expressed as an equality of an 'analytical and a 'topological index, both of which live in a 'differential K-theory group. This version of the Index Theorem represents a common generalization of the K-theory version of the Index Theorem and the Local Index Theorem. This way, differential K-theory groups can be seen as the common place where geometric aspects of the manifold, encoded as index densities of associated geometric operators meet topological aspects encoded in the K-theory class represented by the principal symbol of these operators. Moreover, what physicists call 'abelian Gauge fields' also fits naturally into this context, where, depending on the phycisal setting, K-theory may have to be replaced by another generalized cohomology theory. Talks by Ulrich Bunke, Dan Freed, Mike Hopkins and Greg Moore represented this aspect of the workshop.

Closely related is the geometry of 'gerbes; isomorphism classes of gerbes over a space X are classified by elements of $H^3(X)$ (like complex line bundle are classified by $H^2(X)$. As connections on complex line bundles show up in physics (as 'electromagnetic potential) so do geometric structures on gerbes ('B-fields) which in particular give 3-forms representing the deRham cohomology class of the gerbe. There are also nonabelian version of such gerbes, explained in lectures of Aschieri-Jurco and related to Andre Henriques' talk.

Topological quantum field theory: Recently, there has been much advance in this area of low-dimensional topology. Khovanov homology is a categorification of the Jones polynomial of knots in 3-space and it was used to distinguish the smooth and topological 4-genus of certain Alexander polynomial one knots. This is the first combinatorial, non Gauge theoretic, argument that smooth and topological

4-manifold are very distinct. Dror Bar-Natan gave a survey of this theory and then Sergej Gukov explained an approach that could possibly lead to a categorification of the two variable Homfly-polynomial.

Another very exciting conjecture in this area is the volume conjecture, relating a certain asymptotic behavior of the Jones polynomial at special values to the hyperbolic volume of a knot. Stavros Garoufalidis gave the closing lecture of the workshop on some progress that he and Thang Le made in this area.

The format of the workshop was 4 lectures per day, except for Wednesday afternoon when we arranged a hike as well as a soccer game. We were very happy with the large amount of interaction that was going on inside and across the various groups of researchers. In addition, the relatively large number of young participants contributed a lot of activity through their curiosity and insistence. Finally, we are particularly thankful to the physicists Aschieri, Jurco, Gukov and Moore, as well as to the semi-physicists Freed and Segal for their participation. They were certainly among the most looked after discussion partners.

Workshop: Geometric Topology and Connections with Quantum Field Theory

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Abstracts

The Quantum Field Theory Point of View on Elliptic Cohomology

Graeme Segal

The first step towards elliptic cohomology was the discovery of the "elliptic genus". An R-valued genus Φ — where R is a commutative ring — is a rule that assigns $\Phi_M \in R$ to each closed oriented manifold M so that

(i) $\Phi_M = \Phi_{M'}$ if M and M' are cobordant.
(ii) $\Phi_{M_1 \sqcup M_2} = \Phi_{M_1} + \Phi_{M_2}$.
(iii) $\Phi_{M_1 \times M_2} = \Phi_{M_1}\ \Phi_{M_2}$.

There is plainly a *universal* genus, with values in Thom's oriented cobordism ring $R_{MSO} = MSO^*(\text{point})$. In 1987 Ochanine [4] and others (cf. [3, 5]) singled out a class of $\mathbb{C}$-valued genera with especially strong rigidity properties, and found they were parametrized by elliptic curves. They can be put together to define a universal elliptic genus $\Phi : R_{MSO} \to R$ with values in a subring R of the ring of modular functions (regarded as functions of an elliptic curve). In fact one can take $R = \mathbb{Z}[\frac{1}{2}, \delta, \varepsilon, \Delta^{-1}]$, where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$, and δ and ε are the functions of the curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ which arise when its equation is written in the form

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4.$$

Just as the cobordism ring R_{MSO} is the coefficient ring of a cohomology theory, Landweber proved that $X \mapsto MSO^*(X) \otimes_{R_{MSO}} R$ is a cohomology theory. This was the original definition of elliptic cohomology; it has been extensively developed since then from the perspective of homotopy theory [2].

The subject was made much more interesting by Witten's observation that the elliptic genus of a manifold M can be understood as the index of a differential operator on the free loop space $\mathrm{L}M$ of M. More precisely, the elliptic genus is the Hilbert series $\Sigma q^k \dim(V_k)$ of a (virtual) graded vector space $V = \oplus V_k$ which is the index

$$\ker(\mathcal{D}_{\mathcal{L}M}) - \operatorname{coker}(\mathcal{D}_{\mathcal{L}M})$$

of a differential operator $\mathcal{D}_{\mathcal{L}M}$ defined on $\mathcal{L}M$. Here $q = e^{2\pi i \tau}$, and the grading on the index comes from the action of the circle $\mathbb{T}$ on $\mathcal{L}M$ by rotation of loops.

This loop space perspective, however, does not by itself shed light on the modularity which is the basic property of the elliptic genus. The place where one "naturally" encounters modular forms is as the partition functions of two-dimensional chiral conformal field theories [6], and Witten gave a heuristic explanation of both the modularity and the rigidity properties of the elliptic genus in terms of these field theories (cf [3, 5]).

It would obviously be interesting to be able to represent elements of the elliptic cohomology of a space by some kind of geometric objects, in the way that elements of ordinary K-theory are represented by vector bundles. In my Bourbaki talk [5]

I gave a definition of a chiral conformal field theory of level n "over" a space X, and speculated that such an object represented an elliptic cohomology class of X of dimension n. It was clear, however, that these objects could not by themselves be used to construct the elliptic theory, mainly because they are defined locally on the loop space $\mathbb{L}X$ rather than on X, but also because they seem to be too rigid.

In the last few years Stolz and Teichner [8] have developed a much more sophisticated concept of elliptic object which seems to deal with both of these problems. The main content of this talk is an account of their work.

The first step is to relax the rigidity by replacing chiral conformal field theories by theories which are not chiral, but which are supersymmetric on the antichiral side. This is very closely analogous to replacing finite dimensional vector spaces by Fredholm operators, which provide an effective model of the "category" of "virtual" vector spaces.

The second, more important, step is to make the objects *local*. This is done by extending the standard formalization of a field theory over X to what is called a "three-tier" theory. The point is to give enough extra information to enable one to reconstruct the Hilbert space which a field theory over X associates to a loop in X out of data associated to arbitrarily short paths in X. In fact one gives a von Neumann algebra A_x at each point $x \in X$, and an (A_x, A_y)-bimodule $\mathcal{H}_\gamma$ for each path γ in X from x to y, and one requires concatenation of paths to correspond to the Connes tensor product of bimodules over von Neumann algebras (cf. [9]).

Another attempt to represent elliptic classes geometrically has been made by Baas, Dundas, and Rognes [1]. They aim only to construct classes of dimension 0, and obtain the theory in other dimensions by appealing to the standard "delooping" techniques which are familiar in the construction of algebraic K-theory.

A fuller account of this talk can be found in [7].

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Toward Constructing Elliptic Cohomology by Modularizing K-theory

Po Hu, Igor Kriz

The first part of this talk was based on our joint work [3, 4] and the second part of the talk was based on the joint work of Kriz and Sati [5, 6, 7].

In the first part of this talk, we described the construction of a class of generalized cohomology theories [3] that have certain modularity properties desired in elliptic cohomology theories, in the sense that they come with natural maps

$$E \to K[[q]][q^{-1}]$$

whose images on coefficients we conjecture to be certain modular forms. The theories also enjoy a certain "manifest geometric modularity", which is the basis of our conjectures. The paper [3] also contains a rigorous definition of conformal field theory, based on G. Segal's outline [8]. This construction is further expanded and extended to open sectors in [4].

In the talk, we spent a lot of time on a "genetic introduction" to our construction, which is only marginally addressed in [3]. Roughly, the way the construction came about is as follows. Consider a Hilbert space $\mathcal{H}$ with an action by S^1, which can be also thought of as a $\mathbb{Z}$-grading on $\mathcal{H}$

$$\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n.$$

Suppose the S^1-action has a lowest weight. Our idea was to look for a "manifestly modular" subgroup of the group of S^1-equivariant linear isomorphisms of $\mathcal{H}$. The group of S^1-equivariant isomorphisms are the ones which preserve the grading:

$$GL(\mathcal{H})^{S^1} = \prod_{n\in\mathbb{Z}} GL(\mathcal{H}_n).$$

Taking classifying spaces, we get

$$BGL(\mathcal{H})^{S^1} = \prod_{n\in\mathbb{Z}} BGL(\mathcal{H}_n).$$

This maps to $K[[q]][q^{-1}]$, which connects to M. Ando's interpretation of $K[[q]][q^{-1}]$ as a kind of "elliptic cohomology without modularity" [1]. However, we want generalized cohomology theories that are geometrically modular. An elliptic curve is a torus $S^1 \times S^1$. The first approximation to a construction which would see the other copy of S^1 uses the bar complex: we can try to replace B by the cyclic bar construction B_{cyc}, and recall the fact that the realization of a cyclic bar construction has an S^1-action. Thus, a naive candidate for our object is

$$B_{cyc}(GL(\mathcal{H}))^{S^1\times S^1}. \tag{1}$$

However, the construction (1) doesn't have the properties we want for two reasons. First, it is not true that the S^1-fixed points of B_{cyc} would be B. We cannot replace fixed points by homotopy fixed points, since we are dealing with globally contractible objects. Second, the two copies of S^1 do not play equal roles in (1): the bar degrees of B_{cyc} specify a system of parallel circles on the elliptic curve, breaking modularity.

Nevertheless, there is an object which more or less canonically maps into (1), and does enjoy manifest modularity in the above sense, provided that $\mathcal{H}$ is a 1-conformal field theory, which means a chiral conformal field theory consistent up to worldsheets of genus 1 (some other mild assumptions are needed, details are in [3]). The desired object is our space of *elliptic bundles* [3]. For a conformal field theory $\mathcal{H}$, we defined the idea of stringy $\mathcal{H}$-bundles on surfaces. Telegraphically speaking, an elliptic bundle is a stringy bundle on an elliptic curve E which is equivariant with respect to the action of E on itself, associated with a chiral conformal field theory. Given a class Φ of CFTs closed under $\otimes$, we construct an "elliptic classifying space" of all elliptic bundles $B_{ell}\Phi$. There are two obvious obstructions to modularity of homotopy classes, which we call $p_1/2$ and central charge. We take fibers with respect to them, and then stabilize in the most generic way: take A_∞ multiplicative group completion, then the suspension spectrum, and invert homotopy classes corresponding to suitable modular forms depending on the individual examples (e.g. as the discriminant form). This is our class of generalized cohomology theories, which come by definition with natural maps to $K[[q]][q^{-1}]$. We computed several kinds of examples of homotopy classes of these theories, but haven't fully determined the whole coefficient rings of any of the theories. Based on the examples, it seems reasonable to conjecture that the coefficients of theories constructed are always modular forms, and that the theory therefore is of "elliptic cohomology" type, thus reflecting the geometric modularity of the construction. Even provided that these conjectures are correct, however, we do not know if TMF might arise from such construction directly, or if some refinements are needed (for example some type of fixed point construction).

Our construction has certain striking features which do not appear in G. Segal's proposed approach using elliptic objects [9], [10]. One such feature is that our construction assigns a preferred role to worldsheets of genus 1 (elliptic curves), and in fact can take as inputs genus ≤ 1 sectors of CFT's, or CFT's consistent only up to genus 1 (1-CFT's). Another (related) feature is that our construction clearly seems to involve an elliptic curve not only in the source of maps considered (physically: worldsheet) but also in the target (physically: spacetime). That spacetime is not CFT spacetime, but rather two additional spacetime dimensions.

In the second part of the talk, we outlined new related developments in physics (Kriz and Sati [5], [6], [7]), which arose from independent considerations, but may lead to an explanation of these features. Diaconescu, Moore and Witten [2] defined partition functions for type II string theories using K-theory and showed that the IIA partition function coincides with the M-theory partition function. [5] considers an analogous construction using elliptic cohomology. The origin of this construction was examining the W_7 obstruction to the existence of IIA partition function in [2], which coincides with obstruction to orientability of spacetime with respect to elliptic cohomology. There is a similar construction in the IIB case also ([6, 7]). What is the significance of these elliptic partition functions? Evidence eventually pointed to the possibility that just as the DMW IIA partition function comes from M-theory, the elliptic partition functions coincide with partition functions of 12-dimensional F-theory compactified on an elliptic curve. This elliptic curve seems to explain the two additional spacetime dimensions involved in the construction we discussed in the first part of the talk. The physical conjecture is not yet verified, but can in principle be tested by computing suitable loop versions of E_8 and Rarita-Schwinger indices considered in [2].

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Remarks on the Hamiltonian Formulation of Some Generalized Abelian Gauge Theories

Gregory W. Moore

The talk was concerned with field theories with interesting and subtle connections to topology - at least, they seem subtle to a physicist like the author! The theories in question are related to generalized cohomology theories. To do field theory one needs local fields, and the correct notion for string theory and M-theory appears to be generalized differential cohomology theory. See [1, 2] for an introduction to this subject. By a "generalized abelian gauge theory" (GAGT) we mean a field theory whose space of gauge invariant fields is a differential generalized cohomology group. In supergravity/superstring theory we meet many examples of GAGT's. The following four examples are illustrative.

Example 1: *Generalized Maxwell theories.* Recall the Delgine-Cheeger-Simons cohomology group $\check{H}^\ell(M)$, where M is a spacetime. We take the fiber product of $\Omega^\ell_d(M)$ with $H^\ell(M,\mathbb{Z})$ over $\bar{H}^\ell(M;\mathbb{Z})$ in the appropriate sense. Here $\Omega^\ell_d(M)$ is the space of d-closed forms of degree ℓ and the $^-$on $\bar{H}^\ell$ denotes reduction modulo torsion. Thus $\bar{H}^\ell(M) \cong H^\ell_{DeRham}(M)$. Denoting the setwise fiberproduct by $\mathcal{R}$ we have

$$0 \to H^\ell(M;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \to \check{H}^\ell \to \mathcal{R} \to 0 \tag{1}$$

To a class $[\check{A}]$ we associate its fieldstrength $F \in \Omega^\ell_d(M)$ and its characteristic class $a \in H^\ell(M,\mathbb{Z})$. In a generalized Maxwell theory the fields are taken to be $[\check{A}_i] \in \check{H}^{\ell_i}(M)$ and the action is $S_M := \int \frac{1}{2}\lambda^{-1}_{ij} F_i * F_j$, where λ_{ij} is a positive definite matrix of couplings. Note standard Maxwell theory is the case $\ell = 2$. For further discussion see D. Freed's contribution to these proceedings, and the references.

Example 2: *Generalized Maxwell-Chern-Simons (MCS).* In physical applications we often find Chern-Simons terms in the action. These are formed by combining the Cheeger-Simons product $\check{H}^{\ell_1}(M) \times \check{H}^{\ell_2}(M) \to \check{H}^{\ell_1+\ell_2}(M)$ with the integration: $\int^{\check{H}} : \check{H}^{n+1}(M) \to \mathbb{R}/\mathbb{Z}$, which is defined for M compact, oriented, and $\dim M = n$. We now take the action to be $S = S_M + \pi \int k_{ij} \check{A}_i \cdot \check{A}_j$. A case of special interest, which we call the "self-dual case" occurs when $\dim M = 2p+1$ is odd and all $\ell_i = p$.

In physics, the action $S \mod 2\pi$ must be well-defined. This puts quantization conditions on k_{ij}. For example, if p is odd then k_{ij} should be an even integral symmetric form. However, in physics we often meet fractional values and special things must happen. One example is the case of $p = 1$, which arises in the theory of the fractional quantum Hall effect. In this case k_{ij} must be an *odd* integral form. To make the theory well-defined we must endow M with a spin structure. This defines *spin* Chern-Simons theories.

Example 3: *M-theory.* This is a more dramatic example of fractional k. In this case, very roughly, we must take $[\check{C}] \in \check{H}^4(M)$ where M is an 11-dimensional spin manifold. One of the reasons this is rough is that the quantization law on the fieldstrength is $[G]_{DR} = \bar{a} + \bar{p}_1/4$ where a is an integral class [3]. The shift can be handled in various ways. The important novelty in M-theory is that the action is now $S = \int \frac{1}{2}\lambda^{-1} G * G + \frac{2\pi}{6} \int \check{C} \cdot \check{C} \cdot \check{C} + linear$ and the Chern-Simons term is *cubic.* Amazingly, when one takes into account the shift, the linear term, and the fermion determinants, the effective action of M-theory turns out to be well-defined. See [4] for a detailed explanation of this.

Example 4. *Type II RR fields.* In type II string theory the bosonic fields are of NS type and RR type. The NS type fields include the metric on a 10-dimensional spin manifold M, a scalar (the "dilaton") and a "B-field" $[\check{B}] \in \check{H}^3(M)$. The space of fields is fibered over the NS fields, with the fiber being the differential K-group $[\check{A}] \in \check{K}^{\epsilon}_{\check{B}}(M)$. Here $\epsilon = 0/1$ according to whether the theory is IIA/IIB. This group satisfies properties analogous to those of differential cohomology. Let $\mathcal{R}_K$ denote the setwise fiber product of $\Omega^{\epsilon}_{d_H}(M)$ with $K^{\epsilon}_h(M)$ over $H^{\epsilon}_{d_H}(M)$. Here $\Omega^{\epsilon}_{d_H}(M)$ is the space of $d_H := d - H$ closed forms of parity ϵ, H is the fieldstrength of $\check{B}$, and $H^{\epsilon}_{d_H}(M)$ are the cohomology groups for d_H. The characteristic class of $\check{B}$ is denoted h, and $K^{\epsilon}_h(M)$ is the twisted K-theory. We then have

$$0 \to K^{\epsilon-1}_h(M) \otimes \mathbb{R}/\mathbb{Z} \to \check{K}^{\epsilon}_{\check{B}}(M) \to \mathcal{R}_K \to 0 \tag{2}$$

In the theory of the RR fields there is an important new ingredient: the field must be self-dual. In the Lagrangian picture the action is a period matrix [5, 6]. Note that there is a metric on $\mathcal{R}_K$ defined by $g(F_1, F_2) := \int_M F_1 * F_2$ and a symplectic form $\omega(F_1, F_2) := \int F_1 \bar{F}_2$ where $\bar{R}$ denotes the fieldstrength for the complex-conjugated K-theory class. The metric and symplectic form have a compatible complex structure $J \cdot R = *\bar{R}$. We choose a maximal Lagrangian splitting $\mathcal{R}_K = \Lambda_1 + \Lambda_2$. Then we impose self-duality by requiring that the fields are only valued in Λ_1. Given the above data we have a period matrix, which is a quadratic form on Λ_1. This essentially defines the action. As an example we could take Λ_2 to be the set of fields which are torsion on the 5-skeleton and Λ_1 to be any complementary maximal Lagrangian space. The local degrees of freedom are a one-form and three-form potential, precisely as found in the standard treatments of supergravity. The (Euclidean space) action is, roughly, $S \sim -\pi \int F_0 * F_0 + F_2 * F_2 + F_4 * F_4 + i\pi \int F_0 F_{10} - F_2 F_8 + F_4 F_6$. A forthcomming paper will make this more precise [7]

1. *Hamilton, Hilbert and Gauss*

We now assume spacetime has a collared boundary of the form $X \times \mathbb{R}$ where X is compact and oriented. We would like to describe the Hilbert space of states $\mathcal{H}$. Let us begin by recalling the example of Maxwell theory. Here the configuration space is $\check{H}^2(X)$, phase space is $T^*\check{H}^2(X)$ and hence the natural quantization gives a Hilbert space $\mathcal{H} = L^2(\check{H}^2(X))$. (We ignore subtleties of analysis, but we are confident that they are not too serious for these free field theories.) This is not always the most convenient description of the theory since locality often forces one to work directly with a connection. Classically we choose a line bundle $\mathcal{L}$ for each $c_1 \in H^2(X, \mathbb{Z})$ and consider $\mathcal{A}$, the space of connections on $\mathcal{L}$. Then we can identify $T^*(\mathcal{A}/\mathcal{G}) = \mu^{-1}(0)/\mathcal{G}$ where $\mathcal{G} = Map(X, U(1))$ is the gauge group and $\mu(\epsilon) = \int \epsilon d\Pi$, is the moment map. Here $\Pi \in \Omega^{n-2}(X)$ is the conjugate momentum. Under the Legendre transformation $\Pi = \lambda^{-1} * F|_X$, in the classical theory. Quantum mechanically, we start with $L^2(\mathcal{A})$, so $\hat{\Pi} = -i\frac{\delta}{\delta A}$ and we define $\mathcal{H}$ to be the linear subspace of gauge invariant states $\psi(A) = \psi(g \cdot A)$. The action of the gauge group is $g \cdot A = A + \omega$ where $\omega \in \Omega^1_{\mathbb{Z}}(X)$ and hence physical wavefunctions satisfy $\exp(i \int \omega \hat{\Pi})\psi = \psi$. This is the quantum mechanical version of the restriction to $\mu^{-1}(0)$ in the classical theory. In the classical theory $[F]$ is quantized and $[\Pi] = [\lambda^{-1} * F]$ is not. In the quantum theory $[F]$ and $[\Pi]$ are both quantized.

We would like to give an analogous discussion of the Hamiltonian formulation for all GAGT's. This involves finding a suitable description of a groupoid $\mathcal{A}$ of fields and moreover involves taking into account the effect of Chern-Simons terms. In the presence of these terms the wavefunction is a section of a nontrivial line bundle with connection $L_{CS} \to \mathcal{A}$ and the Gauss law is $\tilde{g}\psi(A) = \psi(g \cdot A)$ where $\tilde{g}$ is a suitable lift of the group action. This has been carried out in detail for generalized Maxwell-Chern-Simons theory for $p = 1$ [8], $p = 2$ [9], and M-theory [10]. The case of type II RR fields is work in progress [7]

There is one general feature which is worth stressing: $Aut(A)$ must act trivially on $L_{CS}|_A$. This means that the wavefunction can only have support on certain *components* of $\mathcal{A}$. For example, for MCS with $p = 1$ we are restricted to $c_1 = 0$, while the M-theory case was discussed in detail in [11]. In physics $Aut(A)$ is the group of global gauge transformations and quite generally $\alpha \cdot \psi = e^{2\pi i \langle \alpha, Q \rangle}\psi$, defines the "total electric charge" in the dual group. This must vanish in a compact universe X.

2. *Explicit Wavefunctions*

We are discussing free theories so we can be explicit about the wavefunctions. Two reasons this is interesting are first, they might be important parts of a Hartle-Hawking wavefunction of the universe, relevant to quantum cosmology in flux compactifications, and second, they give an approach to the mathematical formulation of the partition function of a self-dual field. To explain this latter point, return to MCS for M of dimension $2p + 1$, with $[\check{A}] \in \check{H}^{p+1}(M)$. The groundstate wavefunction in the harmonic sector computes the partition functions in the space of conformal blocks of a self-dual p-form gauge theory in $2p$ dimensions. An heuristic reason for this is that at long distances (which focuses on the groundstate) the action is dominated by the Chern-Simons term. But $\delta S_{CS} = \int_M \delta A F + \int_X \delta A A$ under variation of fields. To get a well-posed boundary value problem we should set $A = *A|_X$. But in Chern-Simons theory the gauge modes in the bulk become dynamical fields ("edge states") on the boundary. In this case the gauge freedom is $A \to A + \omega$, $\omega \in \Omega^p_{\mathbb{Z}}(M)$, so we have a dynamical field with $\omega = *\omega$ on the boundary.[1] The essential point is that there is a kahler structure on the space of harmonic modes. The groundstate wavefunctions are (essentially) holomorphic functions on this space, but the Gauss law implies quasi-periodicity. Together these conditions say that the wavefunction is a theta function. When the wavefunction is properly normalized one also obtains the correct quantum determinants which account for

[1] Indeed, the above approach is really a special case of the AdS/CFT correspondence. This is a profound generalization of the famous Chern-Simons-Witten/Rational CFT correspondence. It would constitute a major step forward in Mathematics if someone could state the AdS/CFT correspondence in a mathematically precise way. Among the many nontrivial implications of such a formulation would be the existence of nontrivial (super) conformal field theories in $d = 2, 3, 4, 5, 6$ dimensions.

small fluctuations of the field. The above approach to formulating the partition function of the self-dual field was carried out explicitly in [8, 9]. One can also recover in this way the partition function of the $M5$ brane as formulated in [12]. This was sketched in [10].

One example of a useful result which can be obtained from this approach is the classification of quantum abelian spin Chern-Simons theories [8]. The classical action is determined by a symmetric integral matrix k_{ij}, that is, by an integral lattice Λ. The quantum theory only depends on Λ through the invariants: 1.) The signature of Λ modulo 24, 2.) The discriminant group $\mathcal{D} = \Lambda^*/\Lambda$ together with its bilinear form $b : \mathcal{D} \times \mathcal{D} \to \mathbb{R}/\mathbb{Z}$ and 3.) The equivalence class of a quadratic refinement q of b given by $q(\gamma) = \frac{1}{2}(\lambda, \lambda - W_2) + \frac{1}{8}(W_2, W_2) \mod 1$ where $[W_2] \in \Lambda^*/2\Lambda^*$ is defined by $W_2(\lambda) = (\lambda, \lambda) \mod 2$, $\forall \lambda \in \Lambda$. Note that this form satisfies the Gauss-Milgram constraint. Conversely, the data (1,2,3) are realized by some Λ. The mod-24 periodicity is related to the mod 24 periodicity that occurs in elliptic cohomology. This theorem also has experimental implications in the fractional quantum Hall effect.

3. *Electric and Magnetic Fluxes Don't Commute*

This section is based on unpublished work with D. Freed and G. Segal.

It is very useful to grade the Hilbert space $L^2(\check{H}^\ell(X))$ of generalized Maxwell theory by the electric and magnetic flux sectors. The magnetic grading is defined by the component group of $\check{H}^\ell(X)$, that is, by $m \in H^\ell(X, \mathbb{Z})$. On the other hand, there is a dual formulation of these theories by $[\check{A}_D] \in \check{H}^{n-\ell}(X)$ (recall $\dim M = n$ and $\dim X = n-1$), so the same Hilbert space is graded by electric flux $e \in H^{n-\ell}(X, \mathbb{Z})$. (Recall that $\frac{1}{2\pi}\Pi = \frac{1}{2\pi\lambda} * F|_X$ had quantized periods in the quantum theory.) A novel point [13] is that one cannot simultaneously grade the Hilbert space by both electric and magnetic flux. The reason is simply that $\check{H}^\ell(X)$ is an abelian group so that $L^2(\check{H}^\ell(X))$ is only a representation of the Heisenberg extension of $\check{H}^\ell(X) \times \check{H}^{n-\ell}(X)$ defined by the Pontryagin pairing. States which are in a definite sector of (the topological class of) the electric flux diagonalize translation by flat fields $a_f \in H^{\ell-1}(X, \mathbb{R}/\mathbb{Z})$, while states of definite (topological class of) magnetic flux diagonalize translation by $H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z})$ in the dual variables. Now, the Pontryagin pairing on $H^{\ell-1}(X, \mathbb{R}/\mathbb{Z}) \times H^{n-\ell-1}(X, \mathbb{R}/\mathbb{Z})$ is nonzero - it is just the torsion pairing on cohomology. Pursuing this line of thought one finds that the Hilbert space can be graded by $\bar{H}^\ell(X) \times \bar{H}^{n-\ell}(X)$ with the Heisenberg extension of $Tors(H^\ell(X)) \times Tors(H^{n-\ell}(X))$ acting to permute the different flux sectors. When these observations and ideas are applied to type II RR fields one arrives at a very surprising and significant conclusion: Since $K(X)$ accounts for the topological classes of *both* electric and magnetic fluxes, the K-theory class of a RR field is not measurable in the quantum theory! We expect this will have interesting applications in string theory.

Acknowledgements

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Differential Geometry of Nonabelian Gerbes

Paolo Aschieri, Branislav Jurco

Abelian bundle gerbes are a higher version of line bundles. Complex line bundles are geometric realizations of the integral 2nd cohomology classes $H^2(M,Z)$ on a manifold, i.e. the first Chern classes. Similarly, abelian (bundle) gerbes are the next level in realizing integral cohomology classes on a manifold, they are geometric realizations of the 3rd cohomology classes $H^3(M,Z)$. One way of thinking about abelian gerbes is in terms of their local transition functions [1]. Local "transition functions" of an abelian gerbe are complex line bundles on double overlaps of open sets satisfying cocycle conditions for tensor products over quadruple overlaps of open sets. The nice notion of abelian bundle gerbe [2] is related to this picture. Abelian gerbes and bundle gerbes can be equipped with additional structures, that of connection 1-form, that of curving (this latter is the 2-form gauge potential that corresponds to the 1-form gauge potential in line bundles) and of curvature (3-form field strength whose de Rham cohomology class is the image in $I\!R$ of the integral third cohomology class of the gerbe).

We study the nonabelian generalization of abelian bundle gerbes and their differential geometry. Nonabelian gerbes arose in the context of nonabelian cohomology [3]. Their differential geometry –from the algebraic geometry point of view– has been recently discussed in [4]. Here we study the subject in the context of differential geometry. We show that nonabelian bundle gerbes connections and curvings are very natural concepts in classical differential geometry. We believe that it is primarily in this context that these structures can have mathematical physics applications. Our results are in agreement with those of [4].

Since local transition functions of an abelian gerbe are complex line bundles (or principal $U(1)$ bundles), nonabelian gerbes should be built gluing appropriate nonabelian principal bundles (bibundles). Bibundles admit a local description in terms of transition functions. This is the starting point for a local description of nonabelian gerbes and of their differential geometry. More importantly we aslo give a global description of these geometric structures. First we define connection one-forms on a bibundle, they are a relaxed version of connections on principal bundles. Nevertheless one can define the exterior covariant derivative and curvature two-form of this connection, and prove a relaxed Cartan structural equation and the Bianchi identity. We then proceed to define and investigate nonabelian bundle gerbes and expecially their connections. Then the nonabelian curving 2-form and the corresponding curvature 3-form compatible with the nonabelian bundle gerbe connection are defined and their relations studied.

We also provide a new way of looking at (twisted) nonabelian gerbes, namely as modules for abelian 2-gerbes. Following the correspondence between line bundles and abelian gerbes, we have that abelian 2-gerbes are geometric realizations of the fourth integral cohomology classes $H^4(M, Z)$. We recall that a twisted nonabelian bundle is a bundle whose cocycle relations holds up to phases. This phases in turn characterize an abelian gerbe. Twisted nonabelian gerbes are a higher version of twisted bundles, we study their properties and show that they are associated with abelian 2-gerbes in the same way that twisted bundles are associated with abelian gerbes.

Using global anomalies cancellation arguments we then see that the geometrical structure underlying a stack of M5-branes is in general indeed that of a twisted nonabelian gerbe. Connections, curvings and curvature for the 2-gerbe and the twisted nonabelian gerbe are also studied and their M5-branes interpretation is discussed. A prominent role is here played by the E_8 group.

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T-Duality for Torus Bundles

Thomas Schick

(joint work with Ulrich Bunke)

The motivation for T-duality is a particular version of mirror symmetry in string theories. Here, if the background space time is a torus bundle E over a base space B, with an element in $h \in H^3(E, Z)$ (representing a flux), the T-dual is expected to be a (usually different) torus bundle E' with $h' \in H^3(E', Z)$. One of the features in T-duality is a duality isomorphism between twisted K-theory groups of E and E'.

We first introduce an axiomatic point of view of twisted homology theories. The guiding principle for our definition of T-duality is that it allows a natural construction of the T-duality transform. T-duality is then actually a relation between pairs as above.

The T-duality transform can be described as follows: given E and a dual bundle $\hat{E}$, form the fiber product $E \times_B \hat{E}$. The transform is then given by pulling back the twisted K-class from E to $E \times_B \hat{E}$, use a isomorphism of twists to change to a different twisted K-theory group, and integrate over the fiber to the twisted K-theory of $\hat{E}$. Observe that some additional structure is needed to construct the twist isomorphism.

The explicit example approach used in the lecture to twisted K-theory is via sections of associated bundles of C^*-algebras. The twists are then given by maps to BPU, the classifying space of bundles whose structure group is the projective unitary group of a Hilbert space. This approach goes back to Raeburn and Rosenberg [1] and is also used in [2].

We discuss existence and uniqueness questions for T-duality pairs, and prove that the T-duality transformation in twisted K-theory and cohomology is an isomorphism. It turns out that in general one doesn't have a T-dual, and if it exists, that it's topological type is not uniquely determined. This is related to the fact that for the definition of T-duality additional structural data is required (producing isomorphisms of twists in appropriate comparison spaces). Our approach to these questions is via the construction and study of appropriate classifying spaces for the structures involved.

The case of 1-dimensional fibers is somewhat special. There we can construct a canonical T-dual for a given pair [5] (i.e. existence and uniqueness are satisfied).

We compare our construction with an approach of Mathai and Rosenberg [6] to T-duality via non-commutative geometry. Some statements of [6] are incorrect, we give counterexamples and proofs of corrected versions.

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Elliptic Cohomology and Derived Algebraic Geometry
Jacob Lurie

Let E be an elliptic curve over a commutative ring R. If certain mild hypotheses are satisfied by E, then Landweber's exact functor theorem ensures the existence of an essentially unique (elliptic) cohomology theory A such that $A(*) \simeq R$ and $A(\mathbf{C}\mathrm{P}^\infty)$ is the ring of functions on the formal completion of the elliptic curve E. In particular, these conditions are satisfied whenever E is classified by an étale map

$$\operatorname{Spec} R \xrightarrow{\phi} \mathcal{M},$$

where $\mathcal{M}$ denotes the moduli stack of elliptic curves; let A_ϕ be the associated cohomology theory.

The assignment

$$\phi \mapsto A_\phi$$

may be viewed as a presheaf of cohomology theories on the moduli stack of elliptic curves. The work of Goerss, Hopkins, and Miller implies that this presheaf of cohomology theories can be refined (in an essentially unique way) to a presheaf of E_∞-ring spectra $\mathcal{O}$ on the moduli stack of elliptic curves. It then makes sense to take the (right-derived functor of) global sections, giving an E_∞-ring spectrum

$$\mathrm{tmf}[\Delta^{-1}] = R\Gamma(\mathcal{M}, \mathcal{O}).$$

A more refined approach (which includes the "point at ∞" on $\mathcal{M}$) yields a spectrum tmf, the spectrum of *topological modular forms*, so named for the existence of a ring homomorphism from $\pi_* \mathrm{tmf}$ to the ring of integral modular forms, which is an isomorphism after inverting 6. The spectrum tmf may be regarded as a universal elliptic cohomology theory, and is a suitable "target" for elliptic invariants such as the Witten genus.

It is natural to think of the presheaf $\mathcal{O}$ as a kind of structure sheaf on the moduli stack $\mathcal{M}$ of elliptic curves. This can be made precise using the language of *derived algebraic geometry*: a generalization of algebraic geometry in which E_∞-ring spectra are allowed to play the role of commutative rings. The pair $(\mathcal{M}, \mathcal{O})$ may naturally be viewed as a Deligne-Mumford stack in the world of derived algebraic geometry, which is a kind of "derived version" of the classical moduli stack of elliptic curves. One may then ask if $(\mathcal{M}, \mathcal{O})$ has some moduli-theoretic significance in derived algebraic geometry; our main result is an affirmative answer to this question.

Given an E_∞-ring spectrum R, there is a natural notion of an *elliptic curve* over R in derived algebraic geometry (which specializes to the usual notion of elliptic curve when R is an ordinary commutative ring). Any elliptic curve E has a formal completion $\hat{E}$; we define an *orientation* of E to be an equivalence $\operatorname{Spf} R^{\mathbf{C}\mathrm{P}^\infty} \simeq \hat{E}$ of formal groups over R. The main result then asserts that there is a natural homotopy equivalence

$$\{ \text{ Oriented Elliptic Curves } E \to \operatorname{Spec} R\} \Leftrightarrow \operatorname{Map}(\operatorname{Spec} R, (\mathcal{M}, \mathcal{O}));$$

in other words, $(\mathcal{M}, \mathcal{O})$ classifies *oriented* elliptic curves in derived algebraic geometry.

This result, and the accompanying ideas, shed light on virtually all aspects of the theory of elliptic cohomology:

(1) The moduli-theoretic approach leads to a new proof of the theorem of Hopkins and Miller described above. Namely, one can begin with the problem of classifying oriented elliptic curves in derived algebraic geometry. Using quite general methods, one can prove the existence of a Deligne-Mumford stack $(\mathcal{M}, \mathcal{O})$ which solves this moduli problem. A delicate (but conceptual and computation-free) analysis reveals that $\mathcal{M}$ is the moduli stack of elliptic curves, and that the presheaf of cohomology theories underling $\mathcal{O}$ agrees with the presheaf defined via Landweber's theorem.

(2) The moduli-theoretic significance of $(\mathcal{M}, \mathcal{O})$ implies the existence of a universal (oriented) elliptic curve $\mathcal{E} \to (\mathcal{M}, \mathcal{O})$. This elliptic curve, together with its orientation, may be naturally viewed as encoding the structure of *equivariant* elliptic cohomology. These equivariant theories were not accessible (at least integrally) by other methods.

(3) The process of generalizing the classical theory of elliptic curves to the derived world can lead to unexpectedly interesting results. For example, in classical algebraic geometry, every elliptic curve is (canonically) self-dual: that is, isomorphic to its dual as an abelian variety. In the derived algebraic geometry, this argument fails; however, it is still true (for a much more subtle reason) if we restrict our attention to *oriented* elliptic curves. The self-duality of the universal curve $\mathcal{E}$ is intimately connected with the theory of 2-*equivariant elliptic cohomology*.

(4) Part of the impetus for the study of elliptic cohomology came from Witten's observation that the Witten genus $f_M(q)$, a certain power series in q that can be associated to any spin manifold M, is actually the q-expansion of a modular form in the case where M admits a *string* structure (that is, the first Pontryagin class of M vanishes). This statement is "explained" by the existence of an orientation (sometimes called the *topological Witten genus*

$$\mathrm{MString} \to \mathrm{tmf};$$

in other words, the universal elliptic cohomology theory is oriented for vector bundles admitting a string structure. Using the moduli-theoretic point of view (in particular, the theory of 2-equivariant elliptic cohomology) one can give a much easier proof of the existence of this orientation, which turns out to be unique if one requires a suitable equivariant version.

(5) When we specialize to the "cusp" of the moduli stack $\mathcal{M}$, elliptic cohomology specializes to K-theory. However, the equivariant story is more interesting. When one computes G-equivariant elliptic cohomology near the cusp, one recovers the representation theory not of G, but of the (positive energy representations of the) loop group of G.

(6) The use of derived algebraic geometry to study cohomology theories associated to algebraic groups is not limited to the elliptic case. Most of the above formalism applies equally well to the multiplicative group, leading to a new method for constructing (equivariant) K-theory, and a new proof of Snaith's result $K \simeq (\Sigma^\infty \mathbf{C}\mathrm{P}^\infty_+)[\beta^{-1}]$. One can also apply the same methods to certain Shimura varieties other than $\mathcal{M}$, and thereby produce "higher chromatic analogues" of elliptic cohomology (though in this case it seems necessary to restrict attention to a fixed prime).

(7) Unfortunately, this approach to elliptic cohomology does not shed much light on the most interesting problem: understanding the geometric significance of tmf. However, the universal property characterizing $(\mathcal{M}, \mathcal{O})$ does lead to a good recognition principle for elliptic cohomology: given a (geometrically defined) cohomology theory Ell, which has sufficiently nice formal properties (E_∞-multiplicative structures, equivariant analogues, and so forth) and which, when computed in certain examples, gives answers of the expected size, one can show that Ell agrees with tmf (in a fashion that is compatible with all of the additional structures).

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Spinors and Twistors on Loop Spaces

Tilmann Wurzbacher

(joint work with Mauro Spera)

This talk reports on joint work of the orator with Mauro Spera on the construction of spinors and twistor spaces over loop spaces, aiming for an analytic approach to elliptic genera via Dirac-type operators on loop spaces extending our work in the flat case to curved riemannian manifolds.

1. Motivation

The "Witten genus" ϕ_W associates a modular form $\phi_W(M)$ to a finite dimensional manifold M. Hypothetically, there should be a "Dirac-Ramond operator" D_K acting on spinor fields over $LM = C^\infty(S^1, M)$, the free loop space of M, such that its S^1-equivariant index equals $\phi_W(M)(q)$, the q-expansion of $\phi_W(M)$, interpreted as a virtual S^1-representation. Killingback and Witten gave a heuristic formula for the operator D_K (see [K] and [Wi] for details):

$$D_K = -i\int_0^1 d\sigma\, \psi^\mu(\sigma)\left[-i\frac{D}{Dx^\mu(\sigma)} + g_{\mu,\nu}\frac{\partial x^\nu}{\partial \sigma}\right].$$

2. Clifford algebras, spinors and Spin^c-groups in infinite dimensions

Given a real, separable Hilbert space (H,g) one associates to its complexification $H^{\mathbf{C}}$ the hermitian extension $\langle,\rangle$ and the complex bilinear extension $B = g^{\mathbf{C}}$ of g. Furthermore, one has the Clifford algebra $Cl(H,g)$ defined via $[\gamma(u),\gamma(v)]_+ = g(u,v)$, its complexification $\mathbf{C}l(H,g) = Cl(H,g)\otimes\mathbf{C}$, and, given a maximal B-isotropic subspace W of $H^{\mathbf{C}}$, a so-called CAR-algebra. The latter algebra $CAR(W)$ is defined via $[a^*(w_1),a(w_2)]_+ = \langle w_1,w_2\rangle$, and is isomorphic to $\mathbf{C}l(H,g)$ as a C*-algebra. The algebra $CAR(W)$ is naturally represented on the "spinor space" $S = S_W := \Lambda W$. Elements of the orthogonal group $O(H,g)$ act as automorphisms on $\mathbf{C}l(H,g)$ and are implemented on S if and only if they are in the "restricted orthogonal group" $O_{res}(H,g;W)$. This yields a central S^1-extension $O^{\sim}_{res}(H,g;W)$ which we also denote as $Pin^c(H,g;W)$ (see [SW2] for details).

3. The flat case

Using von Neumann's theory of "incomplete direct products", we construct in [SW1] the Dirac-Ramond operator on

$$L_0\mathbf{R}^d = \left\{\gamma\in L\mathbf{R}^d \,\middle|\, \int_0^1 \gamma(\sigma)d\sigma = 0\right\}$$

and show that its S^1-equivariant index exists and equals

$$\left(\prod_{n\geq 1}\frac{1}{1-q^n}\right)^d.$$

4. Spinors on loop spaces

Recall that a representation $\rho : Spin(2n,\mathbf{R}) \to O(2n,\mathbf{R})$ yields a homomorphism $M_\rho : LSpin(2n,\mathbf{R}) \to O_{res}(H,g;W)$, where $H = L^2(S^1,\mathbf{R}^n)$ and $H^{\mathbf{C}} \supset H_+ = \{L^2\text{−functions extending holomorphically to the unit disc}\}$, and thus there is an induced central S^1-extension $L^{\sim}Spin(2n,\mathbf{R})$ (see [PS]). Given now a principal bundle $Spin(2n,\mathbf{R}) \to P \to M$ and its loopification

$$(*)\ \ LSpin(2n,\mathbf{R}) \to LP \to LM\,,$$

we define a "string structure on P" to be a principal bundle $L^{\sim}Spin(2n,\mathbf{R}) \to L^{\sim}P \to LM$ covering $(*)$. The associated "spinor bundle" is then the associated vector bundle: $\mathcal{S}(LM) = L^{\sim}P \times_{L^{\sim}Spin(2n,\mathbf{R})} S$, where $S = S_{H_+}$ is the spinor space as above. We observe that the obstruction against its existence is exactly a Dixmier-Douady class $DD(P) \in H^3(LM, \mathbf{Z})$.

5. The tangent bundle of loops in a riemannian manifold

For (M, g) a riemannian manifold with Levi-Civita connection ∇, and for γ in LM, with $T_\gamma LM$ identified with $\Gamma_{C^\infty}(S^1, \gamma^*TM)$, there is a linear endomorphism $\nabla_{\dot\gamma}$, the covariant derivative along γ, acting on $T_\gamma LM$. (This operator plays a prominent role in the symplectic geometry of loop spaces, see e.g. [A] and [Wu].) Completing $T_\gamma LM$ with respect to the natural L^2-structure on it yields a Hilbert space $H(\gamma)$ and spectral subspaces $H_\pm(\gamma)$ of $\frac{1}{2\pi i}\nabla_{\dot\gamma}$. The collection $\{H_+(\gamma) \,|\, \gamma \in LM\}$ does not define a "polarisation of LM" (see [Se]) but "the jumps are at most compact operators", more precisely: using explicit trivializations of $(TLM)^{\mathbf{C}}$ over appropriate open subsets $\mathcal{V}_\theta$ of the loop space LM, we show that for γ_1, γ_2 in $\mathcal{V}_\theta$, the difference of the spectral projectors, $p_+(\gamma_1) - p_+(\gamma_2)$, is a Hilbert-Schmidt operator. (See [SW2] for the details.)

6. Twistors on loop spaces

Given a real Hilbert space H as above, one has $G_{res}(H^{\mathbf{C}}, H_+) = \{p \in B(H) \,|\, p^2 = p = p^*$ and p is Hilbert-Schmidt$\}$ and $I_{res}(H, H_+) = \{p \in G_{res} \,|\, Im(p)$ is maximally B-isotropic$\}$, the "restricted grassmannian" resp. the "restricted isotropic grassmannian" (or "restricted twistor space of (H, g)").

We associate to a d-dimensional riemannian manifold M with frame bundle $Q = O(M)$, the fiber bundles $G_{res}(LM) := LQ \times_{LO(d,\mathbf{R})} G_{res}(H^{\mathbf{C}}, H_+)$ and $I_{res}(LM) := LQ \times_{LO(d,\mathbf{R})} I_{res}(H, H_+)$. Furthermore, we have sets (!) with surjective projection onto LM:

$$\widehat{G}_{res}(LM) = \dot{\bigcup}_{\gamma \in LM} G_{res}(H^{\mathbf{C}}(\gamma), H_+(\gamma)) \text{ and}$$

$$\widehat{I}_{res}(LM) = \dot{\bigcup}_{\gamma \in LM} I_{res}(H(\gamma), H_+(\gamma))\,.$$

Using detailed analysis of $(TLM)^{\mathbf{C}}$, we show:

Theorem ([SW2]**).**

(i) For all riemannian manifolds M, $\widehat{G}_{res}(LM)$ is a smooth, locally trivial fiber bundle and is -as such- isomorphic to $G_{res}(LM)$.

(ii) For all kählerian manifolds M, $\widehat{I}_{res}(LM)$ is a smooth, locally trivial fiber bundle and is -as such- isomorphic to $I_{res}(LM)$.

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A Model for the String Group

André Henriques

The string group $String(n)$ is the 3-connected cover of $Spin(n)$. Given and compact simply connected group G, we will let $String_G$ be its 3-connected cover. The group $String_G$ is only defined up to homotopy, and various models have appeared in the literature. Stephan Stolz and Peter Teichner [7], [6] have a couple of models of $String_G$, one of which, inspired by Anthony Wassermann, is an extension of G by the group of projective unitary operators in a particular Von-Neuman algebra. Jean-Luc Brylinski [4] has a model which is a $U(1)$-gerbe with connection over the group G. More recently, John Baez et al [2] came up with a model of $String_G$ in their quest for a 2-Lie group integrating a given 2-Lie algebra. We show how to produce their model by applying a certain canonical procedure to their 2-Lie algebra.

A 2-Lie algebra is a two step L_∞-algebra. It consists of two vector spaces V_0 and V_1, and three brackets $[\,]$, $[\,,]$, $[\,,\,,]$ acting on $V := V_0 \oplus V_1$. They are of degree -1, 0, and 1 respectively and satisfy various axioms, see [1] for more details.

A 2-group is a group object in a 2-category [3]. It has a multiplication $\mu : G^2 \to G$, and an associator $\alpha : \mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$ satisfying the pentagon axiom. There are strict and weak versions. If the 2-category is that of C^∞ Artin stacks, we get the notion of a 2-Lie group. Since Artin stacks are represented by Lie groupoids, we can think of (strict) 2-Lie group as group objects in Lie groupoids. Equivalently, these are crossed modules in the category of smooth manifolds [3].

It is also good to consider weak 2-groups. The classifying space of a weak 2-group contains (up to homotopy) the same amount of information as the 2-group itself. So we will replace 2-Lie groups with their classifying space. This also allows for an easy way to talk about n-Lie groups. The following definition was inspired by discussions with Jacob Lurie:

Definition 1. *The classifying space of a weak n-Lie group is a simplicial manifold*

$$X_\bullet = \big(X_0 \leftleftarrows X_1 \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; X_2 \cdots \big)$$

satisfying $X_0 = pt$, and the following version of the Kan condition:
Let $\Lambda^{m,j} \subset \partial\Delta^m$ be the jth horn. Then the restriction map

$$X_m = Hom(\Delta^m, X_\bullet) \to Hom(\Lambda^{m,j}, X_\bullet) \tag{1}$$

is a surjective fibration for all $m \leq n$ and a diffeomorphism for all $m > n$.

Given an n-Lie algebra, there exists a canonical procedure that produces the classifying space of an n-Lie group. The main idea goes back to Sullivan's work on rationnal homotopy theory [8]. A variant is further studied in [5].

Definition 2. *Let V be an n-Lie algebra with Chevaley-Eilenberg complex $C^*(V)$. The classifying space of the corresponding n-Lie group is then given by*

$$(\int_n V)_m := Hom_{DGA}\big(C^*(V), \Omega^*(\Delta^m)\big)/\sim, \tag{2}$$

where $\sim$ identifies two m-simplicies if they are simplicially homotopic relative to their $(n-1)$-skeleton.

Example 1. Let $\mathfrak{g}$ be a Lie algebra with corresponding Lie group G. A homomorphism from $C^*(\mathfrak{g})$ to $\Omega^*(\Delta^n)$ is the same thing as a flat connection on the trivial G-bundle $G\times\Delta^n$. These in turn correspond to maps $\Delta^n \to G$ modulo translation. Two n-simplicies are simplicially homotopic relatively to their 0-skeleton if their vertices agree. So we get

$$\left(\int_1 \mathfrak{g}\right)_n = Map\big(sk_0(\Delta^n), G\big)/G = G^n.$$

Therefore $\int_1 \mathfrak{g}$ is the standard simplicial model for BG. We can recover G along with its group structure by taking the simplicial π_1 of this simplicial manifold.

Now let us consider our motivating example. Let $\mathfrak{g}$ be a simple Lie algebra of compact type (defined over $\mathbb{R}$), and let $\langle,\rangle$ be the inner product on $\mathfrak{g}$ such that the norm of the short coroots is 1.

Definition 3. [2] *Let $\mathfrak{g}$ be a simple Lie algebra of compact type. Its string Lie algebra is the 2-Lie algebra $\mathfrak{str} = \mathfrak{str}(\mathfrak{g})$ given by*

$$\mathfrak{str}_0 = \mathfrak{g}, \qquad \mathfrak{str}_1 = \mathbb{R}$$

and brackets

$$[\,] = 0, \quad [(X_1,c_1),(X_2,c_2)] = ([X_1,X_2],0),$$
$$[(X_1,c_1),(X_2,c_2),(X_3,c_3)] = (0,\langle[X_1,X_2],X_3\rangle).$$

The string Lie algebra should be thought as a central extension of the Lie algebra $\mathfrak{g}$, but which is controlled by $H^3(\mathfrak{g},R)$ as opposed to $H^2(\mathfrak{g},\mathbb{R})$. The Chevalley-Eilenberg complex of $\mathfrak{str}$ is then given by

$$C^*(\mathfrak{str}) = \mathbb{R}\oplus\Big[\mathfrak{g}^*\Big]\oplus\Big[\Lambda^2\mathfrak{g}^*\oplus\mathbb{R}\Big]\oplus\Big[\Lambda^3\mathfrak{g}^*\oplus\mathfrak{g}^*\Big]\oplus\Big[\Lambda^4\mathfrak{g}^*\oplus\Lambda^2\mathfrak{g}^*\oplus\mathbb{R}\Big]\oplus\dots$$

Following (2), we study

$$\begin{aligned} Hom_{DGA}\big(C^*(\mathfrak{str}),\Omega^*(\Delta^n)\big) = \big\{&\alpha\in\Omega^1(\Delta^n;\mathfrak{g}), \beta\in\Omega^2(\Delta^n;\mathbb{R})\,\big| \\ & d\alpha+\tfrac{1}{2}[\alpha,\alpha]=0, d\beta+\tfrac{1}{6}[\alpha,\alpha,\alpha]=0\big\}. \end{aligned}\tag{3}$$

The 1-form α satisfies the Maurer Cartan equation, so we can integrate it to a map $f:\Delta^n\to G$, defined up to translation. This map satisfies $f^*(\theta_L)=\alpha$, where

$\theta_L \in \Omega^1(G;\mathfrak{g})$ is the left invariant Maurer Cartan form on G. The 3-form $\frac{1}{6}[\alpha,\alpha,\alpha]$ is then the pullback of the Cartan 3-form

$$\eta = \tfrac{1}{6}\langle[\theta_L,\theta_L],\theta_L\rangle \in \Omega^3(G;\mathbb{R}),$$

which represents the generator of $H^3(G,\mathbb{Z})$. So we can rewrite (3) as

$$\big\{f : \Delta^n \to G, \beta \in \Omega^2(\Delta^n)\,\big|\, d\beta = f^*(\eta)\big\}/G. \tag{4}$$

The set of n-simplices in $\int_2 \mathfrak{str}$ is then the quotient of (4) by the relation of simplicial homotopy relative to the 1-skeleton. Applying this procedure, we get a simplicial manifold whose geometric realization has the homotopy type of $BString_G$ and which is equal to the nerve of the 2-group described in [2]. It is given by

$$\int_2 \mathfrak{str} = \Big[* \leftleftarrows Path(G)/G \;\overset{\leftarrow}{\leftleftarrows}\; \widetilde{Map(\partial\Delta^2}, G)/G \;\overset{\leftleftarrows}{\leftleftarrows}\; Map(\widetilde{sk_1\Delta^3}, G)/G \cdots\Big],$$

where the tilde indicates that the group $Map(sk_1\Delta^i, G)$ has been centrally extended by $S^1 \otimes H_1(sk_1\Delta^i)$. Moreover, its simplicial homotopy groups are given by $\pi_1(\int_2 \mathfrak{str}) = G$ and $\pi_2(\int_2 \mathfrak{str}) = S^1$.

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The Superpolynomial for Knot Homologies

Sergei Gukov

Recently, a number of different knot homology theories have been discovered. Although the details of these theories differ, the basic idea is that for a knot K, one can construct a doubly graded homology theory $H_{i,j}(K)$ whose graded Euler characteristic with respect to one of the gradings gives a particular knot polynomial. Such a theory is referred to as a *categorification* of the knot polynomial. For example, the Jones polynomial J is the graded Euler characteristic of the doubly graded *Khovanov Homology* $H^{Kh}_{i,j}(K)$; that is,

$$J(q) = \sum_{i,j} (-1)^j q^i \dim H^{Kh}_{i,j}(K). \tag{1}$$

Khovanov originally constructed $H^{Kh}_{i,j}$ combinatorially in terms of skein theory [5], but it is conjectured to be essentially the same as Seidel and Smith's *symplectic Khovanov homology* which is defined by considering the Floer homology of a certain pair of Lagrangians [14].

Khovanov's theory was generalized by Khovanov and Rozansky [8] to categorify the quantum $sl(N)$ polynomial invariant $\bar{P}_N(q)$. Their homology $\overline{HKR}^N_{i,j}(K)$ satisfies

$$\bar{P}_N(q) = \sum_{i,j} (-1)^j q^i \dim \overline{HKR}^N_{i,j}(K). \tag{2}$$

For $N = 2$, this theory is expected to be equivalent to the original Khovanov homology. There are also important deformations of the original Khovanov homology [10, 1, 7], as well as of the $sl(N)$ Khovanov-Rozansky homology [3]. In a sense, the deformed theory of Lee [10] also can be regarded as a categorification of the $sl(1)$ polynomial invariant.

Another knot homology theory is knot Floer homology, $\widehat{HFK}_j(K;i)$, introduced in [11, 12]. It provides a categorification of the Alexander polynomial:

$$\Delta(q) = \sum_{i,j} (-1)^j q^i \dim \widehat{HFK}_j(K;i). \tag{3}$$

Unlike Khovanov-Rozansky homology, knot Floer homology is not known to admit a combinatorial definition; in the end, computing $\widehat{HFK}$ involves counting pseudo-holomorphic curves. The polynomials above are closely related as they can all be derived from a single invariant, namely the HOMFLY polynomial $\bar{P}(K)(a = q^N, q)$. While the above homology theories categorify polynomial knot invariants in the same class, their constructions are very different!

In this talk, based on the joint work with Nathan Dunfield and Jacob Rasmussen [2], we propose a framework for unifying the $sl(N)$ Khovanov-Rozansky homology (for all N) with the knot Floer homology. We argue that this unification should be accomplished by a triply graded homology theory which categorifies the HOMFLY polynomial. Moreover, this theory should have an additional formal structure of a family of differentials. Roughly speaking, the triply graded theory by itself captures the large N behavior of the $sl(N)$ homology, and differentials capture non-stable behavior for small N, including knot Floer homology. The differentials themselves should come from another variant of $sl(N)$ homology, namely the deformations of it studied by Gornik [3], building on work of Lee [10].

There are several reasons to hope for this unified theory. Thus, in the joint work with Albert Schwarz and Cumrun Vafa [4], we presented a physical interpretation of the Khovanov-Rozansky homology which naturally led to the unification of the $sl(N)$ homologies, when N is sufficiently large:

Conjecture 1. *There exists a finite polynomial* $\bar{\mathbf{C}}\mathrm{P}(K) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$ *(the "superpolynomial") such that*

$$\overline{KhR}_N(q,t) = \frac{1}{q - q^{-1}} \bar{\mathbf{C}}\mathrm{P}(a = q^N, q, t) \tag{4}$$

for all sufficiently large N.

Here, $\overline{KhR}_N(q,t) := \sum_{i,j} q^i t^j \dim \overline{HKR}^N_{i,j}(K)$ is the graded Poincaré polynomial which encodes the dimensions of the Khovanov-Rozansky homology groups. This conjecture essentially says that, for sufficiently large N, the dimension of the $sl(N)$ knot homology grows linearly in N, and the precise form of this growth can be encoded in a finite set of the integer coefficients. Therefore, if one knows the $sl(N)$ knot homology for two different values of N, both of which are in the "stable range" $N \geq N_0$, one can use (4) to determine the $sl(N)$ knot homology for all other values of $N \geq N_0$. In some examples, it seems that (4) holds true for all values of N, not just large N. In [4], this was used to compute $\bar{\mathbf{C}}\mathrm{P}(K)$ for certain knots. However, this is not always true. The simplest knot for which (4) holds for all $N \geq 3$ but not for $N = 2$ is the 8-crossing knot 8_{19}.

Another motivation for the unified triply graded theory is that, at the small N end, the $sl(2)$ Khovanov homology and $\widehat{HFK}$ seem to be very closely related. For instance, their total ranks are very often (but not always) equal [13]. One hope for our unified theory is that it will explain the mysterious fact that while the connections between $\overline{HKR}_2$ and HFK hold very frequently, they are not universal.

In order to bring knot Floer homology into the picture, we need to consider the *reduced* HOMFLY polynomial $P(K)(a,q)$ of the knot K, determined by the convention that $P(\text{unknot}) = 1$. There is a categorification of $P(K)(a = q^N, q)$ called the *reduced* Khovanov-Rozansky Homology (see [6, §3] and [8, §7]). We use $KhR_N(K)(q,t)$ to denote the Poincaré polynomial of this theory. For this reduced theory, there is also a version of the Conjecture 1. Essentially, it says that, for sufficiently large N, the total dimension of the reduced $sl(N)$ knot homology is independent of N, and the graded dimensions of the homology groups change linearly with N:

Conjecture 2. *There exists a finite polynomial* $\mathbf{C}\mathrm{P}(K) \in \mathbb{Z}_{\geq 0}[a^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$ *such that*

$$KhR_N(q,t) = \mathbf{C}\mathrm{P}(a = q^N, q, t) \tag{5}$$

for all sufficiently large N.

In contrast with the previous case, in the reduced case the superpolynomial is required to have non-negative coefficients. This is forced merely by the form of (5), since for large N distinct terms in $\mathbf{C}\mathrm{P}(a,q,t)$ can't coalesce when we specialize to $a = q^N$. Moreover, one has

$$P(K)(a,q) = \mathbf{C}\mathrm{P}(a, q, t = -1). \tag{6}$$

Thus we will view $\mathbf{C}\mathrm{P}(a,q,t)$ as the Poincaré polynomial of some new triply graded homology theory $\mathcal{H}_{i,j,k}(K)$ categorifying the reduced HOMFLY polynomial.

As with unreduced theory, for some simple cases (5) holds for all $N \geq 2$. However, in general there will be exceptional values of N for which this is not the case. To account for this, we introduce an additional structure on $\mathcal{H}_*(K)$, a family of differentials $\{d_N\}$ for $N > 0$, such that the $sl(N)$ homology is the homology of $\mathcal{H}_*(K)$ with respect to the differential d_N.

$$HFK(q,t) = \mathbf{C}\mathrm{P}(a = t^{-1}, q, t). \tag{7}$$

where we introduced the the Poincaré polynomial $HFK(q,t) := \sum_{i,j} q^i t^j \dim \widehat{HFK}_j(K;i)$.

While we do not give a mathematical definition of the triply graded theory (natural candidates for our proposed theory include a triply graded theory recently introduced by Khovanov and Rozansky [9]) the rich formal structure we propose is powerful enough to make many non-trivial predictions about the existing knot homologies that can then be checked directly. We include many examples where we can exhibit a likely candidate for the triply graded theory, and these demonstrate the internal consistency of our axioms.

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The Baum-Connes Conjecture for Loop Groups

Nitu Kitchloo

This is very much an incomplete account of a possible extension of the Baum-Connes conjecture to Loop groups. The Baum-Connes conjecture is a conjecture for the class of locally compact topological groups. It relates the equivariant K -homology of a certain space related to the group, to the topological K -theory of a C^* -algebra. We now define these objects in more detail:

Definition 0.1. *For a topological group H, the classifying space for proper actions, $\underline{EH}$ is a H-CW complex with the property that all the isotropy subgroups are compact, and give a compact subgroup $K \subseteq H$, the fixed point space $\underline{EH}^K$ is contractible.*

Remark 0.2. *If H is a locally compact group, then one may define the equivariant K-homology $K_H(\underline{EH})$ in a suitable way. This construction has been extended to all topological groups G by Freed-Hopkins-Teleman,* [3].

Remark 0.3. *For a locally compact group, one may also define the reduced C^* -algebra of the group H, $C_r(H)$, given by the norm-closure of H in the space of bounded operators on $L^2(H)$. Let $K(C_r(H))$ denote the topological K-theory of this algebra.*

We have

Conjecture 3. *(Baum-Connes* [1]*) Given a locally compact group H, there is a natural map, μ called the assembly map, which yields an isomorphism*

$$\mu : K_H(\underline{EH}) \longrightarrow K(C_r(H))$$

We would like to extend this framework to the case of a loop group. Let H denote the loop group $H = \mathbb{L}G$, where G is a simply connected compact Lie group, and $\mathbb{L}G$ denotes the universal central extension of the group LG of pointed loops in K. Note that H is not locally compact, however we may define $K_H(\underline{EH})$ as per the work of Freed-Hopkins-Teleman. For this, one would like to understand the space $\underline{EH}$. Consider the following theorem:

Theorem 0.4. *The* **topological affine Tits building**

$$\mathbf{A}(LG) := \mathrm{hocolim}_I \; LG/H_I$$

*of LG is $\mathbb{T}\tilde{\times}LG$-***equivariantly** *contractible. In other words, given any compact subgroup $K \subset \mathbb{T}\tilde{\times}LG$, the fixed point space $\mathbf{A}(LG)^K$ is contractible.*

[Here I runs over certain proper subsets of roots of G, and the H_I are certain **compact** 'parabolic' subgroups of LG.]

Remark 0.5. The natural action of the rotation group $\mathbb{T}$ on LG lifts to $\mathbb{L}G$, and the $\mathbb{T}$-action preserves the subgroups H_I. Hence $\mathbf{A}(LG)$ admits an action of $\mathbb{T}\tilde{\times}\mathbb{L}G$, with the center acting trivially. We can therefore express $\mathbf{A}(LG)$ as

$$\mathbf{A}(LG) = \mathrm{hocolim}_I \; \mathbb{L}G/\mathbb{H}_I$$

where $\mathbb{H}_I$ is the induced central extension of H_I.

Other descriptions of $\mathbf{A}(LG)$

This Tits building has other descriptions as well. For example:

1. $\mathbf{A}(LG)$ can be seen as the classifying space for proper actions with respect to the class of compact Lie subgroups of $\mathbb{T}\tilde{\times}\mathbb{L}G$.

2. It also admits a more differential-geometric description as the smooth infinite dimensional manifold of holonomies on $S^1 \times G$: Let $\mathcal{S}$ denote the subset of the space of smooth maps from $\mathbb{R}$ to G given by

$$\mathcal{S} = \{g(t) : \mathbb{R} \to G,\ g(0) = 1,\ g(t+1) = g(t) \cdot g(1)\} \ ;$$

then $\mathcal{S}$ is homeomorphic to $\mathbf{A}(LG)$. The action of $h(t) \in LG$ on $g(t)$ is given by $hg(t) = h(t) \cdot g(t) \cdot h(0)^{-1}$, where we identify the circle with $\mathbb{R}/\mathbb{Z}$. The action of $x \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is given by $xg(t) = g(t+x) \cdot g(x)^{-1}$.

3. The description given above shows that $\mathbf{A}(LG)$ is equivalent to the affine space $\mathcal{A}(S^1 \times G)$ of connections on the trivial G-bundle $S^1 \times G$. This identification associates to the function $f(t) \in \mathcal{S}$, the connection $f'(t)f(t)^{-1}$. Conversely, the connection ∇_t on $S^1 \times G$ defines the function $f(t)$ given by transporting the element $(0,1) \in \mathbb{R} \times G$ to the point $(t, f(t)) \in \mathbb{R} \times G$ using the connection ∇_t pulled back to the trivial bundle $\mathbb{R} \times G$.

Remark 0.6. These equivalent descriptions have various useful consequences. For example, the model given by the space $\mathcal{S}$ of holonomies says that given a finite cyclic group $H \subset \mathbb{T}$, the fixed point space $\mathcal{S}^H$ is homeomorphic to $\mathcal{S}$. Moreover, this is a homeomorphism of LG-spaces, where we consider $\mathcal{S}^H$ as an LG-space and identify LG with LG^H in the obvious way. Notice also that $\mathcal{S}^{\mathbb{T}}$ is G-homeomorphic to the model of the adjoint representation of G defined by $\mathrm{Hom}(\mathbb{R}, G)$.

Similarly, the map $\mathcal{S} \to G$ given by evaluation at $t = 1$ is a principal ΩG bundle, and the action of $G = LG/\Omega G$ on the base G is given by conjugation. This allows us to relate our work to that of Freed, Hopkins and Teleman in the following section.

Finally, the description of $\mathbf{A}(LG)$ as the affine space $\mathcal{A}(S^1 \times G)$ implies that the fixed point space $\mathbf{A}(LG)^K$ is contractible for any compact subgroup $K \subseteq \mathbb{T}\tilde{\times}\mathbb{L}G$.

It follows from the above remark that

Theorem 0.7. *The space* $\mathbf{A}(LG)$ *is a model for* $\underline{EH}$, *where* $H = \mathbb{L}G$.

The work of Freed-Hopkins-Teleman therefore shows that the equivariant K-theory of $\underline{EH}$ is isomorphic to the Verlinde algebra! Notice at this point, what we cannot define $C_r(H)$, since H is not locally compact. However, we have the following substitute:

Definition 0.8. *Let* $N \subset H$ *denote the normalizer of the maximal torus in* H. *The group* N *is a discrete extension of a (finite rank) torus, and hence is locally compact. We may therefore define the* C^**-algebra* $C_r(N)$.

Consider the following diagram:

$$\begin{array}{ccc} K_N(\underline{EN}) & \xrightarrow{\mu_N} & K(C_r(N)) \\ \downarrow & & \downarrow \\ K_H(\underline{EH}) & \xrightarrow{\mu_H} & K(C_r(H))?? \end{array}$$

where the object depicted on the bottom right corner denotes a hitherto nonexistant group. It is know that the map μ_N above is an isomorphism, and the left vertical map is a split surjection. Therefore, the above diagram shows that the Verlinde algebra is a split summand inside the group $K(C_r(N))!$.

The above diagram indicates that there should be a natural choice of a C^*-algebra $C_r(H)$ that encodes the positive energy representations, whose K-theory would fit in the bottom right hand corner of the above diagram, along with an isomorphism μ_H, making it the diagram commute. The construction of this object is work in progress.

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Geometric Constructions of Smooth Extensions of Cohomology Theories

Ulrich Bunke

(joint work with Thomas Schick)

The potential of the electromagnetic field is a one-form on space-time X. In the case of a non-trivial topological background a potential may only be locally defined and its correct interpretation would be as a connection ω of a $U(1)$-principal bundle $E \to X$. The topological datum of a $U(1)$-principal bundle can be identified with its first Chern class $c_1(E) \in H^2(X, \mathbb{Z})$. The connection ω then appears as a smooth refinement $\hat{c}_1(E, \omega) \in \hat{H}^2(X, \mathbb{Z})$.

The smooth extension of integral cohomology $\hat{H}^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ was introduced by Cheeger-Simons [1] (under the name differential characters and using a shifted indexing). They also defined the refined Chern classes $\hat{c}_i(E, \omega)$ for principal $U(n)$-bundles with connections.

In the talk by D. Freed on this conference we have seen that string theory and its descendents the potentials of fields should have a natural interpretation as classes in smooth extensions of other generalized cohomology theories like K-theory or elliptic cohomology.

Let h be a multiplicative generalized cohomology theory, R be a $\mathbb{Z}$-graded algebra over $\mathbb{R}$, and let H_R denote the ordinary cohomology theory with coefficients in R. The initial datum for a smooth extension is a natural transformation $r : h \to H_R$ of multiplicative cohomology theories.

Examples are

(1) $h := H_{\mathbb{Z}}$, $R := \mathbb{R}$ and $r : H_{\mathbb{Z}} \to H_{\mathbb{R}}$ the map induced by the inclusion $\mathbb{Z} \to \mathbb{R}$.
(2) $h := K$, complex K-theory, $R := \mathbb{R}[z]$, $\deg(z) := -2$, and $r := \mathbf{ch} : K \to H_R$, the Chern character.
(3) $h := MSO$, oriented bordism, $\phi : MSO \to H_R$ a transformation associated to a formal power $\phi(x)$ series with coefficents in R of total degree 0, where $\deg(x) = -2$.

In the talk we presented a set of axioms that any smooth extension $\hat{h} \to h$ of $r : h \to H_R$ should satisfy. Roughly speaking the relation of $\hat{h}$ with h and R-valued differential forms is the same as the relation of $\hat{H}_{\mathbb{Z}}$ with $H_{\mathbb{Z}}$ and real valued forms known from the work of Cheeger-Simons. For details we refer to the forthcoming paper [3].

It is now a natural question whether a smooth extension of $r : h \to H_R$ exists, and under which conditions such an extension is unique. The first question was answered in the positive by the homotopy theoretic construction of smooth extensions by Hopkins-Singer [2].

The status of the uniqueness problem is unclear at the moment. On the other hand it is of particular interest in cases where we have different constructions of smooth extensions say by geometric or analytic means. On the one hand it is clear from the axioms by a five Lemma argument that a natural transformation between two extensions of the same initial datum is an isomorphism of groups automatically. But even in the case if integral cohomology theory it is not clear that it must be compatible with the multiplicative structure.

On the other hand, it is not obvious in general how to construct natural transformations between the different models.

A smooth extension $\hat{h}$ of the cohomology theory h can be considered as a sort of categorification of h. For example, it is not possible to talk about connections on classes in $H^2(X, \mathbb{Z})$, but classes in $\hat{H}^2(X, \mathbb{Z})$ contain such geometric information.

As further explained in the talk by D. Freed the categorification $\hat{h}$ of h is usually to crude for applications. This motivates to construct nice geometric or analytic models for the smooth extension $\hat{h}$ which can be the basis for a finer categorification. In the case integral cohomology this amounts to the observation that classes of $\hat{H}^2(X, \mathbb{Z})$ are isomorphism classes of $U(1)$-bundles with connection. In the applications one would like to talk about sections, so the finer categorification would amount to consider the $U(1)$-bundles with connections themselves.

In the talk we explained a geometric/analytic construction of the smooth extension $\hat{K} \to K$ associated to $\mathbf{ch} : K \to H_R$, and geometric constructions of smooth extensions of bordism theories. These constructions will appear in [3] and [4].

It is a natural question whether cohomology operations can be lifted to the smooth extensions. In [4] we will show that the Adams operations on K-theory can be lifted to $\hat{K}$. Another example discussed in [4] is the lift of the Chern character $\hat{\mathbf{ch}} : \hat{K} \to \hat{H}_{\mathbb{Q}[z]}$. We have a lift of the Grothendick-Riemann-Roch theorem

$$\begin{array}{ccc} \hat{K}(X) & \stackrel{\hat{\mathbf{ch}}}{\to} & \hat{H}_{\mathbb{Q}[z]}(X) \\ \hat{\pi}_! \downarrow & & \downarrow \int \hat{\hat{A}} \cup \dots \\ \hat{K}(Y) & \stackrel{\hat{\mathbf{ch}}}{\to} & \hat{H}_{Q[z]}(Y) \end{array} .$$

Here $\hat{\pi} : X \to Y$ is a proper submersion which is oriented for $\hat{K}$-theory (this a refinement of a K-orientation to the smoothly extended world which discussed at length in [3]), and $\hat{\hat{A}} \in H^0_{\mathbb{Q}[z]}(Y)$ is a lift of the $\hat{\mathbf{A}}$-class of the vertical bundle which depends on the data of $\hat{\pi}$.

References

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Progress on the Volume Conjecture

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The *Volume Conjecture* of a knot relates quantum topology (namely the Jones polynomial) to hyperbolic geometry (namely the volume function of representations of the knot complement in $SL(2, C)$). More precisely, given a knot K in the 3-sphere, one considers the n-th colored Jones polynomial $J_n(q) \in Z[q, q^{-1}]$. The volume conjecture states that if the knot is hyperbolic, then

$$\lim_n \frac{1}{n} \log |J_{K,n}(e^{2\pi i/n})| = \frac{1}{2\pi} \mathrm{vol}(\rho_{2\pi i})$$

where

$$\rho_{2\pi i} : \pi_1(S^3 - K) \to SL(2, C)$$

is the discrete faithful representation.

There are two statements in the volume conjecture:

- To prove that the limit of the left hand side exists.
- To identify the limit of the left hand side with a geometric quantity of the right hand side.

The volume conjecture asks for a natural generalization. Namely, let us denote for a complex number a,

$$f(n, a) = \frac{1}{n} \log |J_{K,n}(e^{2\pi i a/n})|.$$

Then, we may ask for the existence and identification of the limit

$$\lim_n f(n, a)$$

with a geometric quantity when a is a fixed complex number. The situation is not as simple as one might think. For every knot K, the character variety of $SL(2, C)$ representations of its knot complement is an affine algebraic set, with a real valued volume function. The character variety is not irreducible, and we will formulate the *Generalized Volume Conjecture* for a near 0 and a near $2\pi i$.

When a is near zero, the Generalized Volume Conjecture states that

$$\lim_n f(n, a) = c_a \mathrm{vol}(\rho_a)$$

where

$$\rho_a : \pi_1(S^3 - K) \to SL(2, C)$$

is the unique abelian representation near $\rho_0 = I$ which satisfies

$$\rho_a(\text{meridian}) = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}.$$

Now, $\mathrm{vol}(\rho_a) = 0$ for the above representations.

When a is near $2\pi i$ and $a/(2\pi i)$ is irrational or 1, then the volume conjecture states that

$$\lim_n f(n, a) = c_a \mathrm{vol}(\rho_a)$$

where

$$\rho_a : \pi_1(S^3 - K) \to SL(2, C)$$

is the unique abelian representation near $\rho_{2\pi i}$ which satisfies

$$\rho_a(\text{meridian}) = \rho_a(\text{meridian}) = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}.$$

On the other hand, when $a/(2\pi i)$ is a rational number near 1 (but not 1), then

$$\begin{aligned} \liminf_n f(n, a) &= 0 \\ \limsup_n f(n, a) &= c_a \mathrm{vol}(\rho_a) \end{aligned}$$

where ρ_a is as above.

Now, our results are the following:

Theorem 1. For every knot K there exists a neighborhood U_K of $0 \in C$ such that the following limit exists when $a \in U_K$:

$$\lim_n J_{K,n}(e^{2\pi i a/n}) = \frac{1}{\Delta(e^a)}$$

where Δ is the Alexander polynomial. Moreover, convergence is uniform on compact sets.

This implies the Volume Conjecture for small complex angles.

Actually, there is an improved theorem which proves the volume conjecture *to all orders* for small complex angles. In order to formulate it, one needs to use the existence of a sequence

$$R_n(q) = P_n(q)/\Delta^{2n+1}(q) \in Q(q)$$

of rational functions where $P_0(q) = 1$. The nth such function is the n-th loop term in the *loop expansion* of the colored Jones function.

Theorem 2. For every knot K there exists a neighborhood U_K of $0 \in C$ such that the following asymptotic expansion exists when $a \in U_K$:

$$\lim_n J_{K,n}(e^{2\pi i a/n}) \sim \sum_{k=0}^{\infty} R_k(e^a) \left(\frac{a}{n}\right)^k$$

Moreover, convergence is uniform on compact sets. In other words, for every $N \geq 0$ the following limit exists, uniformly on compact sets:

$$\lim_n \left(J_{K,n}(e^{2\pi i a/n}) - \sum_{k=0}^{N} R_k(e^a) \left(\frac{a}{n}\right)^k \right) \left(\frac{n}{a}\right)^N = 0.$$

The next theorem is the following:

Theorem 3. For every knot K and every $a \in C$,

$$\limsup_n f(n,a) < \infty.$$

When $a = 2\pi i$, we can give a better bound. Namely,

Theorem 4. For every knot K with $c+2$ crossings,

$$\limsup_n f(n, 2\pi i) < \frac{v_8}{2\pi} c$$

where $v_8 = 3.669...$ is the volume of the regular ideal octahedron. Combined with the volume conjecture, this leads to an estimate

$$\mathrm{vol}(\rho_{2\pi i}) \leq c v_8$$

which is true and asymptotically optimal. The asymptotic optimality uses work of Agol-Storm-W.Thurston which in turn uses work of Pelerman.

Two more results when $a/(2\pi i)$ is rational near 1 but not equal to 1.

Theorem 4. For every knot K there exist a neighborhood V_K of $1 \in C$ such that when $a/(2\pi i) \in V_K$ is rational and not equal to 1, then

$$\liminf_n f(n,a) \leq 0.$$

Our last result states that the volume conjecture can only be barely true.

Theorem 5. For every knot K and every fixed $m \neq 0$

$$\lim_n \frac{1}{n} \log |J_{K,n+m}(e^{2\pi i/n})| = 0.$$

Our proofs will use everything that we currently know about the colored Jones function. Namely, its Vassiliev invariant expansion, its the loop expansion, its cyclotomic expansion, its state-sum formulas, q-holonomicity and a tiny dose of hyperbolic geometry. Are you interested?

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