June 21: Signature invariants

Is there a classification of 1-conn. top. 4-mfds?

$\Updownarrow$ Freedman

—— " —— of unimodular forms over $\mathbb{Z}$?

Answer: Yes, for indefinite forms, in terms of rank, signature and type (even/odd)

Def.: • $r \in R$ is positive if $r = \sum_{i=1}^{n} r_i \overline{r_i} \neq 0$

• λ is positive if $\lambda(m,m)$ positive $\forall\, m \in M$.

Ex.: • $R = (\mathbb{C}, \bar{\ })$ gives $R_{>0} \subseteq \mathbb{R} \subseteq \mathbb{C}$, don't need sum.

• $R = \mathbb{Z}$ also gives $\mathbb{Z}_{>0} \subseteq \mathbb{Z}$. Need sum here!

λ is called indefinite if it is neither positive nor negative, for example, if $\exists\, m \neq 0$ s.t. $\lambda(m,m) = 0$. For some rings (e.g. $\mathbb{C}, \mathbb{Z}$) this latter cond. is equivalent to being indefinit.

Def.:

$$\begin{array}{ccc} h & \in & \mathrm{Herm}_n(\mathbb{C}) := \text{hermitian } n\times n \text{ matrices over } \mathbb{C} \\ \downarrow & & \updownarrow \langle , \rangle \\ \langle h(m), n \rangle & & \text{hermitian forms on } \mathbb{C}^n \end{array}$$

induces $\mathbb{C}^n = V_0 \perp V_- \perp V_+$ 0-, resp. pos., neg. Eigenspaces of h

Signature $(h) := \dim V_+ - \dim V_-$

$\quad\parallel$

Lemma: signature $(a^* h a)$ $\forall a \in GL_n\mathbb{C}$, i.e. it only depends on the isometry class of the underlying hermitian form.

Proof: $v \in a^{-1} V_+(h) \Rightarrow \langle (a^* h a) v, v \rangle = \langle h(av), av \rangle > 0$

(and similarly for V_0, V_-)

So $a^* h a /_{a^{-1} V_+(h)}$ is positive definit, hence $\dim V_+(h) \leq \dim V_+(a^* h a)$

Simil $\leq$ for V_- and V_0 and therefore $=$ ■

Thm.:

$$\left\{ \begin{array}{c} \text{non-deg. herm.} \\ \text{forms on fin.} \\ \text{dim. } \mathbb{C}\text{-vectorspaces} \end{array} \right\} \Big/ \text{isometry} \quad \xleftrightarrow[\text{signature}]{\text{rank}} \quad \left\{ (r,s) \in \mathbb{N}^2 \;\middle|\; r \overset{(2)}{\equiv} s \right\}$$

Here $sg(\lambda) = \#\,\text{positive } \lambda_i - \#\,\text{negative } \lambda_i$

$rk(\lambda) = \quad -\,"\,- \quad + \quad -\,"\,-$.

As a consequence, in every rank there is a unique positive (and negative) form.

$R = \mathbb{Z}$: In each rank, there are finitely many isom. classes of positive (definite) forms but the number of these grows exponentially!

If λ is indefinite (neither positive nor negative) then it contains an x with $\lambda(x,x) = 0$ and splits of $\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix}$. In fact $\lambda \cong 1^m \perp (-1)^n$ if λ is odd

$\cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $*$ odd. $\quad \lambda \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m \perp (\pm E_8)^n \quad \lambda$ is even

For $h \in \mathrm{Herm}_n(NG)$, consider the bounded self-adj. operator $h : \underbrace{(\ell^2 G)^n}_{= V} \circlearrowleft$

It has a compact spectrum

$$\mathbb{R} \supseteq \sigma(h) := \{ z \in \mathbb{C} \mid h - z \cdot id \text{ not invertible} \}$$

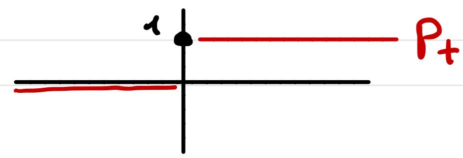
Thm.: The subalgebra of $B(V)$ generated by h (weakly closed, i.e. von Neumann alg.) is given by $L^\infty(\sigma(h))$ = bounded measurable fct. to $\mathbb{C}$

$$h \leftrightarrow id : \sigma(h) \subseteq \mathbb{C}$$

$$p(h) \leftrightarrow p \text{ polynomial in } \mathbb{C}[z]$$

$$f(h) \leftrightarrow f = \lim_{n \to \infty} p_n \quad \text{weak limit}.$$

$V_+ := p_+(h)$ where



Similarly, define V_- and V_0 and since $p_0 + p_- + p_+ = 1$ we get $V = V_+ \perp V_- \perp V_0$

Def.: $\operatorname{sign}_G(h) := \dim_{\mathcal{N}G} V_+ - \dim_{\mathcal{N}G} V_-$ von Neumann signature

Lemma: $\operatorname{sign}_G(h) = \operatorname{sign}_G(a^* h a) \quad \forall\, a \in GL_n(\mathcal{N}G)$

Proof exactly as above, using that $M_n(\mathcal{N}\Gamma) \subset B(V)$ is closed subalgebra, containing $p_{\pm, 0}(h)$ and hence $V_{\pm, 0}$ are $\mathcal{N}G$-modules. $\blacksquare$

Example: $h = \begin{pmatrix} z & 1 \\ 1 & z \end{pmatrix}$ for $z = t + t^{-1} - 2 \in \mathbb{C}[t^{\pm 1}]$

$\mathbb{C}[t^{\pm 1}] \subset \mathrm{Pol}(S^1) \subseteq L^\infty S^1 = \mathcal{N}\mathbb{Z}$

For each $u \in S^1$ get $h(u) \in \mathrm{Herm}_n \mathbb{C}$

$$\operatorname{sign}_{\mathbb{Z}}(h) = \int_{u \in S^1} \operatorname{sign} h(u)$$