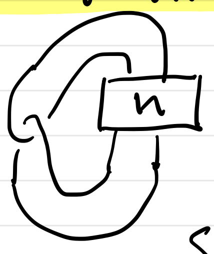
June 28 : L^2- signatures for knots

Recall that we want to prove that for a genus 1 surface $F \subseteq S^3$ s.t. $K = \partial F$ is top. slice, there is a circle $\gamma \subseteq F$ with
- $S_F(\gamma, \gamma) = 0$
- $\sigma_{\mathbb{Z}}(\gamma) = 0$.

Here $\sigma_{\mathbb{Z}}(\gamma) = \int_{S^1} \sigma_\omega(\gamma)$

where $\sigma_\omega(\gamma) := \text{signature}\left((1-\bar\omega) S_\gamma + (1-\omega) S_\gamma^t \right) \in \mathbb{Z}$

is the **Levine-Tristram signature**, S_γ = Seifert form for γ.

Ex.: $K_n =$ 

top. slice $\Rightarrow 4n+1 = \ell^2$ and γ is a $\left(\frac{\ell-1}{2}, \frac{\ell+1}{2}\right)$- torus knot.

twist knots, $n \in \mathbb{Z}$

So $\sigma_{\mathbb{Z}}(\gamma) = 0 \iff \ell = 1$ or 3
$\iff n = 0$ or 2 ∎

To motivate the new obstructions, recall the following criterion.

Slice-criterion: K is top. slice $\Longleftrightarrow$

$\exists$ W^4 with (0) $\partial W = S^0(K) =$ 0-surgery on K

compact, oriented, connected

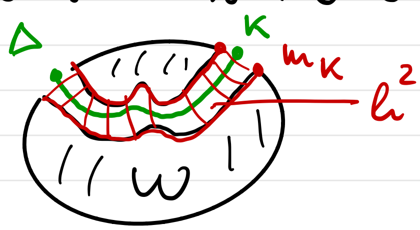
(1) $\pi_1 W$ is normally gen. by meridian of K,

(2) $H_2 W = 0$, $H_1 W \cong \mathbb{Z}$.

Proof: K slice via $\Delta \Rightarrow W := D^4 \setminus \mathring{\nu}(\Delta)$ satisfies properties by Alex. duality. 3

Given W, form $B := W \cup_{m_K} h^2$, a 4-mfd. with $\partial = S^3$.

By assumption, B is 1-connected and $H_2 B = 0$. By the Poincaré-conjecture $B \approx D^4$ and the cocore of h^2 provides a slice disk Δ for K ∎

$B =$ 

Remark: Everything we do below works for Poincaré complexes satisfying (0)–(2). This has not yet been explored!

Consider a c.c.o. 3-mfd. M^3 and $\phi : \pi_1 M \to \Gamma$

Assume that $M = \partial W^4$ and that $\exists$ $\pi_1 W \overset{\tilde{\phi}}{\dashrightarrow} \Gamma$

Thm.: The **reduced L^2-signature**

[Atiyah] $\sigma(W, \tilde{\phi}) := \sigma_\Gamma^{(2)}(W, \tilde{\phi}) - \sigma(W) \in \mathbb{R}$

only depends on the boundary (M, ϕ).

In fact, picking a metric g on M gives

$$\sigma_\Gamma^{(2)}(W, \tilde{\phi}) - \sigma(W) = \eta(M, g) - \eta_\Gamma^{(2)}(M, g, \phi)$$

As a consequence, this is a top. invariant, denoted by $\rho_\Gamma(M, \phi)$ independant of W, g !!

Using this result to obstruct sliceness is made difficult by the unknown group $\pi_1 W$, where W satisfies (0) - (2) in our Slice-criterion:

- For which $\pi_1 \mathring{S}(K) \xrightarrow{\phi} \Gamma$ should we compute ϱ_Γ ?
- We only know $H_2 W = 0$ (so $\sigma(W) = 0$) but need some homological algebra to show $\sigma^{(2)}_\Gamma W = 0$.

Let's start with $\Gamma = \mathbb{Z}$ for which we have a canonical epim. $\pi_1 \mathring{S} K \overset{\phi}{\twoheadrightarrow} \mathbb{Z}$.

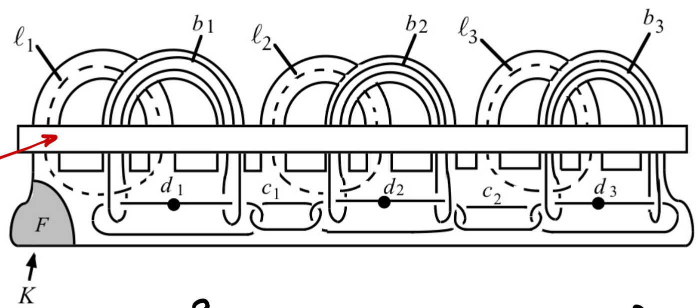
Lemma: $\varrho_{\mathbb{Z}}(\mathring{S}K, \phi) = \int_{w \in S^1} \sigma_w(K) \in \mathbb{R}$

Moreover, if K is alg. slice $\sigma_w(K) = 0$ a.e., more precisely, away from zeros of Alex. pol.

Proof: To compute $\rho_{\mathfrak{A}}$, we will find a 4-mfd. W with $\partial W = S^0K$ and such that $\partial W \hookrightarrow W$ induces iso. on H_1 ($\Rightarrow \tilde{\phi}$ ex.)

Start with a Seifert surface F for K and let

$W_F :=$



glue a pure string link with $2g$ comp.

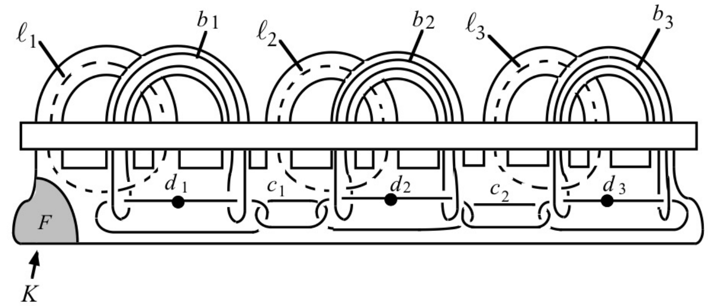
2-handle b_i are 0-framed, ℓ_i arb. acc. to F.

c_i not really needed: cancel with d_{i+1}, d_1 stays.

$\simeq S^1 \vee 2g \cdot S^2$, $\quad \partial W_F = S^0K$

λ_{W_F} is computed from null-homotopies of ℓ_i (into board) and b_i (out of board). The arising intersections only depend on the Seifert form S_F, i.e. linking numbers of bands.

$$S_F = \left(\begin{array}{c|c} l_i \cdot l_j & l_i \cdot b_j \\ \hline b_i \cdot l_j & b_i \cdot b_j \end{array}\right)$$



integral matrix

with $S_F - S_F^t = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)$. Then we have 3 blocks to compute:

$$\lambda_F(L_i, L_j) = S_F(l_i \cdot l_j) \text{ with } l_i\text{-framings on the diagonal.}$$

$$\lambda_F(B_i, L_j) = \delta_{ij} + (1-t) \cdot S_F(b_i \cdot l_j)$$

$$\lambda_F(B_i, B_j) = (1-t)(1-t^{-1}) \cdot S_F(b_i \cdot b_j)$$

Substituting $\omega \in S^1$ for t, it turns out that signature $\lambda_F(\omega) = $ signature $(1-\bar{\omega}) S_F + (1-\omega) S_F^t$ ∎