June 26 : C^*- and Von Neumann algebras

Let $(R, *)$ be a $\mathbb{C}$-alg. with 1 & involution (= complex conj. on $\mathbb{C}$)

If $r \in R$ define its spectral radius as

$$sr(r) := \sup_{z \in \mathbb{C}} \{ |z| \mid \underbrace{r - z \cdot 1 \text{ not invertible}}_{z \in \sigma(r)} \}$$

condition on $(R, *)$

Def.: R is a C^*-algebra if $\sqrt{sr(rr^*)}$ defines a complete norm on R.

Ex.: (i) $R \subseteq B(H)$ closed in norm = sr - topology

(ii) $R = C^0(X; \mathbb{C})$ for X a compact Hausdorff space.
Here sr is just the supremum norm.

Consequence: If $rr^* = r^*r$ the $f(r) \in R$ exists for any continuous $f: \sigma(r) \to \mathbb{C}$, not just for polynomials!

Thm.: (i) Any C^*-alg. is isom. to Ex. (i).
(ii) $\{\text{compact Hausdorff spaces}\} \xrightarrow[C^0]{\sim} \{\text{comm. } C^*\text{-alg.}\}$

Def.: A C^*-alg. $(R, *)$ is von Neumann if the underlying Banach space has a predual L, i.e. $L^* = R$
Banach space.

again just a property of R !

Ex.: (i) $R \subseteq B(H)$ closed in strong (or weak) topology

(ii) $R = L^\infty(X; \mathbb{C})$ for X a localizable measurable space.
Here $\|\ \|$ is just the supremum norm

Same consequence as above, except that $f \in L^\infty(\sigma(r); \mathbb{C})$
with spectral measure

Here X is equipped with

- a σ-alg. M of measurable sets
- a σ-ideal N of negligible sets

$\Big\}$ measurable

s.t. M/N admits arbitrary suprema (localizable)
$\iff$ $L^\infty(X; \mathbb{C})$ is a v.N. alg.

Thm.: (i) Any v.N.-alg. is isom. to Ex. (i).

(ii) $\left\{ \begin{matrix} \text{localizable} \\ \text{measurable spaces} \end{matrix} \right\} \xrightarrow[L^\infty]{\sim} \{\text{comm. v.N. alg.}\}$

This equivalence of categories uses the following notion of morphisms on the left hand side:

Consider $f: X \to X'$ s.t. $f^{-1}(M') \subseteq M$, $f^{-1}(N') \subseteq N$

Then a morphism is an equivalence class of such f where $f \sim g \;:\Leftrightarrow\; \{x \in X \mid f(x) \neq g(x)\} \in N$.

Note $[f] \in L^\infty(X;\mathbb{C})$ is not always represented by a morphism $X \to \mathbb{C}$ because we don't require $f^{-1}(\text{null}) \subseteq N$!

We still require $f^{-1}(\text{Borel set}) \subseteq M$ and take equiv. classes.

Remark: To embed $L^\infty(X)$ into $B(H)$ we choose a measure μ on X, comp. with M, N and consider the mult. action of $L^\infty(X)$ on $H := L^2(X,\mu)$.

If $L^\infty(X) \subseteq B(\text{separable } H)$ then X is isom. to exactly one measurable space in the following list: $X = \{1,\dots,n\}$, $\mathbb{Z}$, S^1, $S^1 \amalg \{1,\dots,n\}$, $S^1 \amalg \mathbb{Z}$ $\subseteq \mathbb{C}$

Example : $L^\infty(S^1)$ acts on $L^2(S^1)$ such that

$$\bar{z} = z^{-1} \quad z : S^1 \hookrightarrow \mathbb{C} \qquad \overset{\shortparallel}{H} \qquad \mathbb{C}[z^{\pm 1}] \subseteq L^\infty(S^1)$$

$\Rightarrow$ $L^\infty(S^1) \subseteq B(H)$ is generated as v.N. alg. by one unitary element z. $\qquad \mathbb{C}[\mathbb{Z}]$ dense in sup-norm

$\Rightarrow$ $\mathcal{N}\mathbb{Z} = L^\infty(S^1)$ using $\ell^2(\mathbb{Z}) \cong L^2(S^1)$

ONB $z^i \longleftrightarrow z^i$ polynomials

$$\begin{array}{ccc} \mathcal{N}\mathbb{Z} & & f \\ \downarrow{\scriptstyle tr_{\mathbb{Z}}} & & \downarrow \\ \mathbb{C} & & \langle f(\Omega), \Omega \rangle = \int_{S^1} f \cdot 1 \cdot \bar{1} = \int_{S^1} f \end{array}$$

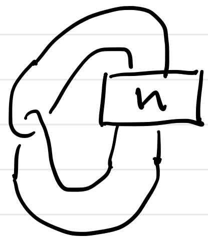
So the v.N. trace is a non-comm. version of integration. Also $\mathcal{N}\mathbb{Z}^k = L^\infty(T^k)$

$$\begin{array}{ccc} \mathcal{N}\mathbb{Z}^k & & f \\ \downarrow{\scriptstyle tr_{\mathbb{Z}^k}} & & \downarrow \\ \mathbb{C} & & \int_{T^k} f \end{array}$$

Key point : $H \leq G \Rightarrow \mathcal{N}H \subseteq \mathcal{N}G$

$$\begin{tikzcd} \mathcal{N}H \arrow[r, phantom, "\subseteq"] \arrow[dr, "tr_H"'] & \mathcal{N}G \arrow[d, "tr_G"] \\ & \mathbb{C} \end{tikzcd}$$

Consider the family of twist knots

K_n = 

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n full twists, $n \in \mathbb{Z}$.

Theorem [Casson-Gordon-Fintushel-Stern]:
ribbon obstructions - smooth Gauge theory obstr.

K_n is slice $\iff$ $n = 0, 2$

$\Updownarrow$

K_n bounds a smoothly embedded disk in D^4.

Our proof [Cochran-Orr-T.] works for topological (or even homotopical) slices.

Idea of proof: "$\Leftarrow$" Show that K_2 is ribbon.

"$\Rightarrow$" If K_n is slice then Alexander polynomial

$$A(K_n) = f(t) \cdot f(t^{-1}) \implies 4n+1 \text{ is a square}$$

i.p. $n > 0$.

In this case, there are exactly two circles $\gamma \subseteq F$ = genus 1 Seifert surface with self-linking $S(\gamma, \gamma) = 0$.

Thm. [COT2]: If K is a topologically slice knot with $A(K) \neq 1$ and a genus 1 Seifert surface F then $\exists$ circle $\gamma \subseteq F$ with $S(\gamma, \gamma) = 0$ and $0 = \int_{S^1} \sigma_\omega(\gamma)$, where σ_ω are Levine-Tristram signatures.

Recall that $\int_{S^1} \sigma_\omega(\gamma) = \sigma_{\mathbb{Z}}\left((1-z)S_\gamma + (1-z^{-1})S_\gamma^t\right)$

and to show that this is determined by K (not γ) we identify it with $\sigma_\Gamma(\lambda_K)$ for a metabelian group $\Gamma \twoheadrightarrow \mathbb{Z}$, and $\mathbb{Z}\Gamma$-form λ_K.

Thm.: Let $K : S^{4n-3} \hookrightarrow S^{4n-1}$ be a knot, $n>1$.
Then K is slice, i.e. $\exists$ $\begin{array}{ccc} S^{4n-3} & \hookrightarrow & S^{4n-1} \\ \cap & & \cap \\ D^{4n-2} & \hookrightarrow & D^{4n} \end{array}$ iff
K is algebraically slice, i.e. the
Seifert form S has a metabolizer

$$\begin{array}{cc} \frac{1}{2}\dim & " \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right)} \end{array}$$

Note: Slice $\Rightarrow$ alg. slice also holds for $n = 1$.

Outline of proof: Ambient surgery on $\begin{array}{ccc} S^{k} & \hookrightarrow & F^{4n-2} \subseteq S^{4n-1} \\ \cap & & \cap \\ D^{k+1} & \hookrightarrow & D^{4n} \end{array}$
for $k = 1, 2, \ldots, 2n-1$ turns F into D^{4n-2}.
Hardest case is to find $D^{2n} \xrightarrow{j_i} D^{4n}$ disjoint for $i = 1, \ldots, r$.
Need metabolizer to find Whitney disks ■