Outline for May 17 Top. invariants of 4-manifolds

1) $H_* M$ computed from the handle chain complex

where $C_k(M, \partial_- M) :=$ free abelian group on k-handles

and $\partial \downarrow$

$C_{k-1}(M, \partial_- M)$, $\partial h_k^i : h_{k-1}^j = \#$ transverse intersections $\varphi_i(S^{k-1} \times 0) \pitchfork \varphi_j(0 \times S^{d-k})$

1') $H_* \tilde{M}$ computed from equiv. in $\partial_+ M^{(k-1)}$.

version over $\mathbb{Z}[\pi_1 M]$. Tricky in small degree.

2) Intersection form $\lambda : H_2 M \otimes H_2 M \longrightarrow \mathbb{Z}$

given by linking numbers (& framings) of linear combinations of 2-handles.

3) Many examples, def. of linking number.

Cool group theory - smooth manifolds facts:

$$\frac{\left\{ F_g \twoheadleftarrow \pi_1\Sigma_g \twoheadrightarrow F_g \right\}_{g\geq 1}}{\text{automorphism} \\ + \text{ stabilization}} \cong \frac{\left\{ \begin{array}{c} \text{closed connected} \\ \text{oriented 3-mfds} \end{array} \right\}}{\text{diffeomorphism}}$$

$$\frac{\left\{ \begin{array}{ccc} & F_g \twoheadleftarrow \pi_1\Sigma_g \twoheadrightarrow F_g & \\ \text{s.t. all 3 pushouts are free} & \downarrow & \\ & F_g & \end{array} \right\}_{g\geq 1}}{\text{automorphisms} \\ + \text{ stabilization}} \cong \frac{\left\{ \begin{array}{c} \text{closed connected} \\ \text{oriented 4-mfds} \end{array} \right\}}{\text{diffeomorphism}}$$

Group & 4-manifold trisections

Remark: Not useful, so far, for Poincaré conjecture!

Attaching a 2-handle: $W^4 = M^4 \cup D^2 \times D^2$

$\varphi : S^1 \times D^2 \hookrightarrow \partial M$

φ is specified up to isotopy by

- $\varphi/_{S^1 \times 0}$ and $\varphi/_{S^1 \times 1}$ (framing $\hat{=}$ one normal vector)

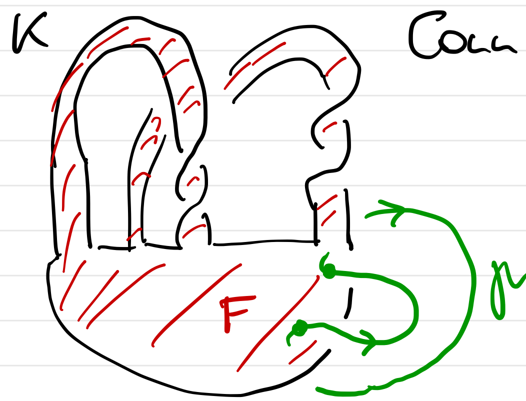
If $\varphi/_{S^1 \times 0}$ has a preferred parallel in ∂M, e.g. linking number 0 in S^3, then $\varphi/_{S^1 \times 1}$ can be specified by an integer $n \in \mathbb{Z} \cong \pi_1 SO(2)$ $\hat{=} \pi_0 \mathrm{Diff}_+(S^1 \times D^2)$.

On the boundary, we get $\partial W = \partial M \setminus \varphi(S^1 \times \mathring{D}^2) \cup \underbrace{D^2 \times S^1}_{S^1 \times S^1 \,=}$ surgery on φ

which is determined by

$\varphi(S^1 \times 1)$ bounding a disk in ∂W: 

Seifert surface F for a knot K gives a homomorphism $\pi_1(S^3 \setminus K) \twoheadrightarrow \mathbb{Z}$ as before:



needs orientations

Count transverse intersections with F

This does not depend on base point for μ and is equal to

$$\mu \longmapsto lk(K, \mu) \in \mathbb{Z}$$

Alternatively (and more symmetrically),

$lk(K_1, K_2) = \# F_1 \pitchfork F_2$ where $F_i \subseteq D^4$, $\partial F_i = K_i \subseteq S^3$

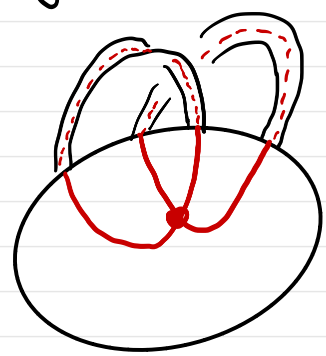
Well defined because ......

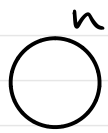
Q: When is there a $\pi_1(S^3 \setminus (K_1, K_2)) \twoheadrightarrow$ free group?

Scholem: If $\underset{\parallel}{L}$ $(\ell_1,n_1),\dots,(\ell_m,n_m)$ is a framed link (no dots!) then the intersection form λ_{M_L} on $H_2 M_L \cong \mathbb{Z}^m$ is given by

$$\lambda_{M_L}(i,j) = \begin{cases} lk(\ell_i,\ell_j) & i\neq j \\ n_i & i=j \end{cases}$$

Proof: $M_L \simeq \bigvee^m S^2$ with free generators of H_2 given by $F_i \cup_{\ell_i} \mathrm{core}(h_i)$



Examples: (i) 

(ii) 

$$\begin{tikzcd} H_2 M_L \arrow[r] \arrow[d, "\lambda_{M_L}"'] & H_2(M_L,\partial M_L) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & H_1(\partial M_L) \\ \mathrm{Hom}(H_2 M_L,\mathbb{Z}) \cong H^2 M_L & & \end{tikzcd}$$

$H_1(\partial M_L) = \mathbb{Z}/n$ in (i) and 0 in (ii).

Freedman's classification theorem:

$$\left\{\begin{array}{c}\text{closed 1-connected}\\ \text{top. 4-manifolds}\end{array}\right\}\Big/\text{homeomorphism} \ \underset{\lambda + KS}{\cong}\ \left\{\begin{array}{c}\text{unimodular sym. forms}\\ \text{on f.g. free abelian groups}\end{array}\right\}\Big/\pm\text{isometry}$$

$\times\ \mathbb{Z}/2$ for odd forms

$$KS(M) = 0 \iff M \times \mathbb{R} \text{ smooth}$$

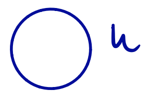
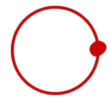
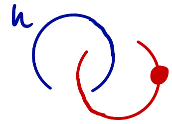
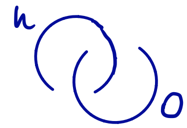
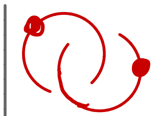
$$\iff M \# \#^k S^2 \times S^2 \text{ is smooth}$$

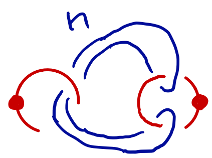
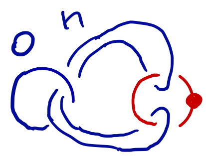
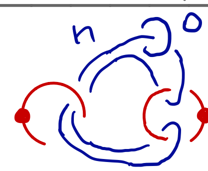
$$\iff \tau_M(c) \underset{(2)}{\equiv} \frac{\lambda_M(c,c) - \text{signature}(\lambda_M)}{8}$$

secondary intersection invariant for c.

for a characteristic class

$$H_2 M \ni c : \lambda_M(c,x) \underset{(2)}{\equiv} \lambda_M(x,x)\ \forall x$$

L	$\emptyset$					
M_L	D^4	$S^2 \tilde{\times}_n D^2$	$S^1 \times D^3$	D^4	$S^2 \tilde{\times}_n S^2 \setminus \mathring{D}^4$	does
∂M_L	S^3	$L_{n,1} = S^2 \tilde{\times}_n S^1$	$S^1 \times S^2$	S^3	S^3	not
$\hat{M}_L$	S^4	$n = 0, \pm 1$ $S^4, \pm \mathbb{C}P^2$	$S^1 \times S^3$	S^4	$S^2 \tilde{\times}_n S^2$	exist

L					???
M_L	n.o.h	$(S^1 \times S^1) \tilde{\times}_n D^2$	n.o.w	n.o.h	$K3 \setminus \mathring{D}^4$
∂M_L	$n \neq \pm 1$ n.o.h	$(S^1 \times S^1) \tilde{\times}_n S^1$	$(S^1 \times S^1) \tilde{\times}_n S^1$	$\#^2 S^1 \times S^2$	S^3
$\hat{M}_L$	d.n.e	d.n.e	d.n.e.	$(S^1 \times S^1) \tilde{\times}_n S^2$	K3-surface