Link homotopy and non-repeating Whitney towers in the 4-ball

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Milnor's link-homotopy classification of almost-trivial links in S^3.

Theorem:

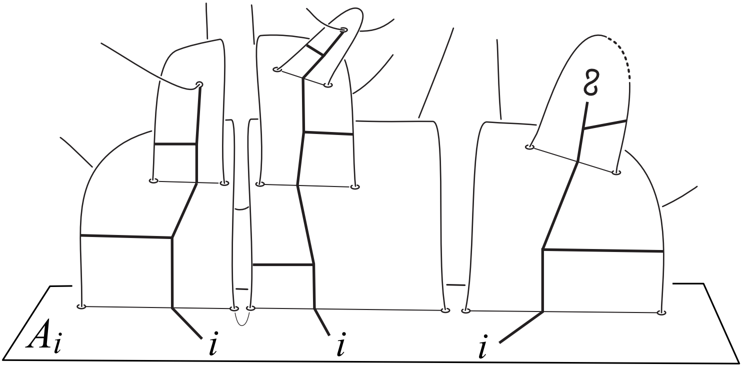
(1) A link $L = L_1 \cup L_2 \cup \cdots L_m \subset S^3$ is almost trivial if and only if L bounds an order $m-2$ non-repeating Whitney tower $\mathcal{W} \subset B^4$.

(2) The Milnor invariant $\mu_L \in \mathbb{Z}[\mathcal{S}_{m-2}]$ (as in Pete's talk) is the image of the non-repeating intersection invariant $\lambda_{m-2}(\mathcal{W}) \in \Lambda_{m-2}(m)$ under a specific 'choice of basis' isomorphism $\Lambda_{m-2}(m) \xrightarrow{\cong} \mathbb{Z}[\mathcal{S}_{m-2}]$.

Corollary: For almost-trivial links L and L' the following statements are equivalent:
(i) L and L' are link-homotopic.
(ii) $\mu_L = \mu_{L'} \in \mathbb{Z}[\mathcal{S}_{m-2}]$.
(iii) $\lambda_{m-2}(\mathcal{W}) = \lambda_{m-2}(\mathcal{W}') \in \Lambda_{m-2}(m)$ for any order $m-2$ non-repeating Whitney towers $\mathcal{W}$ and $\mathcal{W}'$ bounded by L and L', repspectively. ...**definitions/proof sketches later**...

Recall: The *intersection forest* multiset $t(\mathcal{W})$ of a Whitney tower $\mathcal{W}$

$$\mathcal{W} \mapsto t(\mathcal{W}) = \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^\infty$$



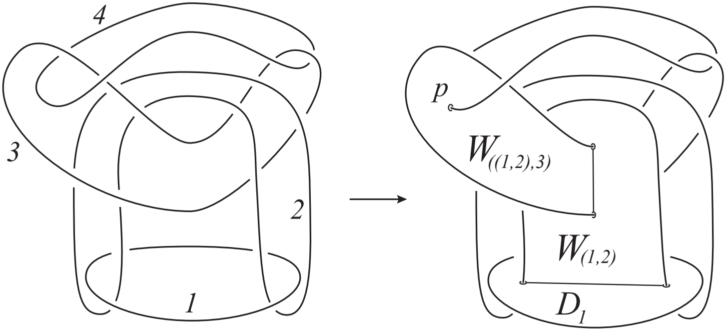
'framed tree' $t_p \leftarrow\!\shortmid\ p$ unpaired intersection with sign $\epsilon_p = \pm 1$,
'twisted tree' $J^\infty := J \text{ --- } \infty \leftarrow\!\shortmid\ W_J$ with twisting $\omega(W_J) \neq 0 \in \mathbb{Z}$.

Definition

- The *order* of a tree is the number of trivalent vertices.
- The *order* of a Whitney disk or an intersection point is the order of the corresponding tree.

$\mathcal{W}$ is an order n *non-repeating* Whitney tower if all $t_p \in t(\mathcal{W})$ having distinctly-labeled vertices are of order $\geq n$.

Example: The Bing double of the Hopf link bounds an order 2 non-repeating Whitney tower $\mathcal{W}$.



Exercise: Draw $t_p \subset \mathcal{W}$ and check $\mathcal{W}$ is order 2 non-repeating.

Non-repeating order n Whitney towers

$\mathcal{W}$ is an order n *non-repeating* Whitney tower if all $t_p \in t(\mathcal{W})$ having distinctly-labeled vertices are of order $\geq n$.

Note: Can ignore both repeating Whitney disks and twisted Whitney disks in this non-repeating setting.

Non-repeating Whitney towers characterize being able to 'pull apart' components:

Theorem (Pulling apart surfaces)

$$A = \cup_{i=1}^{m} A_i \looparrowright X \textit{ admits an order } m-1 \textit{ non-repeating } \mathcal{W}$$

if and only if

$$A \textit{ is homotopic (rel } \partial) \textit{ to } A' = \cup_{i=1}^{m} A'_i \textit{ with } A'_i \cap A'_j = \emptyset \textit{ for all } i \neq j.$$

Theorem: $A = \cup_{i=1}^{m} A_i \looparrowright X$ admits an order $m-1$ non-repeating $\mathcal{W}$ if and only if A is homotopic (rel ∂) to $A' = \cup_{i=1}^{m} A'_i$ with $A'_i \cap A'_j = \emptyset$ for all $i \neq j$.

Corollary: Two m-component links in S^3 are link homotopic (homotopy preserves disjointness at all times) if and only if they cobound immersed annuli in $S^3 \times I$ admitting an order $m-1$ non-repeating Whitney tower.

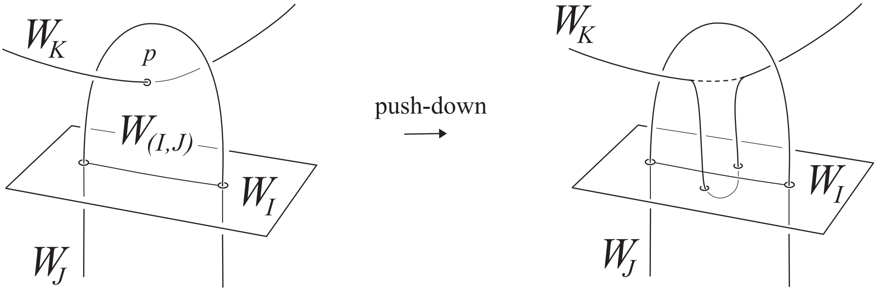
In particular, an m-component link is link-homotopically trivial if it bounds immersed disks admitting an order $m-1$ non-repeating Whitney tower in B^4.

Proof of Corollary: "Singular concordance implies link homotopy" – Giffen, Goldsmith, (and PT for higher dim co-dimension 2).

Theorem: $A = \cup_{i=1}^{m} A_i \looparrowright X$ admits order $m-1$ non-repeating $\mathcal{W}$ iff A is homotopic (rel ∂) to $A' = \cup_{i=1}^{m} A'_i$ with $A'_i \cap A'_j = \emptyset$ for all $i \neq j$.

The "if" direction is true by definition, since disjoint order 0 surfaces form a non-repeating Whitney tower of any order.

The "only if" direction uses 'pushing down' to clean up all Whitney disks:

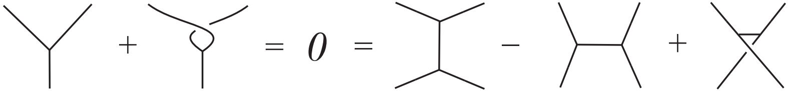


Theorem: $A = \cup_{i=1}^{m} A_i \looparrowright X$ admits order $m-1$ non-repeating $\mathcal{W}$ iff A is homotopic to $A' = \cup_{i=1}^{m} A'_i$ with $A'_i \cap A'_j = \emptyset$ for all $i \neq j$.

Proof sketch of "only if" direction (see 'Pulling apart 2-spheres in 4-manifolds' arXiv:1210.5534 [math.GT]):
If $\mathcal{W}$ contains no Whitney disks, then the A_i are pairwise disjoint.
Consider a Whitney disk $W_{(I,J)}$ in $\mathcal{W}$ of maximal order.
If $W_{(I,J)}$ is *clean*, then do the $W_{(I,J)}$-Whitney move on W_I or W_J.
If $W_{(I,J)}$ is not clean, then for any $p \in W_{(I,J)} \cap W_K$, at least one of (I,K) or (J,K) is a repeating bracket, so can push p down off of $W_{(I,J)}$ at cost of only creating repeating intersections.
Repeating this procedure on all maximal order Whitney disks eventually yields the desired order $m-1$ non-repeating Whitney tower with no Whitney disks (ie. disjoint order 0 surfaces A'_i).

Non-repeating obstruction theory

$\Lambda_n(m) :=$ free abelian group on order n framed trees, each having univalent vertices labeled by distinct indices from $\{1, 2, \ldots, m\}$, modulo local *antisymmetry* (AS) and *Jacobi* (IHX) relations:



Definition: If $\mathcal{W}$ is an order n non-repeating Whitney tower, the order n *non-repeating intersection invariant* $\lambda_n(\mathcal{W})$ is defined by

$$\lambda_n(\mathcal{W}) := [\sum \text{sign}(p) \cdot t_p] \in \Lambda_n$$

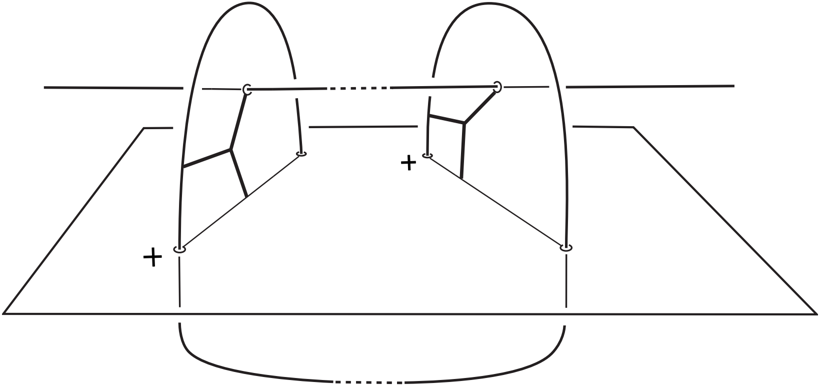
where the sum is over all order n non-repeating intersections $p \in \mathcal{W}$.

Theorem (non-repeating order-raising)

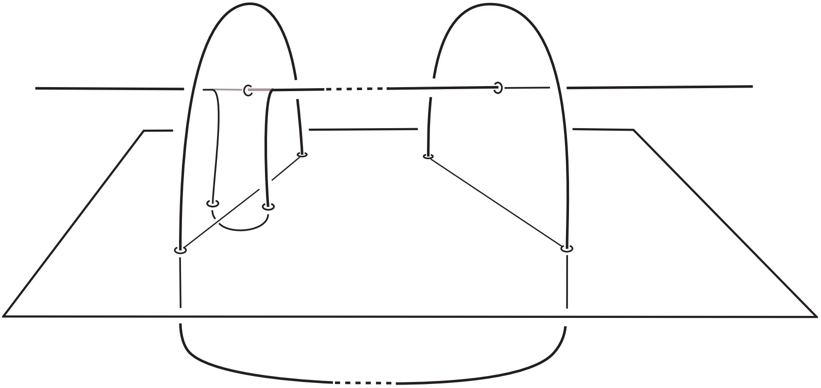
$A \looparrowright X$ admits a non-repeating Whitney tower $\mathcal{W}$ of order n with $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n$ if and only if A admits an order $(n+1)$ non-repeating Whitney tower. □

Non-repeating obstruction theory

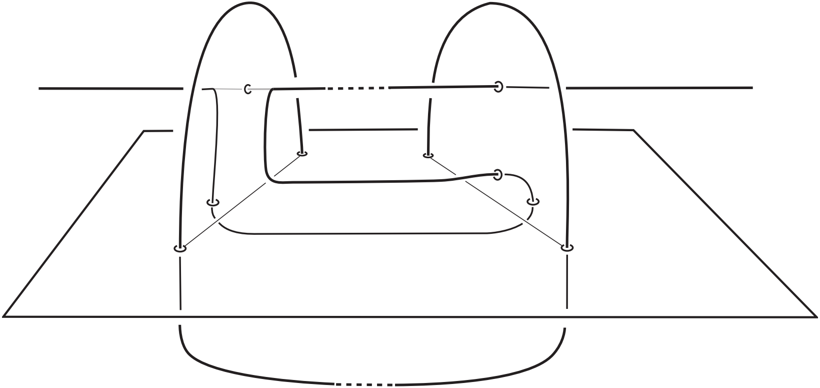
Proof of order-raising uses geometric realizations of IHX relations and ‘transfer moves’ to convert algebraically canceling trees into geometric canceling trees (intersections paired by Whitney disks)”, modulo creating higher-order trees.



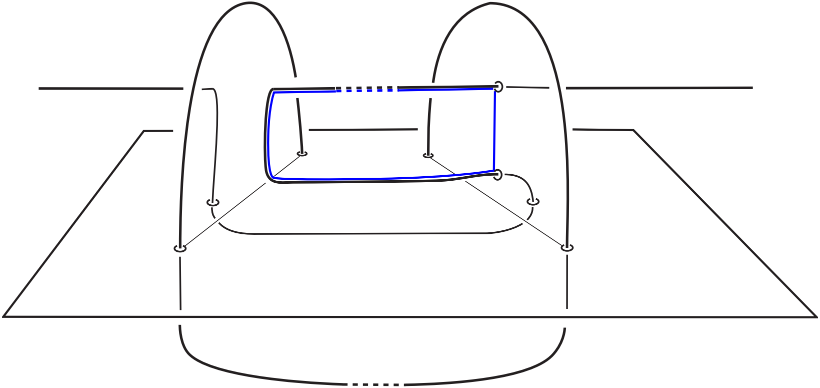
Transfer move



Transfer move

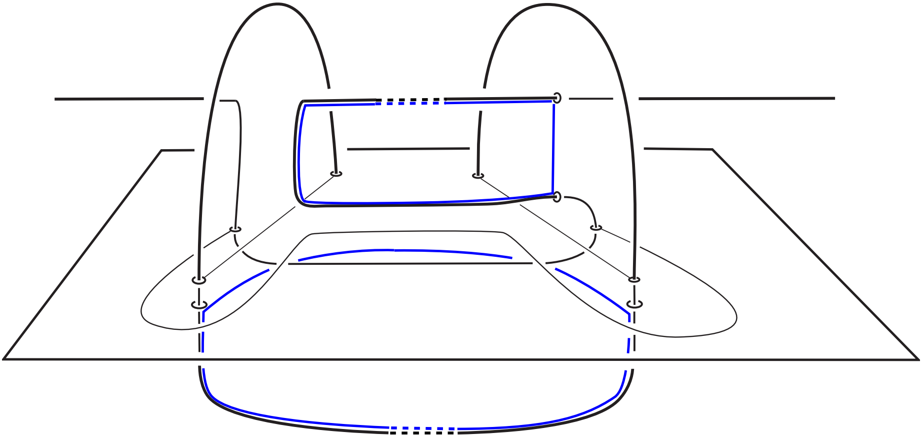


Transfer move



Transfer move

New higher-order Whitney disks are uncontrolled (they can only contribute higher-order intersections). Construction is supported near Whitney disks union an arc in original Whitney tower.

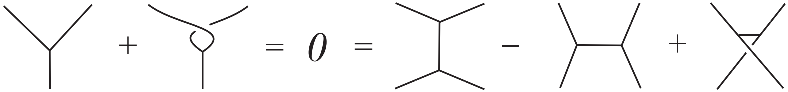


For details on the order-raising intersection/obstruction theory proof, including general order n Whitney towers ('repeating' labels allowed), see:

Section 4 of 'Whitney tower concordance of classical links'
arXiv:1202.3463 [math.GT]
(includes twisted Whitney towers)

Section 4 of 'Whitney towers and the Kontsevich integral'
arXiv:math/0401441 [math.GT]
(uses some slightly different notation)

$\Lambda_n(m) :=$ free abelian group on order n framed trees, each having univalent vertices labeled by distinct indices from $\{1, 2, \ldots, m\}$, modulo local *antisymmetry* (AS) and *Jacobi* (IHX) relations:



The relations are homogeneous in labels, and order n trees have $n + 2$ univalent vertices, so choosing $(n + 2)$-element subsets of distinct indices decomposes $\Lambda_n(m)$ into the direct sum of $\binom{m}{n+2}$-many isomorphic 'copies' of $\Lambda_n(n + 2)$.

So suffices to understand the groups $\Lambda_n(n + 2)$, for $1 \leq n \leq m - 2$.

Example: $\Lambda_0(4)$

Have $\binom{4}{0+2} = 6$ two-element subsets of $\{1,2,3,4\}$:

$$\Lambda_0(4) = \Lambda_0(1,2) \oplus \Lambda_0(1,3) \oplus \Lambda_0(1,4) \oplus \Lambda_0(2,3) \oplus \Lambda_0(2,4) \oplus \Lambda_0(3,4)$$

where $\Lambda_0(i,j)$ denotes the non-repeating tree group on order 0 trees labeled distinctly from $\{i,j\}$.

So $\Lambda_0(4) \cong \mathbb{Z}^6$ is the $\mathbb{Z}$-span of the six order 0 trees i —— j for distinct labels $i \neq j$ from $\{1,2,3,4\}$.

In the setting of link homotopy $\lambda_0(L) := \lambda_0(\mathcal{W}) \in \Lambda_0(4)$ measures the pairwise linking of components of a 4-component link L, where $\mathcal{W}$ is any order 0 non-repeating Whitney tower bounded by L (immersed disks bounded by components).

Example: $\Lambda_1(4)$

Have $\binom{4}{1+2} = 4$ three-element subsets of $\{1,2,3,4\}$:

$$\Lambda_1(4) = \Lambda_1(1,2,3) \oplus \Lambda_1(1,2,4) \oplus \Lambda_1(1,3,4) \oplus \Lambda_1(2,3,4)$$

So $\Lambda_1(4) \cong \mathbb{Z}^4$ is the $\mathbb{Z}$-span of the four order 1 trees $i \,\text{---}\!\!<^{k}_{j}$ with distinct labels i, j, k from $\{1,2,3,4\}$.

We may take these generating trees to be canonically oriented (at the trivalent vertex) using the ordering of the labels.

In the setting of link homotopy $\lambda_1(\mathcal{W}) \in \Lambda_1(4)$ corresponds to Milnor's 'triple linking numbers' $\mu_{ijk}(L)$ for 3-component sublinks of a 4-component link L, where $\mathcal{W}$ is any order 1 non-repeating Whitney tower bounded by L (which exists iff $\lambda_0(L) = 0$).

Example: $\Lambda_2(4)$

$\Lambda_2(4)$ is the highest order (non-trivial) non-repeating group for four components, with a single $\binom{4}{2+2}$ four-element subset of $\{1,2,3,4\}$. Since the order is ≥ 2 the IHX relations come into play:

$$\Lambda_2(4) = \langle\, {}^{3}_{1}{>\!\!-\!\!<}^{4}_{2}\,,\ \ {}^{3}_{2}{>\!\!-\!\!<}^{4}_{1} \rangle \cong \mathbb{Z}^2$$

since by the IHX relation we have

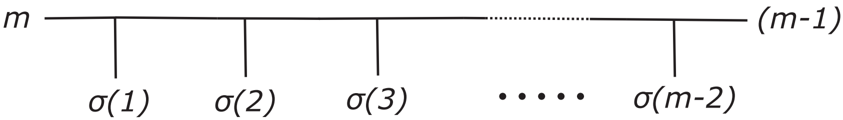
$${}^{3}_{4}{>\!\!-\!\!<}^{1}_{2} = {}^{3}_{1}{>\!\!-\!\!<}^{4}_{2} + {}^{3}_{2}{>\!\!-\!\!<}^{4}_{1}$$

Subsequent slides will find a basis for $\Lambda_{m-2}(m) \cong \mathbb{Z}^{(m-2)!}$

Will also describe relationship between Milnor invariants and $\lambda_n(L) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ for $\mathcal{W}$ any order n non-repeating Whitney tower bounded by an m-component link L.

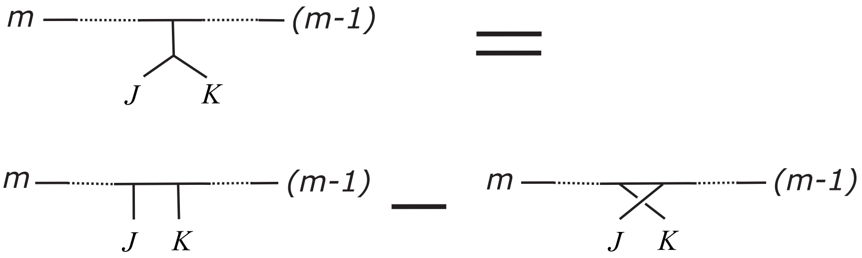
‘Maximal’ order $n = m - 2$ non-repeating tree groups

Will see that $\Lambda_{m-2}(m) \cong \mathbb{Z}[\mathcal{S}_{m-2}]$,
where $\mathcal{S}_{m-2}$ is the symmetric group on $\{1, 2, \ldots, m-2\}$,
with a basis given by the ‘simple’ trees $t(\sigma)$ for $\sigma \in \mathcal{S}_{m-2}$:



Simple trees span $\Lambda_{m-2}(m)$

If the geodesic between the m-vertex and the $(m-1)$-vertex has length less than $m-1$, apply an IHX relation: $\mathrm{I} = \mathrm{H} - \mathrm{X}$:



Eventually get length $m-1$ geodesics between the m-vertex and the $(m-1)$-vertex in each tree.

Placing a root at the *m*-vertex of each tree gives an isomorphism from $\Lambda_{m-2}(m)$ to the degree $m-1$ *reduced* free Lie algebra $\mathrm{RL}_{m-1}(m-1)$ which is the subgroup of non-repeating length $m-1$ brackets in the free Lie algebra (over $\mathbb{Z}$) on $m-1$ generators, with AS and IHX relations going to skew-symmetry relations and Jacobi identities.

The rank of $\mathrm{RL}_{m-1}(m-1)$ is $(m-2)!$, by Theorem 5.11 of Magnus, Karass and Solitar's book 'Combinatorial group theory' Dover Publications, Inc. (1976). See also Sections 4–5 of Milnor's 'Link Groups' Annals of Math. 59 (1954), and/or Pete's posted notes.

So the rank of $\Lambda_{m-2}(m)$ is $(m-2)!$, and the simple trees $t(\sigma)$ are linearly independent.

Recall:

A *link-homotopy* of an m-component link $L = L_1 \cup L_2 \cup \cdots \cup L_m$ in the 3–sphere is a homotopy of L which preserves disjointness of the link components, i.e. during the homotopy only self-intersections of the L_i are allowed.

The *Milnor group* $\mathcal{M}(L)$ of $L = \cup_{i=1}^{m} L_i \subset S^3$ has a presentation

$$\mathcal{M}(L) = \langle x_1, x_2, \ldots, x_m \,|\, [\ell_i, x_i],\ [x_j, x_j^h] \rangle$$

where each x_i is represented by a meridian (one for each component), and the ℓ_i are words in the x_i determined by the link longitudes.

The *free Milnor group* $\mathcal{M}(m)$ is given by setting all $\ell_i = 1$ in this presentation.

Reduced ('non-repeating') free Lie algebra

The *reduced* free Lie algebra $\mathrm{RL}(m) = \oplus_{n=1}^{m} \mathrm{RL}_n(m)$ is the subgroup of the free $\mathbb{Z}$-Lie algebra on generators $X_1, X_2, \ldots, X_m$ spanned by iterated Lie brackets on distinct generators.

$x_i^{\pm 1} \mapsto \pm X_i$ induces $\mathcal{M}(m)_{(n)}/\mathcal{M}(m)_{(n+1)} \cong \mathrm{RL}_n(m)$

This isomorphism takes a product of length n commutators in distinct x_i to a sum of length n Lie brackets in distinct X_i.

In particular, $\mathrm{RL}_n(m) = 0$ for $n > m$.

Define $\mathcal{M}^i(L) := \mathcal{M}(L)/\{x_i = 1\}$

If longitudes $[\ell_i] \in \mathcal{M}^i(L)_{(n+1)}$ for all i, then we have isomorphisms:

$$\mathcal{M}(L)_{(n+1)}/\mathcal{M}(L)_{(n+2)} \cong \mathcal{M}(m)_{(n+1)}/\mathcal{M}(m)_{(n+2)} \cong \mathsf{RL}_{(n+1)}(m).$$

Definition

The elements $\mu^i_n(L) \in \mathsf{RL}^i_{(n+1)}(m)$ determined by the longitudes ℓ_i are the *non-repeating Milnor-invariants* of *order n*. Here $\mathsf{RL}^i(m)$ is the reduced free Lie algebra on the $m-1$ generators X_j, for $j \neq i$.

Note that *degree $n+1 \leftrightarrow$ order n*:
Via non-associative bracketings $\leftrightarrow$ binary trees, have $\mathsf{RL}_{(n+1)}(m) \leftrightarrow$ the abelian group on order n *rooted* non-repeating trees modulo IHX and antisymmetry relations.

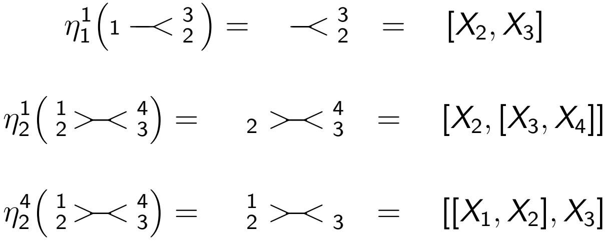
Eta-maps connecting $\Lambda_n(m)$ and $\mathsf{RL}^i_{(n+1)}(m)$

For each i, define a map

$$\eta^i_n : \Lambda_n(m) \to \mathsf{RL}^i_{(n+1)}(m)$$

by sending a tree t which has an i-labeled univalent vertex v_i to the iterated bracketing determined by t with a root at v_i. Trees without an i-labeled vertex are sent to zero.

Examples:



Lemma: $\sum_{i=1}^{m} \eta_n^i : \Lambda_n(m) \longrightarrow \oplus_{i=1}^{m} \mathsf{RL}^i_{(n+1)}(m)$ is a monomorphism.

Proof sketch:

Putting an i-label in place of the root in a tree corresponding to a Lie bracket in $\mathsf{RL}^i_{(n+1)}(m)$ gives a left inverse to η_n^i.

For the top degree $n+2=m$, this is an inverse because every index i appears exactly once in a tree t of order $n=m-2$.

For arbitrary n, composing the sum of these left inverse maps with $\sum_{i=1}^{m} \eta_n^i$ is multiplication by $n+2$ on $\Lambda_n(m)$.

Since $\Lambda_n(m)$ is torsion-free, it follows that $\sum_{i=1}^{m} \eta_n^i$ is injective.

Theorem ("$\lambda(\mathcal{W}) = \mu(L)$")

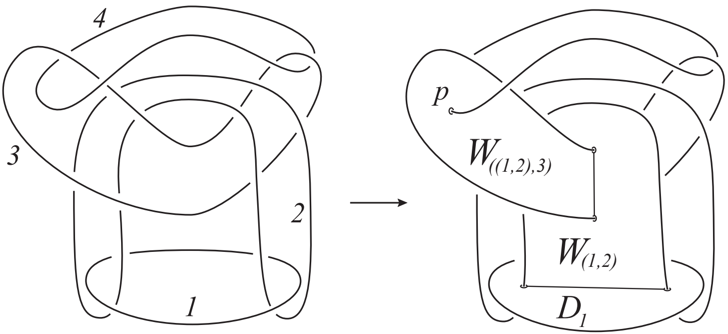
If an m-component link $L \subset S^3$ bounds a non-repeating Whitney tower $\mathcal{W}$ of order n on immersed disks $D = \cup_{i=1}^m D_i^2 \looparrowright B^4$, then for each i the longitude ℓ_i lies in $\mathcal{M}^i(L)_{(n+1)}$, and

$$\eta_n^i(\lambda_n(\mathcal{W})) = \mu_n^i(L) \ \in \mathsf{RL}^i_{(n+1)}(m)$$

Since the sum of the η_n^i is injective, the intersection invariant $\lambda_n(\mathcal{W}) \in \Lambda_n(m)$ does not depend on the Whitney tower $\mathcal{W}$ and is a link homotopy invariant of L, denoted by $\lambda_n(L)$.

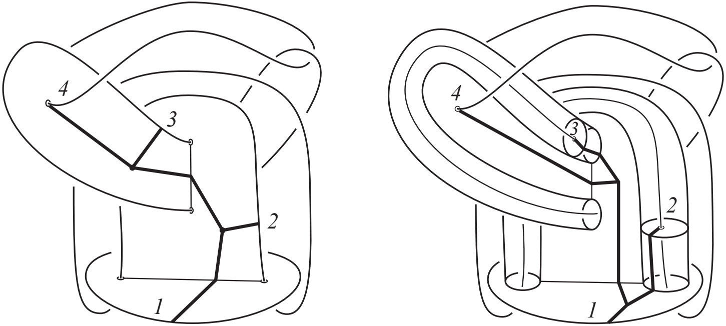
Corollary: L is link homotopically trivial, if and only if $\lambda_n(L) = 0$ for $1 \leq n \leq m-2$, if and only if L has all vanishing Milnor invariants.

Example: Bing double of Hopf link



Example: Bing double of Hopf link

$$\lambda_2(L) = {}^{1}_{2}\!>\!\!-\!\!<\!{}^{4}_{3} \rightsquigarrow \mu^1_2(L) = \eta^1_2(\,{}^{1}_{2}\!>\!\!-\!\!<\!{}^{4}_{3}) = {}_{2}\!>\!\!-\!\!<\!{}^{4}_{3} = [X_2,[X_3,X_4]]$$



To read ith longitude $L_i = \partial D_i$, convert D_i to a *grope* $G_i \subset B^4 \setminus \mathcal{W}^i$, where $\mathcal{W}^i$ is formed from $\mathcal{W}$ by deleting every Whitney disk whose tree contains an i-labeled vertex.

Then G_i displays $L_i = \partial G_i$ as iterated commutator (bracket).

Outline of proof that $\eta^i_n(\lambda_n(\mathcal{W})) = \mu^i_n(L) \in \mathrm{RL}^i_{(n+1)}(m)$

1. Arrange (using splitting, pushing down, and deleting repeating Whitney disks) that the only repeating intersections in $\mathcal{W}$ are self-intersections in the order 0 disks D_j.
2. Convert the order 0 disk D_i to a *grope* G_i of class $n+1$ bounded by L_i, such that G_i is in the complement $B^4 \smallsetminus \mathcal{W}^i$, where $\mathcal{W}^i$ is the result of deleting from $\mathcal{W}$ the disk D_i and each Whitney disk whose tree contain an i-labeled vertex. Then G_i will display the longitude ℓ_i in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$ as a product of $(n+1)$-fold commutators of meridians to the order 0 surfaces $D^i := \cup_{j \neq i} D_j$ of $\mathcal{W}^i$ by the same formula as in the definition of the map η^i_n.
3. Use *Whitney tower-grope duality* and Dwyer–Freedman–Teichner's theorem to show that $S^3 \smallsetminus \partial D^i \to B^4 \smallsetminus \mathcal{W}^i$ induces an isomorphism on the Milnor groups modulo the $(n+2)$th terms of the lower central series, so $\mu^i_n(L)$ can be computed in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$.

1. Arrange (using splitting, pushing down, and deleting repeating Whitney disks) that the only repeating intersections in $\mathcal{W}$ are self-intersections in the order 0 disks D_j.

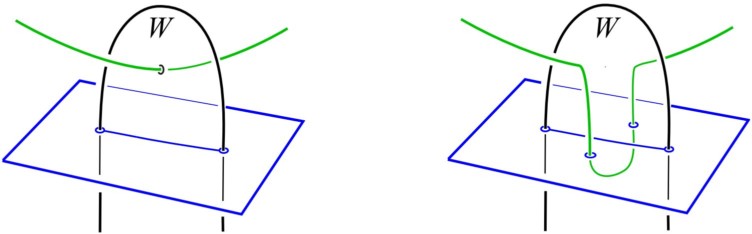


Figure: 'Pushing down' an intersection.

Warm-up exercise for Step 2

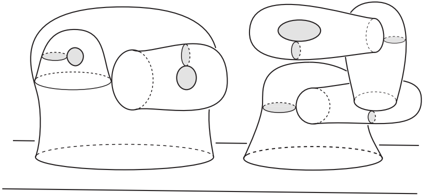
Suppose that $\mathcal{W}$ is an order n non-repeating split Whitney tower on $A = A_1 \cup A_2 \cup \cdots \cup A_m \looparrowright X^4$.

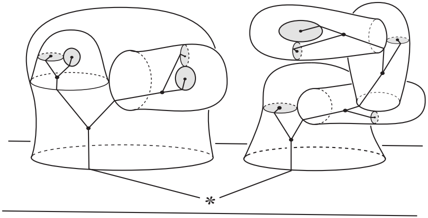
For any $i \in \{1, 2, \ldots, m\}$ denote by $\mathcal{W}^i$ the Whitney tower which is the result of deleting from $\mathcal{W}$ the order 0 surface component A_i and each Whitney disk whose tree contains an i-labeled vertex.

Exercise: Check that $\mathcal{W}^i$ is an order n non-repeating Whitney tower on the $(m-1)$-component order 0 surface $A \setminus A_i$.

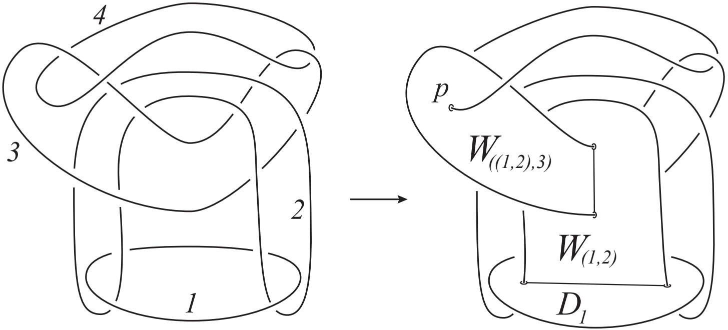
HINT: Recall that the interior of any Whitney disk in a *split* Whitney tower $\mathcal{W}$ either contains a single un-paired intersection, or a single boundary arc of a higher-order Whitney disk, or does not contain any singularities (is embedded and disjoint from the rest of $\mathcal{W}$).

Step 2 of proof: Gropes (dyadic, capped, with trees)



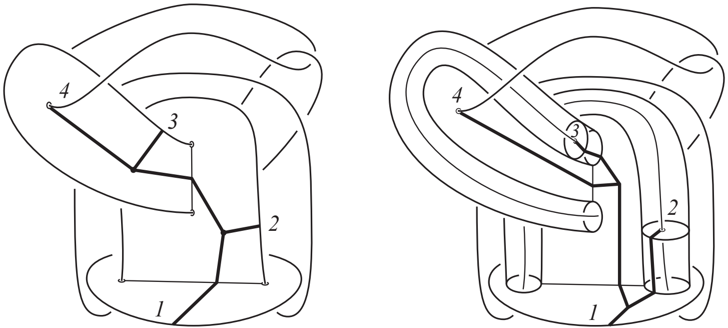


Example of case $i = 1$:



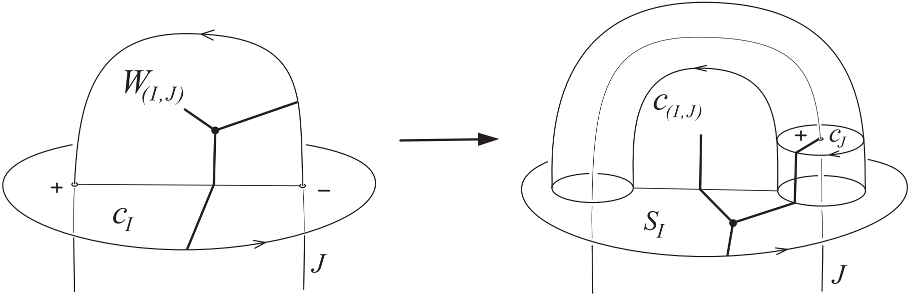
Whitney tower-to-grope construction: $D_i \mapsto G_i$ (Step 2 of proof)

Example of case $i = 1$:



Converting a Whitney tower to a grope

The ‘tree-preserving’ surgery step at a trivalent vertex.



See ‘Whitney towers and gropes in 4-manifolds’
arXiv:math/0310303 [math.GT]

Want to show that $S^3 \smallsetminus \partial D^i \to B^4 \smallsetminus \mathcal{W}^i$ induces an isomorphism on Milnor groups modulo the $(n+2)$th terms of the lower central series, so that $\mu_n^i(L)$ can be computed in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$.

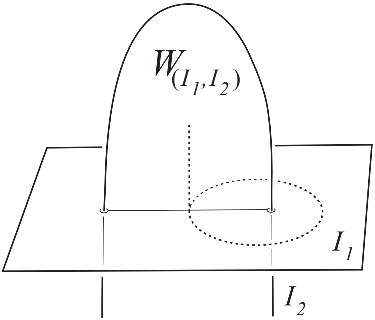
Will use the following consequence of **Dwyer–Freedman–Teichner's theorem** ('4-manifold topology II: Dwyer's filtration and surgery kernels' Inventiones 122 (1995)):

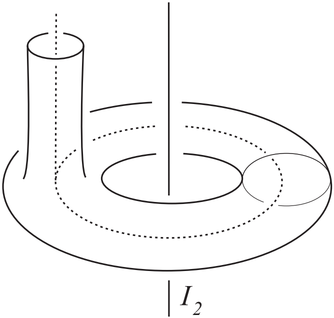
Thm: If the inclusion $Y \subset X$ induces an isomorphism $H_1 Y \cong H_1 X$, and $H_2(X)$ is generated by class $n+2$ gropes, then $Y \subset X$ induces $\pi_1 Y/(\pi_1 Y)_{n+2} \cong \pi_1 X/(\pi_1 X)_{n+2}$.

Proposition

If $\mathcal{V}$ is a split Whitney tower on $A : \cup A_j \looparrowright X^4$, where each order 0 surface A_j is a sphere $S^2 \to X$ or a disk $(D^2, \partial D^2) \to (X, \partial X)$, then there exist dyadic gropes $G_k \subset X \setminus \mathcal{V}$ such that the G_k are geometrically dual to a generating set for the relative first homology group $H_1(\mathcal{V}, \partial A)$. Furthermore, the tree $t(G_k)$ associated to each G_k is obtained by attaching a rooted edge to the interior of an edge of a tree t_p associated to an unpaired intersection p of $\mathcal{V}$.

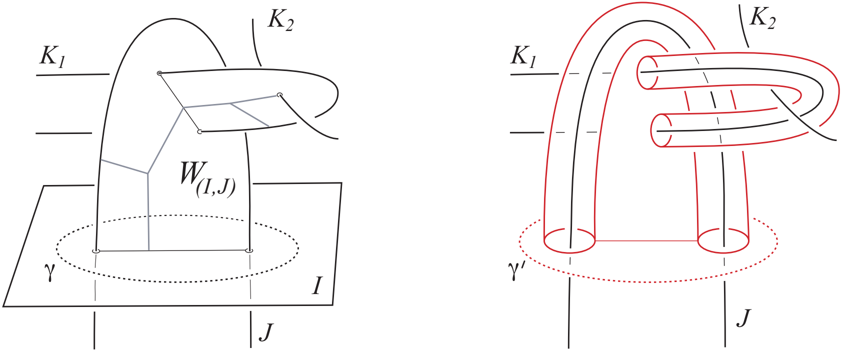
Here *geometrically dual* means that the bottom stage surface of each G_k bounds a 3–manifold which intersects exactly one generating curve of $H_1(\mathcal{V}, \partial A)$ transversely in a single point, and is disjoint from the other generators. In particular, there are as many gropes G_k as free generators of $H_1(\mathcal{V}, \partial A)$. Note that it follows from the last sentence of the proposition that if $\mathcal{V}$ is order n, then each G_k is class $n + 2$.





Lemma

Any meridian to a Whitney disk $W_{(I_1,I_2)}$ in a Whitney tower $\mathcal{V} \subset X$ bounds a grope $G_{(I_1,I_2)} \subset X \smallsetminus \mathcal{V}$ such that $t(G_{(I_1,I_2)}) = (I_1, I_2)$.



Lemma

Let $W_{(I,J)}$ be a Whitney disk in a split Whitney tower $\mathcal{V}$ such that $W_{(I,J)}$ contains a trivalent vertex of a tree $t_p = \langle (I,J), K \rangle$ associated to an unpaired intersection point $p \in \mathcal{V}$. If $\gamma \subset W_I$ is the boundary of a regular neighborhood in W_I of $\partial W_{(I,J)} \subset W_{(I,J)} \subset \mathcal{V}$; then the normal circle bundle T to W_I over γ is the bottom stage of a dyadic grope $G \subset (X \smallsetminus \mathcal{V})$, such that $t(G) = (I,(J,K))$.

Theorem:

(1) A link $L = L_1 \cup L_2 \cup \cdots L_m \subset S^3$ is almost trivial if and only if L bounds an order $m-2$ non-repeating Whitney tower $\mathcal{W} \subset B^4$.

Proof sketch of the 'if' direction: If L bounds an order $m-2$ non-repeating Whitney tower $\mathcal{W} \subset B^4$, then for each i the $(m-1)$-component link $L^i := L \setminus L_i$ bounds the order $m-2$ non-repeating Whitney tower $\mathcal{W}^i$ formed by deleting from $\mathcal{W}$ the order 0 disk D_i bounded by L_i and every Whitney disk whose tree contains a i-vertex. Hence L^i is link-homotopically trivial by the corollary to the 'Pulling apart surfaces' theorem.

Theorem:

(1) A link $L = L_1 \cup L_2 \cup \cdots L_m \subset S^3$ is almost trivial if and only if L bounds an order $m-2$ non-repeating Whitney tower $\mathcal{W} \subset B^4$.

Proof sketch of the 'only if' direction: Starting with an order 0 non-repeating Whitney tower (immersed disks) bounded by L, raise the order inductively to $m-2$ via the non-repeating intersection/obstruction theory, using that proper sublinks of L are homotopically trivial and that the non-repeating intersection invariant target $\Lambda_n(m)$ decomposes as a direct sum for each $n < m-2$. (Details on next slides.)

Proof: L almost-trivial $\implies$ L bounds order $m-2$ non-rep $\mathcal{W}$

Assume $L = L_1 \cup L_2 \cup \cdots \cup L_m \subset S^3$ is almost-trivial.
Will consider fixed $m \geq 3$.
(Case $m = 2$ follows from next observation.)

Observe that L bounds an order 0 non-repeating Whitney tower (link components bound immersed disks into B^4).

Proceed by induction on order: Assume that L bounds an order n non-repeating Whitney tower $\mathcal{W}_n$ for $0 \leq n \leq m-2$.

If $n = m-2$ then we're done.

For $n < m-2$, it will suffice to show that $\lambda_n(\mathcal{W}_n) = 0 \in \Lambda_n(m)$ to get an order $n+1$ non-repeating Whitney tower bounded by L (by Theorem 'non-repeating order-raising'). See next slide.

Proof: L almost-trivial $\implies$ L bounds order $m-2$ non-rep $\mathcal{W}$

For any $n+2$-element subset $s \subset \{1, 2, \ldots, m\}$ of distinct elements denote by $L(s) \subset L$ the sublink of components with labels in s. Let $\mathcal{W}_n^{s^*}$ denote the Whitney tower formed by deleting from $\mathcal{W}_n$ the order 0 disks labeled by elements of $s^* := \{1, 2, \ldots, m\} \setminus s$, and deleting any Whitney disk in $\mathcal{W}_n$ whose tree has at least one vertex labeled by an element of s^*. Then $\mathcal{W}_n^{s^*}$ is an order n non-repeating Whitney tower bounded by $L(s)$. Denote by $\Lambda_n(s)$ the order n non-repeating tree group on trees with distinct labels in s.

Since L is almost trivial, each L^s is homotopically trivial, so for each s we have $\lambda_n(\mathcal{W}_n^s) = 0 \in \Lambda_n(s^*)$ (by Thm/Cor "$\lambda(\mathcal{W}) = \mu(L)$").

Since $\Lambda_n(m)$ is the direct sum of the $\Lambda_n(s^*)$, and $\lambda_n(\mathcal{W}_n)$ is the sum of the $\lambda_n(\mathcal{W}_n^s)$ it follows that $\lambda_n(\mathcal{W}_n) = 0 \in \Lambda_n(m)$.

Theorem:
(2) The Milnor invariant μ_L (as in Pete's talk) is the image of the non-repeating intersection invariant $\lambda_{m-2}(\mathcal{W}) \in \Lambda_{m-2}(m)$ under projection to a direct summand of $\Lambda_{m-2}(m)$ isomorphic to $\mathbb{Z}[\mathcal{S}_{m-2}]$.

Proof sketch: Simple trees form a basis, and compute longitudes. Can use "Whitney move IHX" construction to arrange all trees in an order $m-2$ non-repeating $\mathcal{W}$ to have a m-labeled vertex at one end.

NOTE: The Whitney move IHX construction changes a Whitney tower by locally replacing a Whitney disk W_I with Whitney disks $W_H - W_X$, where the rooted trees I, H and X form an IHX relation. See Section 4.4 of 'Introduction to Whitney towers' arXiv:2012.01475 [math.GT]

Corollary: For almost-trivial links L and L' the following statements are equivalent:
(i) L and L' are link-homotopic.
(ii) $\mu_L = \mu_{L'} \in \mathbb{Z}[\mathcal{S}_{m-2}]$.
(iii) $\lambda_{m-2}(\mathcal{W}) = \lambda_{m-2}(\mathcal{W}') \in \Lambda_{m-2}(m)$ for any order $m-2$ non-repeating Whitney towers $\mathcal{W}$ and $\mathcal{W}'$ bounded by L and L', respectively.
Proof sketch:
(i) implies (ii), since μ_L is invariant under link homotopy.
(ii) implies (iii), by above Theorem "$\lambda(\mathcal{W}) = \mu(L)$" identifying Milnor invariants $\in \mathbb{Z}[\mathcal{S}_{m-2}]$ with $\lambda_{m-2}(\mathcal{W}) \in \Lambda_{m-2}(m)$.
(iii) implies (i), since can tube together $\mathcal{W}$ and $\mathcal{W}'$ to get immersed annuli in $S^3 \times I$ admitting an order $m-2$ non-repeating Whitney tower with vanishing λ_{m-2}, hence can be pulled apart.