July 12 : Casson-Gordon invariants via L^2-signatures

We are about to show that a twist knot K_n is top. slice $\iff n = 0, 2$.

We will need finer & finer versions of the following Lagrangian Lemma:

Let W^{2n+1} be compact oriented. Then $L := \ker\left(H_n(\partial W; k) \to H_n(W; k)\right)$

This means that is a Lagrangian in $H_n(\partial W; k)$

1) The intersection form $\lambda_{\partial W / L} \equiv 0$

2) $\dim_k L = \frac{1}{2} \cdot \dim_k H_n(\partial W; k)$ for any coeff. field k

Proof: 1) If $\partial N_i \to \partial W$ represent elements in L, $i = 1, 2$

$\overset{\cap}{\text{oriented, compact}}$ $N_i^{n+1} \to W^{2n+1}$ then $\partial N_1 \pitchfork_{\partial W} \partial N_2 = \partial(N_1 \pitchfork_W N_2)$

$\leftrightarrow$:= same dim$_k$

2) $0 \to H_{2n+1} W \dots \to H_{n+1}(W, \partial) \to H_n \partial W \to H_n W \to H_n(W, \partial) \to \dots \to H_0(W, \partial) \to 0$

$L \qquad H_n \partial W / L$

Poincaré duality ∎

Cor.: Top. slice knots are alg. slice

Proof: $\begin{matrix} K & \subseteq & F^2 & \subseteq & S^3 \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ \Delta^2 & \subseteq & W^3 & \subseteq & D^4 \end{matrix}$ F is a Seifert surface and $H_1 F \cong H_1(F \cup \Delta^2)$ carries the Seifert form $S_F(\gamma_1, \gamma_2) = lk_{S^3}(\gamma_1, \gamma_2^{\uparrow})$. Let $k = \mathbb{Q}$ and $L \subseteq H_1 F$ consist of curves s.t. a multiple lies in $\ker(H_1(F \cup \Delta; \mathbb{Q}) \to H_1(W; \mathbb{Q}))$. Then $S_F/_L \equiv 0$:

$L \ni \gamma_i = \partial N_i$, $N_i^2 \subseteq W^3$. $W^3 \subseteq D^4$ is oriented so its normal bdl. is trivial $\Rightarrow$ $\gamma_i^{\uparrow} = \partial(N_i^{\uparrow}) \subseteq N_i^{\uparrow} \subseteq W^{\uparrow}$ i.e. $N_1 \cap_{D^4} N_2^{\uparrow} = \emptyset$

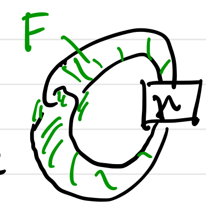
$\Rightarrow lk_{S^3}(\gamma_1, \gamma_2^{\uparrow}) = 0$ and L is a Lagrangian as required ∎

Let's come back to a genus 1 knot $K \subseteq F \subseteq S^3$ and assume it's top. slice via a disk $\Delta \xrightarrow{\text{flat}} D^4$.

Let $\mu \subseteq F$ represent the above Lagrangian for S_F.

Thm. A: If Alex. pol. $A_K \neq 1$ then $\sigma^{(2)}_{\mathbb{Z}}(\mu) = 0$

Rem.: The assumption is necessary since any knot μ (i.e. with $\sigma^{(2)}_{\mathbb{Z}}(\mu) \neq 0$) can arise from the above setting with K = Whitehead double of μ :=



These have $A_K = 1 \xRightarrow[\text{Freedman}]{}$ They are top. slice ∎

The proof of Thm. A is surprisingly tricky but it implies the slice result for twist knots K_n, so it can't be obvious!

Def.: A group Γ is PTFA "poly-torsion free-abelian"

if $\exists$ normal series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = \Gamma$ s.t.

the G_{i+1}/G_i are torsion free abelian.

Ex.: Γ is torsion free nilpotent, $\Gamma = \dfrac{\pi_1(S^3 \setminus K)}{n\text{-th term in derived series}}$

Key properties: (i) $\mathbb{Z}\Gamma$ has no zero-divisors and satisfies the Ore condition, i.e.

$\mathcal{K}\Gamma := \{ a \cdot b^{-1} \mid a \in \mathbb{Z}\Gamma,\ b \in \mathbb{Z}\Gamma \setminus 0 \}$ is a skew-field!

Note: $\forall\, a_1, b_1 \neq 0 \quad \exists\, a_2, b_2 \neq 0 : \quad b_1^{-1} a_1 = a_2 b_2^{-1} \iff a_1 b_2 = b_1 a_2$

(ii) Let $\partial : P_2 \to P_1$ be a hom. of projective $\mathbb{Z}\Gamma$-modules.

If $\partial \otimes_{\mathbb{Z}\Gamma} \mathbb{Q}$ is injective, so is ∂ (and hence $\partial \otimes_{\mathbb{Z}\Gamma} \mathcal{K}\Gamma$)

"locally indicable" groups have this property.

Outline of the prove of Thm. A: Let $W := D^4 \setminus \nu\Delta \Rightarrow \partial W = S_0 K$

Step 1: If $\pi_1 W \xrightarrow{\alpha} \Gamma$ is a non-trivial homom. to a PTFA then

homological algebra

$$H_2(W; \mathcal{K}\Gamma) = 0 \quad \text{and} \quad 0 = \sigma^{(2)}_\Gamma(W,\alpha) = \rho^{(2)}_\Gamma(S_0 K, \alpha|_{\dots})$$

Step 2: In the setting of Thm. A $\exists\, \alpha : \pi_1 W \to \Gamma$ s.t.

needs new idea

$$\rho^{(2)}_\Gamma(S_0 K, \alpha) = \rho^{(2)}_{\mathbb{Z}}(S_0 \mu, \text{proj.}) \quad \blacksquare$$

general property for $\mathcal{N}\Gamma$ + specific cob. over Γ from $S_0 K$ to $S_0 \mu$

The main problem is to construct Γ and a non-trivial homom. from the unknown group $\pi_1 W$.

Def.: A sequence $\Gamma_{n+1} \twoheadrightarrow \Gamma_n \twoheadrightarrow \dots \twoheadrightarrow \Gamma_0 = \mathbb{Z}$ of PTFA groups is defined inductively as follows: $\Gamma_{n+1} := \frac{\mathcal{K}\Gamma_n}{\mathcal{R}\Gamma_n} \rtimes \Gamma_n$

Here $\mathcal{R}\Gamma_n := \mathcal{K}[\Gamma_n, \Gamma_n][t^{\pm 1}]$ are (non-comm.) pid's, skew polynomial rings.

ex.: $H_1 \Gamma_{n+1} \cong \mathbb{Z}$

rational functions

Example: $\Gamma = \mathbb{Z} \implies \mathbb{Z}\Gamma = \mathbb{Z}[t^{\pm 1}]$

$\Gamma_1 = \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}] \rtimes \mathbb{Z} \impliedby \mathcal{R}\Gamma = \mathbb{Q}[t^{\pm 1}] \subseteq \mathcal{K}\Gamma = \mathbb{Q}(t)$

This is the group which we will use to get $\overset{1}{\overset{\#}{\alpha}} : \pi_1 W \to \Gamma_1$