Outline for May 8:

1) Recall handle decomposition for compact, connected mfds with $\partial_- M = \emptyset$
2) Apply to class. of d-mfds for $d = 1, 2, 3$
3) Explain dotted unlink as 1-handles $\subseteq M^4$
4) Read off 4-manifold invariants from the dotted framed link presentation of M^4:
   - $\pi_1 M$, $H_* M$, $H_* \tilde{M}$
   - intersection form on H_2 via linking numbers (first invariant!)
   - equivariant intersection form on π_2.

May 8: The classification of c.c.o. d-manifolds for $d = 1, 2, 3$: $\partial M =: \partial_- M$

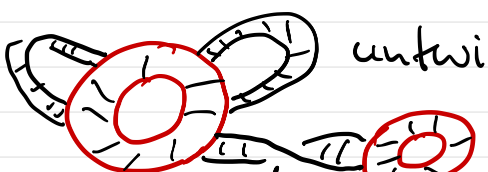
$d = 1$: M^1 compact connected $\overset{(2)}{\Longrightarrow}$

(i) $\partial M \neq \emptyset$, only need 1-handles $\cap$, and M connected implies $M \cong [0,1]$

(ii) $\partial M = \emptyset$, have one 0-handle $\cup$ and one 1-handle $\cap$, i.e. $M \cong S^1$ ∎

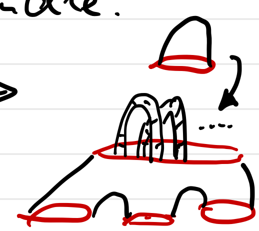
$d = 2$: $\partial M = \coprod^n S^1$ and as above we have zero $(n>0)$ or one $(n=0)$ 0-handle.

twisted 1-handle



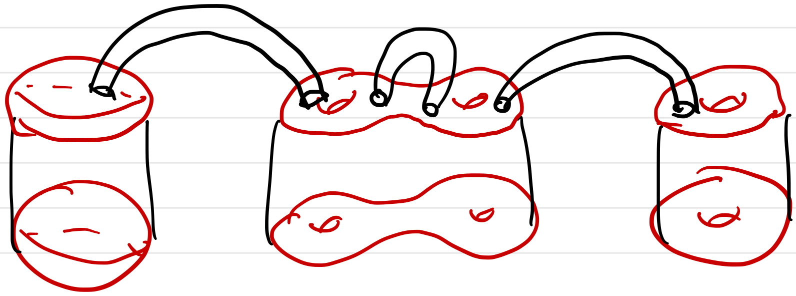
undefined sign

untwisted. M c.c.o. $\Rightarrow$ $M \cong \Sigma_g \setminus n$ disks $\cong$



$d=3$: $\partial M = \partial_- M = \coprod_{i=1}^{n} \Sigma_{g_i}$ and zero one 0-handle

one 3-handle



$= M^{(1)}$, $\partial_+ M^{(1)} = \Sigma_g$

where $g \geq \sum_{i=1}^{n} g_i$

2-handle attachments $\varphi_i : S^1 \times D^1 \hookrightarrow \partial_+ M^{(1)}$ are given

by $\underset{\displaystyle g}{\underset{\shortparallel}{|I_2|}}$ disjointly embedded simple closed curves in Σ_g

Σ_g

Surgery on $\varphi_1, \dots, \varphi_g$ gives $S^2 = \partial_-(3\text{-handle})$ $\blacksquare$

Starting with $\partial_- M^3 = \emptyset$, we get the following presentation in the oriented case:

$h^0 \cup n\, h^1 \cong$ 

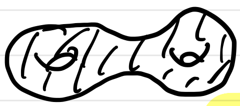
Then 2-handles are attached along a link in $\partial_+ M^{(1)} = \Sigma_n$. New boundary $\partial_+ M^{(2)}$ = surgery on $\alpha_1, \dots, \alpha_m$ is ∂M (if $\neq \emptyset$).
If $\partial M = \emptyset$ then $m = n$ and $\partial_+ M^{(2)} \cong S^2$ which is (uniquely) filled by a 3-handle.

Corollary: $\pi_1 M^3 \cong (x_1, \dots, x_n \mid \alpha_1, \dots, \alpha_m)$ and $m-n$ can be computed from ∂M, e.g. $m = n$ for $\partial M = \emptyset$.

Special case: **Heegaard decomposition** of a c.c.o. 3-manifold $M =$

$$(0\text{-handle} \cup g \text{ 1-handles}) \cup_{\Sigma_g} (g \text{ 2-handles} \cup \text{3-handle})$$

$\qquad\quad \| \qquad\qquad\qquad\qquad\qquad\qquad \|$

 glued along a **diffeomorphism of Σ_g** 

Gluing is determined by $\varphi_1^c, \dots, \varphi_g^c : S^1 \hookrightarrow \Sigma_g$ ■

Remark: We could have assumed $\partial M = \emptyset$ and hence zero one 3-handle and a single 0-handle. Then 1-handles give $M^{(1)} = \natural^g S^1 \times D^2$ and we have $|I_2| \leq g$ with $\varphi_i : S^1 \times D^1 \hookrightarrow \partial M^{(1)} \cong \Sigma_g$ ■