May 3 : Introduction to knots & links

or

How too see low-dimensional manifolds

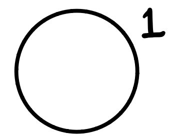
Goal for today :

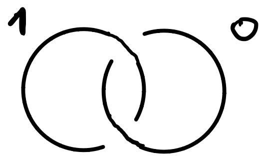
Discuss how a framed, dotted link L in $\mathbb{R}^3$

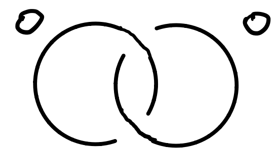
- defines a (compact, connected, oriented) 4-mfd. M_L^4
- also defines the (closed, -"-, -"-) 3-mfd ∂M_L

and which operations on L preserve the diffeomorphism class of M_L .

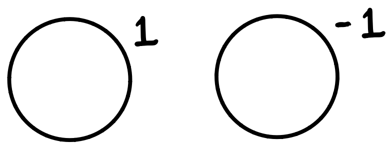
Note : ∂M_L is the result of surgery on $L \subseteq S^3$ and the above ∂-operations are called Kirby moves.

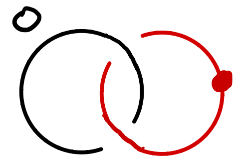
Examples: $S^4 \hat{=} \emptyset$, $\mathbb{CP}^2 \hat{=}$ 

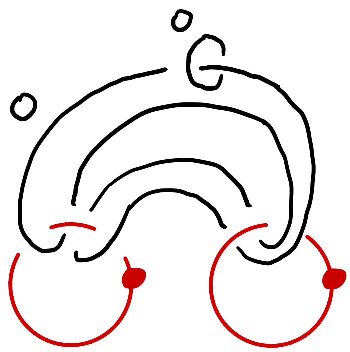
$S^2 \tilde{\times} S^2 \hat{=}$ 

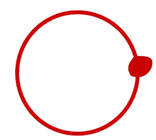
$S^2 \times S^2 \hat{=}$ 

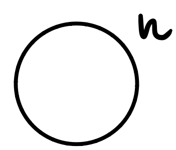
$\cong$

$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \hat{=}$ 

$S^4 \hat{=}$ 

$S^1 \times S^1 \times S^2 \hat{=}$ 

$S^1 \times S^3 \hat{=}$ 

$L^3_{n,1} \hat{=} \partial$ 

Def.: • Knot, link, unlink, isotopy
- connected sum, long knots, prime decomposition
- Link complements, surgery on framed links
- Link group, homology, $\mathbb{Z}^m$-cover
- Projections, Reidemeister moves & singularities
- Wirtinger presentation, word problem

Morse theory: $M \xrightarrow{f} \mathbb{R}$ generic $\Leftrightarrow$ f is a Morse fct. ($\Rightarrow$ crit. pts discrete) with distinct crit. values

Key point: f + gradient-like vector field for f determines a handle decomposition of M:

Def.: A **handle decomposition** of $(M^d, \partial_- M, \partial_+ M)$ is

$$\partial_- M \times D^1 =: M^{(-1)} \subseteq M^{(0)} \subseteq M^{(1)} \subseteq \dots \subseteq M^{(d)} = M \quad \text{with}$$

$$M^{(k)} \simeq M^{(k-1)} \cup I_k \text{ handles of index } k$$

$$= M^{(k-1)} \cup_{\coprod_{i \in I_k} S^{k-1} \times D^{d-k} \xrightarrow{\varphi_i} \partial_+ M^{(k-1)}} \coprod_{I_k} D^k \times D^{n-k}$$

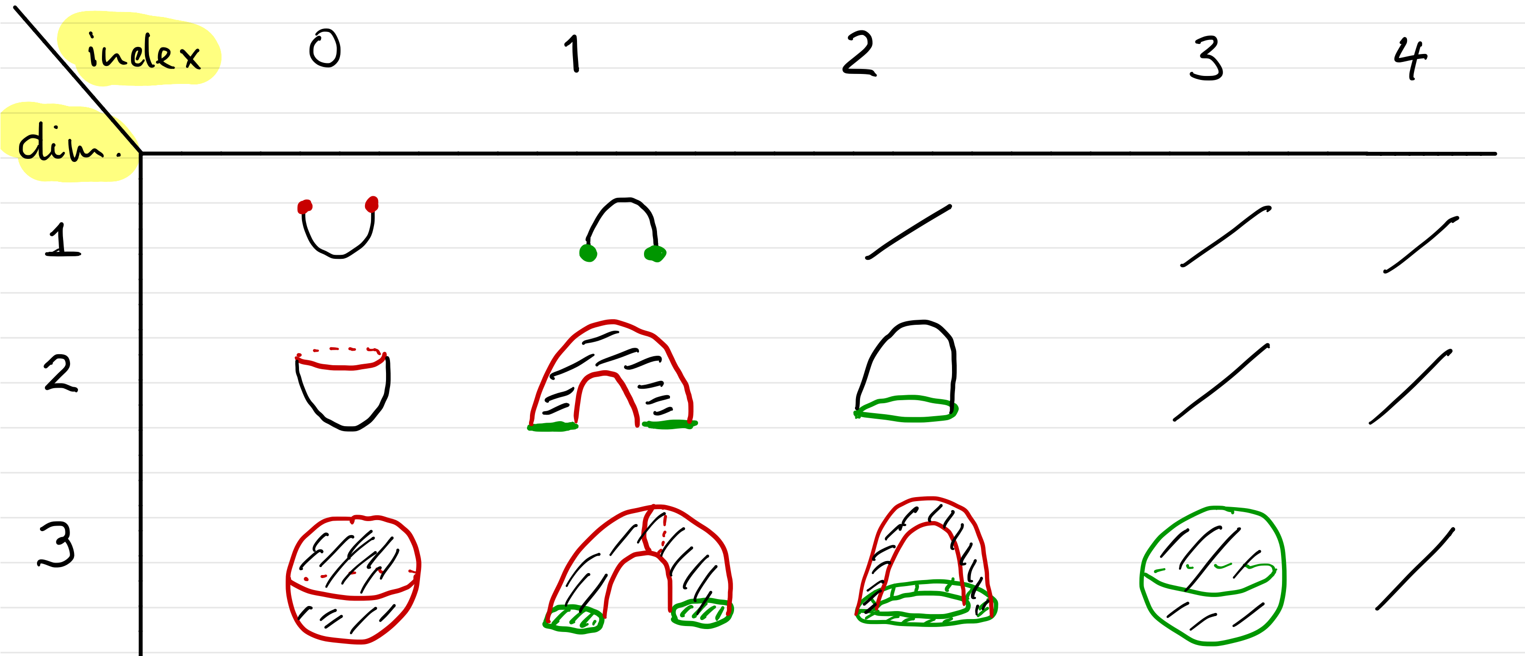
Here we assume that $\partial M = \partial_- M \sqcup \partial_+ M$ and

$$\partial_+ M^{(-1)} := \partial_- M \times \{+1\} \qquad \partial_- M \overset{''}{} \quad \partial_- M^{(-1)} := \partial_- M \times \{-1\}$$

By construction, we see that $\quad = \partial_- M^{(k)} \ \forall k$

$\partial_+ M^{(k)}$ is obtained from $\partial_+ M^{(k-1)}$ by surgery on I_k framed $(k-1)$-spheres $S^{k-1} \times D^{d-k} \xrightarrow{\varphi_i} \partial_+ M^{(k-1)}$.

Handles are the building blocks of manifolds



4 $(D^4, S^3, \emptyset)$ $(D^1 \times D^3, D^1 \times S^2, S^0 \times D^3)$ $(D^2 \times D^2, D^2 \times S^1, S^1 \times D^2)$ $(D^3 \times D^1, D^3 \times S^0, S^2 \times D^1)$ $(D^4, \emptyset, S^3)$

Important facts (from Morse/Cerf theory):

- handle decompositions exist in the smooth and PL-categories: The attaching maps φ_i, $i \in I_k$, are smooth resp. PL, inducing the respective structures on M (straightening of corners).
- for dim. $d \neq 4$, they also exist for topological mfd.s
- M^4 has a handle decomp. $\Leftrightarrow$ M is smooth

  $\Leftrightarrow$ M is PL
- handle decompositions are unique up to

(i) isotopy of φ_i in $\partial_+ M^{(d-1)} \cup (I_k \setminus i)$ handles of index k

(ii) birth/death "handle cancellation" "includes "handle slides"

Exercise for smooth manifolds M with $\partial_- M$ compact

(1) a handle decomp. has finitely many handles (i.e. $\left| \coprod_{k=0}^{d} I_k \right| < \infty$) $\Leftrightarrow$ M is compact

(2) M has a handle decomposition with $\left| \pi_0 M \setminus \mathrm{image}(\pi_0 \partial_- M) \right|$ many 0-handles.

(3) M is orientable $\Leftrightarrow$ all attaching maps $\varphi_i : S^0 \times D^{d-1} \hookrightarrow \partial M^{(0)}$ for 1-handles are untwisted, i.e. induce "same" sign on $\varphi_i / \{\pm 1\} \times D^{d-1}$ if it is defined!

(4) Turning a handle decomp. of M upside down interchange $\partial_+ M$ & $\partial_- M$ and I_k & I_{d-k}.