June 5: Intersection forms in algebra & topology

Hermitian forms over an ring R with involution $r \mapsto \bar{r}$

$\lambda : M \times \overline{M} \to R$ is a pairing if
- M is a left R-mod
- λ is bi-additive
- $\lambda(rm, n) = r \cdot \lambda(m,n)$, $\lambda(m, n \cdot r) = \lambda(m,n) \cdot \bar{r}$

where $\overline{M} := M$ is right R-mod. given by $n \cdot r := \bar{r} n$

Def.: $\lambda^*(m,n) := \overline{\lambda(n,m)}$ defines another pairing and we call λ
- hermitian if $\lambda = \lambda^*$
- quadratic if $\lambda = q + q^*$

$m \mapsto \left(m' \mapsto \lambda(m,m')\right)$

There is an adjoint $\tilde{\lambda} : M \to \overline{M}^* = \mathrm{Hom}_R(\overline{M}, R)$ which is a homom. of left R-modules. We say that
- λ is non-degenerate if $\tilde{\lambda}$ is injective
- — " — non-singular — " — bijective "unimodular"

Examples of rings with involution: • R comm., $\bar{r} = r$

• $R \subseteq \mathbb{C}$ with complex conj. • $R = \mathbb{Z}[G]$ $\bar{g} = g^{-1}$ or $\omega : G \xrightarrow{\text{hom.}} \{\pm 1\}$ $\bar{g} = \omega(g) \cdot g^{-1}$

Example of hermitian forms:

• $M = R$ $\lambda(r, s) := r \cdot \bar{s}$ "identity form" $\tilde{\lambda} \hat{=} \mathrm{id}_R$

• $M = N \oplus N^*$ $\lambda(n + \varphi, n' + \varphi') = \varphi(n') \pm \overline{\varphi'(n)}$ "hyperbolic form"

• X compact connected $2n$-dim. mfd. "intersection form"

$M = H_n(X; \mathbb{Z}/2)$ $\lambda(m, n) = \langle PD(m), n \rangle \in \mathbb{Z}/2$

In oriented case, may use $R = \mathbb{Z}$. If m, n are represented by maps from closed n-mfds,

$m = f_*[M]$ then $\lambda(m, n) =$ (signed) number of transverse intersections

$n = g_*[N]$, $f \pitchfork g$ $= \Delta^{-1}(f \times g)$

$n = 1$:

Classification of surfaces reformulated

$$\frac{\left\{\begin{array}{l}\text{compact connected} \\ \text{surfaces w. boundary} \neq \emptyset\end{array}\right\}}{\text{diffeom.}} \xleftrightarrow[\text{and } \lambda]{H_1(-;\mathbb{Z}/2)} \frac{\left\{\begin{array}{l}\text{hermitian forms} \\ \text{on fin. dim.} \\ \mathbb{Z}/2\text{-modules}\end{array}\right\}}{\text{isom.}}$$

$$\frac{\left\{\begin{array}{l}\text{closed connected} \\ \text{surfaces}\end{array}\right\}}{\text{diffeom.}} \xleftrightarrow[\text{and } \lambda]{H_1(-;\mathbb{Z}/2)} \frac{\left\{\begin{array}{l}\text{non-singular} \\ \text{hermitian forms} \\ \text{on fin. dim.} \\ \mathbb{Z}/2\text{-modules}\end{array}\right\}}{\text{isom.}}$$

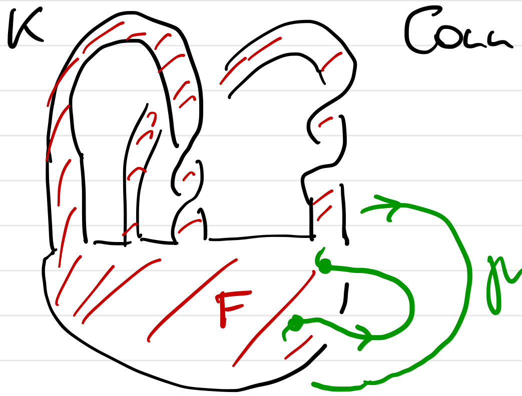
Remark: A closed $2n$-mfd. X has the same intersection form as $X \setminus \mathring{D}^{2n}$ $\longleftrightarrow$ $\partial = S^{2n-1}$.

$\mathrm{Rad}(\lambda) :=$ $\{ m \in M \mid \lambda(m,n) = 0 \ \forall n \in M \}$ **"radical"**

If $\lambda = \lambda_{\text{surface } F}$ then $\dim_{\mathbb{Z}/2} \mathrm{Rad}(\lambda) = |\pi_0(\partial F)| - 1$

Seifert surface F for a knot K gives a homomorphism $\pi_1(S^3 \setminus K) \twoheadrightarrow \mathbb{Z}$ as before:

needs orientations



Count transverse intersections with F

This does not depend on base point for μ and is equal to

$$\mu \longmapsto lk(K, \mu) \in \mathbb{Z}$$

Alternatively (and more symmetrically),

$$lk(K_1, K_2) = \# F_1 \pitchfork F_2 \quad \text{where} \quad F_i \subseteq D^4, \quad \partial F_i = K_i \subseteq S^3$$

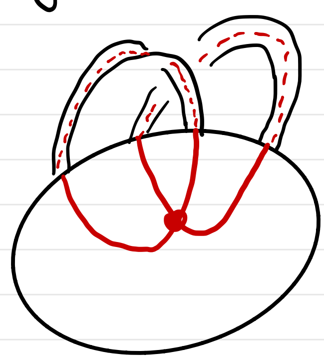
Well defined because ......

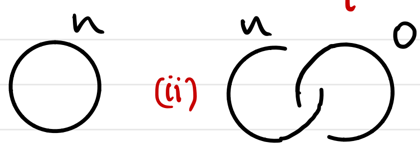
Q: When is there a $\pi_1(S^3 \setminus (K_1, K_2)) \twoheadrightarrow$ free group?

Scholium: If $L = (\ell_1, n_1), \dots, (\ell_m, n_m)$ is a framed link (no dots!) then the intersection form λ_{M_L} on $H_2 M_L \cong \mathbb{Z}^m$ is given by

$$\lambda_{M_L}(i,j) = \begin{cases} lk(\ell_i, \ell_j) & i \neq j \\ n_i & i = j \end{cases}$$

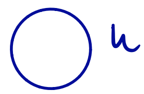
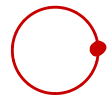
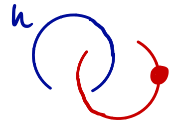
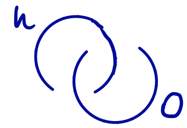
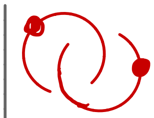
Proof: $M_L \simeq \bigvee^m S^2$ with free generators of H_2 given by $F_i \cup_{\ell_i} \mathrm{core}(h_i)$

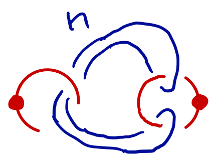
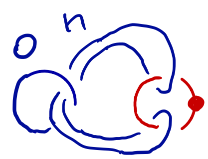
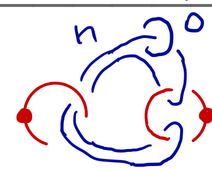


Examples: (i)  (ii)

$$\begin{array}{ccccc} H_2 M_L & \longrightarrow & H_2(M_L, \partial M_L) & \twoheadrightarrow & H_1(\partial M_L) \\ \lambda_{M_L} \downarrow & & \text{\rotatebox{90}{$\cong$}} & & \| \\ \mathrm{Hom}(H_2 M_L, \mathbb{Z}) & \cong & H^2 M_L & & \end{array}$$

$\mathbb{Z}/n$ in (i) and 0 in (ii).

L	$\varnothing$					
M_L	D^4	$S^2 \underset{n}{\tilde{\times}} D^2$	$S^1 \times D^3$	D^4	$S^2 \underset{n}{\tilde{\times}} S^2 \setminus \mathring{D}^4$	does
∂M_L	S^3	$L_{n,1} = S^2 \underset{n}{\tilde{\times}} S^1$	$S^1 \times S^2$	S^3	S^3	not
$\hat{M}_L$	S^4	$n=0,\pm 1$: $S^4, \pm\mathbb{C}P^2$	$S^1 \times S^3$	S^4	$S^2 \underset{n}{\tilde{\times}} S^2$	exist

L					???
M_L	n.o.h	$(S^1 \times S^1) \underset{n}{\tilde{\times}} D^2$	n.o.h	n.o.h	$K3 \setminus \mathring{D}^4$
∂M_L	$n \neq \pm 1$ n.o.h	$(S^1 \times S^1) \underset{n}{\tilde{\times}} S^1$	$(S^1 \times S^1) \underset{n}{\tilde{\times}} S^1$	$\#^2\, S^1 \times S^2$	S^3
$\hat{M}_L$	d.n.e	d.n.e	d.n.e.	$(S^1 \times S^1) \underset{n}{\tilde{\times}} S^2$	K3-surface

Freedman's classification theorem:

$$\left\{\begin{array}{l}\text{closed 1-connected}\\ \text{top. 4-manifolds}\end{array}\right\} \Big/ \text{homeomorphism} \;\underset{\lambda + KS}{\cong}\; \left\{\begin{array}{l}\text{unimodular sym. forms}\\ \text{on f.g. free abelian groups}\end{array}\right\} \Big/ \pm\text{isometry}$$

$\times\, \mathbb{Z}/2$ for odd forms

$$KS(M) = 0 \iff M \times \mathbb{R} \text{ smooth}$$

$$\iff M \# \#^{k} S^2 \times S^2 \text{ is smooth}$$

$$\iff \tau_M(c) \underset{(2)}{\equiv} \frac{\lambda_M(c,c) - \text{signature}(\lambda_M)}{8}$$

for a characteristic class $H_2 M \ni c : \lambda_M(c,x) \underset{(2)}{\equiv} \lambda_M(x,x) \;\forall x$

secondary intersection invariant for c.