July 17 : Filtrations of the knot concordance group

Recall the def. of $\Gamma_0 = \mathbb{Z}$, $\Gamma_{n+1} := \mathcal{K}\Gamma_n / \mathcal{R}\Gamma_n \rtimes \Gamma_n$

$$\begin{tikzcd}
 & & \Gamma_0 \\
Bl_0(x_0, -) : \pi_1 S_0 K \arrow[urr, "x_{-1}"] \arrow[rr, "x_0"] \arrow[drr, "x_1"] \arrow[ddrr, "x_2"] & & \Gamma_1 \arrow[u] \\
 & & \Gamma_2 \arrow[u] \\
 & & \vdots
\end{tikzcd}$$

$$\begin{tikzcd}
x_n \in H_1(S_0K; \mathcal{R}\Gamma_n) \arrow[d, "Bl_n", "\cong"'] \\
x_n \in \operatorname{Hom}_{\mathcal{R}\Gamma_n}\left(H_1(S_0K; \mathcal{R}\Gamma_n), \frac{\mathcal{K}\Gamma_n}{\mathcal{R}\Gamma_n}\right)
\end{tikzcd}$$

identify

Def.: K is rationally n-solvable, $n \in \frac{1}{2}\cdot\mathbb{N}_0$, if

$n = 0$: $\exists$ spin-4-mfd. W_0 s.t. $H_1(W_0)$ freely gen. by m_K: x_{-1} extends

$n = \frac{1}{2}$: –"– $S_0K \overset{j_{\frac{1}{2}}}{\hookrightarrow} W_{\frac{1}{2}}$ –"– giving Lagrangian $= \operatorname{Ker} j_{\frac{1}{2}}$ of $Bl_0 \Rightarrow \sigma^{(2)}_{\Gamma_0}(W_{\frac{1}{2}}, \tilde{x}_{-1}) = 0$

$n = 1$: –"– $S_0K \overset{j_1}{\hookrightarrow} W_1$ –"– $\forall x_0 \in \operatorname{Ker} j_1 = L_1$ the map x_0 extends to $\pi_1 W_1 \xrightarrow{\tilde{x}_0} \Gamma_1$

$n = 1.5$: –"– $W_{1.5}$ –"– giving Lagr. $L_2(x_0)$ for $Bl_1(x_0)$ $\forall x_0 \in L_1$

$\Longrightarrow \sigma^{(2)}_{\Gamma_1}(W_{1.5}, \tilde{x}_0) = 0 \ \forall x_0 \in L_1$.

etc. see COT 1.

It is clear that K slice $\Rightarrow$ K rat. n-solvable $\forall n \in \frac{1}{2}\cdot\mathbb{N}_0$

and that in the genus 1 case

we really proved that $\sigma^{(2)}_{\mathcal{Z}}(p) \neq 0 \Rightarrow K$ is not 1.5-solvable.

We actually get a filtration of the knot concordance group $\mathcal{C}$ that can be further enhanced in 2 steps.

Def.: K is n-solvable if $\exists$ n-solution W

with $\partial W = S_0 K$, $H_1 W$ freely gen by m_K

$n \in \mathbb{N}_0$: $H_2(W^{(n)}) \ni \langle L_1, \dots, L_g, D_1, \dots, D_g \rangle$ freely gen. $H_2 W$.

n-th derived cover, equiv. intersection form is hyperbolic

$n \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$: same, except that L_i are $(n+\frac{1}{2})$-surfaces
D_i are $(n-\frac{1}{2})$-surfaces

(n)-surfaces, lift by def. to $W^{(n)}$.

Thm: $\forall n \in \frac{1}{2}\cdot\mathbb{N}_0$ and K a knot

(0) K 0-solvable $\iff$ Arf $K = 0$

($\frac{1}{2}$) K $\frac{1}{2}$-solvable $\iff$ K alg. slice

(1.5) K 1.5-solvable $\Rightarrow$ all Casson-Gordon invariants of K vanish

(i) K n-solvable $\Rightarrow$ K rat. n-solvable

(ii) K bounds a symmetric grope of height $(n+2)$ in D^4 $\Rightarrow$ K is n-solvable

Moreover, $\forall n \in \mathbb{N}$ $\exists$ knots (Cochran-Harvey-Leidy ∞ indep. in $\mathcal{C}$) that bound $(n+2)$ gropes in D^4 but are not (rationally ?) n-solvable, see Cochran-T. Finally:

Lemma: $\exists$ Whitney-tower interpretations of (ii)