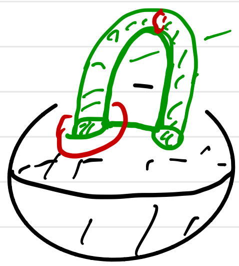
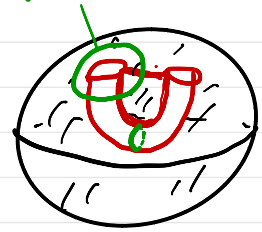
May 15: The diffeo. $D^3 \cup$ 1-handle $\cong D^3 \setminus$ 1-handle

runs once over 1-handle!



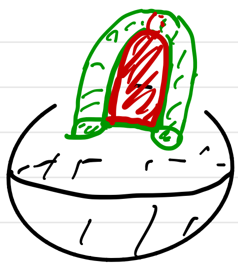
$\cong$



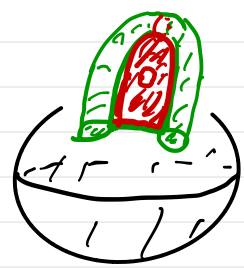
handle cancellation

generalizes to $D^d \cup$ 1-handle $\cup$ 2-handle $\cong D^d$

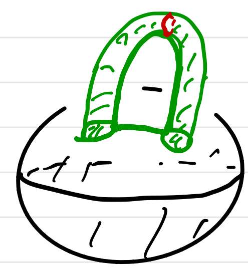
$$\Rightarrow D^d \underset{S^0\times D^{d-1}\hookrightarrow S^{d-1}}{\cup} D^1\times D^{d-1} \cong \left(D^d \underset{S^0\times D^{d-1}\hookrightarrow S^{d-1}}{\cup} D^1\times D^{d-1} \underset{S^1\times D^{d-2}\hookrightarrow S^1\times S^{d-2}}{\cup} D^2\times D^{d-2} \right) \setminus \overset{(d-2)\text{-handle}}{D^2\times D^{d-2}_\varepsilon}$$



$\setminus$ 1-handle $\cong$



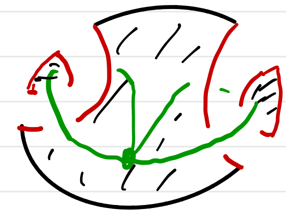
$\cong$



In dim. $d=4$, we use

unlink on boundary!

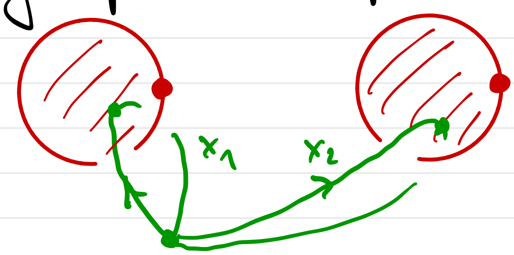
$h^0 \cup n\, h^1 \cong D^4 \setminus n \cdot h^2 \;\hat{=}$



where we can see the boundary:

The group isomorphism $\pi_1(h^0 \cup n\, h^1) \cong \pi_1(\bigvee^n S^1) \cong F_n :=$ free group on $x_1, \dots, x_n$

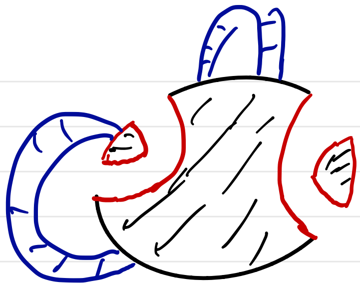
$S^3 \supseteq \mathbb{R}^3 \supseteq$



is given by reading off words of intersections with spanning disks for the dotted unlink.

Corollary: We get a presentation of

$\pi_1(D^4 \cup n\, h^1 \cup m\, h^2 \cup h^3_i \cup h^4_j) \cong (x_1, \dots, x_n \,/\, r_1, \dots, r_m)$.



$$= M^4_L = D^4 \setminus \underbrace{\text{2-handles}}_{\text{dotted part of } L} \cup \underbrace{\text{2-handles}}_{\text{framed part of } L}$$

$$\cong \text{0-handle} \cup \underbrace{\text{untwisted 1-handles}} \cup \text{2-handles}$$

∂M_L = surgery on (S^3, L^0) where $L^0 = L$ with dots replaced by 0 framings.

Thm. [Laudenbach]: $\partial M_L \cong \#^k S^1 \times S^2$ iff $M^{(2)}$ extends to a handle dec. of $M = M_L \cup \natural^k S^1 \times D^3$ a closed mfd. Moreover, there is a unique way of attaching these 3- & 4-handles.

Cor.: There is a well-defined surjective map

$$\left\{\begin{array}{l}\text{dotted, framed links } L \\ \text{with } \partial M_L \cong \# S^1 \times S^2\end{array}\right\} \twoheadrightarrow \left\{\begin{array}{l}\text{diff. classes of closed, con-} \\ \text{nected, oriented 4-manifolds}\end{array}\right\}$$

Cor. [Kirby calculus]: Every cl.c.o. 3-mfd. can be obtained by surgery on a framed link. Moreover, $\partial M^4_L \cong \partial M^4_{L'} \iff L$ & L' differ by Kirby moves

$\parallel$

handle slides and deletion/creation of $\bigcirc^{\pm 1}$

Proof: Given N^3 we choose any co.c.o. M^4 with $\partial M = N$ (using $\Omega_3 = 0$). $\Longrightarrow$

$M = M_L \cup$ 3-handles. φ_i are non-separating since ∂M is connected

$\coprod_{I^3} \varphi_i : S^2 \times D^1 \hookrightarrow \partial M_L$

$\Longrightarrow \partial M_L = \partial M \# \#^{I^3} S^1 \times S^2$

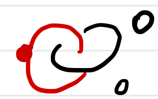
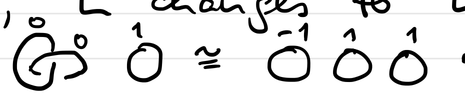
and $\partial M =$ surgery on $|I^3|$ 0-framed S^1's in ∂M_L

$=$ surgery on $(S^3, L^0 \cup |I^3|$ circles$)$.

This proves existence of surgery descriptions. Idea for uniqueness: Assume that $S(L) \cong S(L')$ for framed links L, L' in S^3. Then form the 1-connected 4-mfds M_L, $M_{L'}$ with their given boundary. $\mathbb{CP}^2$-stable classification shows that

$$M_L \# \#^{n_1} \mathbb{CP}^2 \# \#^{n_2} \overline{\mathbb{CP}}^2 \cong M_{L'} \# \#^{n_1'} \mathbb{CP}^2 \# \#^{n_2'} \overline{\mathbb{CP}}^2 .$$

So adding (± 1)-framed unlinks to L & L', we assume $M_L \cong M_{L'}$. Cerf theory shows that L' is obtained from L by
- handle slides
- birth / death

For 1-2 cancelling pair, L changes to $L \cup$ 
and we know that  . $\hat{=}\ \partial$  ∎
2-3 handle pairs more tricky!

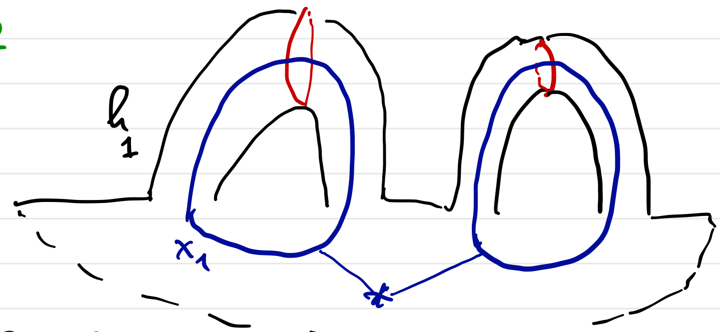
How to cancel 1-handles for simply-connected M^d, $d \geq 5$? 3D-P.C.!

Step 1: Create a 2-3 handle pair $h^2 \cup h^3$.

Step 2: Given a 1-handle h^1_1, there are (original) h^2_i

s.t. in $\partial_+\big(h^0 \cup (h^1_1 \cup \dots \cup h^1_n)\big) = \#^n S^1 \times S^{d-2}$ $d \geq 4$ $\Leftrightarrow$ 1-conn.

$\varphi_i : S^1 \times D^{d-2}$



Free generator x_1 of $\pi_1 \partial_+$ is a word in $[\varphi_i]^{g_i}$ and we can slide the new h^2 over h^2_i along g_i to get $[\partial h^2] = x_1 \in \pi_1 \partial_+$.

If $d \geq 5$ then homotopy implies isotopy in ∂_+ so h^2 can be isotoped s.t. $\partial h^2 \cap S^{d-2}_1 = \{pt\} \Rightarrow h^1_1$ can be cancelled ■

Corollary: M^5 1-connected, closed. $\Longrightarrow$

$$M^5 = \text{0-handle} \cup n\cdot\text{2-handles} \cup n\cdot\text{3-handles} \cup \text{5-handle}$$

$$= \natural^n S^2 \tilde{\times} D^3 \underset{\# S^2 \tilde{\times} S^2}{\cup} \natural^n S^2 \tilde{\times} D^3$$

where $\tilde{\times}$ denotes one of the two oriented D^3-bundles over S^2 $\left(\pi_1 SO(3) = \mathbb{Z}/2 \right)$.

M^5 is spin $\Longleftrightarrow$ all $\tilde{\times} = \times$. Exercise:

Thm [Wall]: M^5 spin $\Rightarrow$ diffeom. type is completely determined by isom. type of $H_2 M$!

$lk/_{\mathrm{Tors}\, H_2 M}$ is skew $\Rightarrow$ torsion $H_2 M \cong A \oplus A$ ∎