July 12 : Casson-Gordon invariants via L^2-signatures

We are about to show that a twist knot K_n is top. slice $\iff n = 0, 2$.

We will need finer & finer versions of the following Lagrangian Lemma:

Let W^{2n+1} be compact oriented. Then $L := \ker\left(H_n(\partial W; k) \to H_n(W; k)\right)$

This means that is a Lagrangian in $H_n(\partial W; k)$

1) The intersection form $\lambda_{\partial W}/_L \equiv 0$

2) $\dim_k L = \frac{1}{2} \cdot \dim_k H_n(\partial W; k)$ for any coeff. field k

Proof : 1) If $\partial N_i \to \partial W$ represent elements in L, $i = 1,2$

$\qquad\qquad\quad \cap\!\!\!\!\!\shortmid \qquad\quad \cap\!\!\!\!\!\shortmid$

oriented, compact $N_i^{n+1} \longrightarrow W^{2n+1}$ then $\partial N_1 \pitchfork_{\partial W} \partial N_2 = \partial(N_1 \pitchfork_W N_2)$

$\longleftrightarrow$:= same $\dim_k$

2) $0 \to H_{2n+1} W \dots \to H_{n+1}(W, \partial) \to H_n \partial W \to H_n W \to H_n(W, \partial) \to \dots \to H_0(W, \partial) \to 0$

$L \qquad\qquad H_n \partial W / L$

Poincaré duality ■

Cor.: Top. slice knots are alg. slice

Proof:
$$\begin{array}{ccccc} K & \subseteq & F^2 & \subseteq & S^3 \\ \cap\!\shortparallel & & \cap\!\shortparallel & & \cap\!\shortparallel \\ \Delta^2 & \subseteq & W^3 & \subseteq & D^4 \end{array}$$

F is a Seifert surface and $H_1 F \cong H_1(F \cup \Delta^2)$ carries the Seifert form $S_F(\gamma_1, \gamma_2) = \mathrm{lk}_{S^3}(\gamma_1, \gamma_2^{\uparrow})$. Let $k = \mathbb{Q}$ and $L \subseteq H_1 F$ consist of curves s.t. a multiple lies in $\ker(H_1(F \cup \Delta; \mathbb{Q}) \to H_1(W; \mathbb{Q}))$. Then $S_{F/L} \equiv 0$:

$L \ni \gamma_i = \partial N_i$, $N_i^2 \subseteq W^3$. $W^3 \subseteq D^4$ is oriented so its normal bdl. is trivial $\Rightarrow$ $\gamma_i^{\uparrow} = \partial(N_i^{\uparrow}) \subseteq N_i^{\uparrow} \subseteq W^{\uparrow}$ i.e. $N_1 \cap_{D^4} N_2^{\uparrow} = \emptyset$

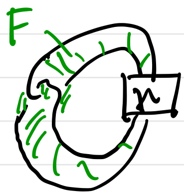
$\Rightarrow \mathrm{lk}_{S^3}(\gamma_1, \gamma_2^{\uparrow}) = 0$ and L is a Lagrangian as required ∎

Let's come back to a genus 1 knot $K \subseteq F \subseteq S^3$

and assume it's top. slice via a disk $\Delta \overset{\text{flat}}{\longleftrightarrow} D^4$.

Let $\gamma \subseteq F$ represent the above Lagrangian for S_F.

Thm. A: If Alex. pol. $\Delta_K \neq 1$ then $\sigma^{(2)}_{\mathbb{Z}}(\gamma) = 0$

Rem.: The assumption is necessary since any knot γ (i.p. with $\sigma^{(2)}_{\mathbb{Z}}(\gamma) \neq 0$) can arise from the above setting with K = Whitehead double of γ := 

.

These have $\Delta_K = 1 \xRightarrow[\text{Freedman}]{}$ They are top. slice ∎

The proof of Thm. A is surprisingly tricky but it implies the slice result for twist knots K_n, so it can't be obvious!

Def.: A group Γ is PTFA "poly-torsion free-abelian"

if $\exists$ normal series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = \Gamma$ s.t.

the G_{i+1}/G_i are torsion free abelian.

Ex.: Γ is torsionfree nilpotent, $\Gamma = \dfrac{\pi_1(S^3 \setminus K)}{\text{n-th term in derived series}}$

Key properties: (i) $\mathbb{Z}\Gamma$ has no zero-divisors and satisfies the Ore condition, i.e.

$\mathcal{K}\Gamma := \{ a \cdot b^{-1} \mid a \in \mathbb{Z}\Gamma,\ b \in \mathbb{Z}\Gamma \setminus 0 \}$ is a skew-field!

Note: $\forall\, a_1, b_1 \neq 0 \ \exists\, a_2, b_2 \neq 0 : \ b_1^{-1} a_1 = a_2 b_2^{-1} \iff a_1 b_2 = b_1 a_2$

(ii) Let $\partial : P_2 \longrightarrow P_1$ be a hom. of projective $\mathbb{Z}\Gamma$-modules.

If $\partial \otimes_{\mathbb{Z}\Gamma} \mathbb{Q}$ is injective, so is ∂ (and hence $\partial \otimes_{\mathbb{Z}\Gamma} \mathcal{K}\Gamma$)

"locally indicable" groups have this property.

Outline of the prove of Thm. A: Let $W := D^4 \setminus \nu\Delta \Rightarrow \partial W = S_0 K$

Step 1: If $\pi_1 W \xrightarrow{\alpha} \Gamma$ is a non-trivial homom. to a PTFA then

homological algebra

$$H_2(W; \mathbb{K}\Gamma) = 0 \text{ and } 0 = \sigma^{(2)}_\Gamma(W, \alpha) = \rho^{(2)}_\Gamma(S_0 K, \alpha|_{\dots})$$

Step 2: In the setting of Thm. A $\exists\, \alpha : \pi_1 W \to \Gamma$ s.t.

needs new idea

$$\rho^{(2)}_\Gamma(S_0 K, \alpha) = \rho^{(2)}_{\mathbb{Z}}(S_0 \mu, \text{proj.}) \quad \blacksquare$$

general property for $\mathcal{N}\Gamma$ + specific cob. over Γ from $S_0 K$ to $S_0 \mu$

The main problem is to construct Γ and a non-trivial homom. from the unknown group $\pi_1 W$.

Def.: A sequence $\Gamma_{n+1} \twoheadrightarrow \Gamma_n \twoheadrightarrow \dots \twoheadrightarrow \Gamma_0 = \mathbb{Z}$ of PTFA groups is defined inductively as follows: $\Gamma_{n+1} := \dfrac{\mathbb{K}\Gamma_n}{\mathcal{R}\Gamma_n} \rtimes \Gamma_n$

ex.: $H_1 \Gamma_{n+1} \cong \mathbb{Z}$

Here $\mathcal{R}\Gamma_n := \mathbb{K}[\Gamma_n, \Gamma_n][t^{\pm 1}]$ are (non-comm.) pid's, skew polynomial rings.

Example: $\Gamma = \mathbb{Z} \implies \mathbb{Z}\Gamma = \mathbb{Z}[t^{\pm 1}]$

rational functions

$\Gamma_1 = \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}] \rtimes \mathbb{Z} \Leftarrow \mathcal{R}\Gamma \cong \mathbb{Q}[t^{\pm 1}] \subseteq \mathcal{K}\Gamma = \mathbb{Q}(t)$

Let M^3 be closed oriented, $\pi_1 M \xrightarrow{\alpha \neq 1} \Gamma$ PTFA

s.t. $H_*(M;\mathcal{K}) = 0$, where $\mathcal{K} = \mathcal{K}\Gamma \supseteq \overset{\text{p.i.d.}}{\mathcal{R}} \supseteq \mathbb{Z}\Gamma$

Def.: Higher Blanchfield pairing of (M,α) is

$$Bl : H_1(M;\mathcal{R}) \xleftarrow[\cong]{\beta} H_2(M;\mathcal{K}/\mathcal{R}) \xleftarrow[\cong]{\text{P.D.}} H^1(M;\mathcal{K}/\mathcal{R}) \xrightarrow[\cong]{\mathcal{K}} H_1(M;\mathcal{R})^{\#}$$

$$H_1(M;\mathcal{R})^{\#} = \mathrm{Hom}_{\mathcal{R}}(H_1(M;\mathcal{R}), \mathcal{K}/\mathcal{R})$$

This is a non-singular hermitian linking form

$Bl(x,y) = \overline{Bl(y,x)}$ in analogy to $\mathbb{Q}/\mathbb{Z}$-valued linking form on $\mathbb{Z}$-Torsion$(H_1 M)$.

Note: $0 = H_1(M;\mathcal{K}) = H_1(M;\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{K} \iff H_1(M;\mathcal{R})$ is $\mathcal{R}$-torsion.

Let's construct & use Blanchfield forms for knots:

We are given a knot $K \subseteq S^3$ with slice complement W^4,

$$\begin{array}{ccc} K & \subseteq & S^3 \\ \cup\!\!\mid & & \cup\!\!\mid \\ \Delta & \subseteq & D^4 \end{array} \qquad \partial W = S_0 K$$

Homological Lemma: Let $X \in \{ S^3 \setminus K,\ S_0 K,\ W \}$ be a finite CW-complex, $\pi_1 X \xrightarrow{\alpha \neq 1} \Gamma$ PTFA.

Then $H_*(X; \mathcal{K}\Gamma) = 0$. If $\mathbb{Z}\Gamma \subseteq \overset{\text{p.i.d.}}{\mathcal{R}} \subseteq \mathcal{K}\Gamma$

$\ker\big(H_1(S_0 K; \mathcal{R}) \to H_1(W; \mathcal{R}) \big) =: P$ is a Lagrangian for the Blanchfield form: $P^\perp = P$.

Proof: $H_i(X; \mathcal{K}\Gamma) = H_i(X; \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{K}$ 10

- $H_0(X; \mathcal{K}\Gamma) = \mathbb{Z} \otimes \mathcal{K}\Gamma = 0$ since $\alpha \neq 1$
- $\mathbb{Q}$-Euler char. of $X^{\mathbb{Z}\Gamma}$ = $\mathcal{K}\Gamma$-Euler char. of X

(i) Start with $X = S^3 \setminus K$, a finite 2-complex $\Rightarrow$

$C_* := C_*(X_\Gamma)$ is a fin. gen. free $\mathbb{Z}\Gamma$-complex

$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

$H_2 X = 0 \Rightarrow \partial \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}$ is injective $\Rightarrow \partial \otimes_{\mathbb{Z}\Gamma} \mathcal{K}\Gamma$ is inj.

$\Rightarrow H_2(X; \mathcal{K}\Gamma) = 0$. But $H_1(X; \mathcal{K}) = 0$ now follows from $\mathcal{K}\Gamma$-Euler char $(X) = 0$ by dimension count ∎

Lemma: If a map $Y \to X$ (both finite complexes) is an n-equivalence on $H_*(-; \mathbb{Z})$ then also on $H_*(-; \mathcal{K}\Gamma)$ ∎

(ii) $Y = S^3 \setminus K \subseteq X = S_0 K$ is a 1-equiv. on $H\mathbb{Z}$

$\Leftrightarrow$ onto on $H_0, H_1 \Rightarrow H_i(X; \mathcal{K}\Gamma) = 0 \quad i = 0, 1$

This is also true for $i = 2, 3$ by P.D. + univ. coeff. ∎

(iii) $Y = S^3 \setminus K \subseteq X = W^4$ is a 2-equivalence

$\Rightarrow \quad H_i(W; \mathcal{K}\Gamma) = 0$ for $i = 0, 1, 2$.

Also true for $i = 3, 4$ by P.D. and (ii) $\blacksquare$

Finally, we need to show that $P = P^\perp$.

Consider the foll. diagram with $\mathcal{R}$-coeff., $M := S_0 K$: ($\mathcal{R}$-torsion!!)

$$\begin{tikzcd}
y \in H_2(W, M) \arrow[r, "\partial"] & H_1 M \arrow[r, "j_*"] & H_1 W \\
H_3(W, M; \mathcal{K}\Gamma/\mathcal{R}) \arrow[u, "\cong\ \beta"] \arrow[r, "\partial"] & H_2(M; \mathcal{K}\Gamma/\mathcal{R}) \arrow[u, "\cong\ \beta"] & \\
H^1(W; \mathcal{K}\Gamma/\mathcal{R}) \arrow[u, "\cong\ \mathrm{P.D}"] \arrow[r, "j^*"] \arrow[d, "\cong\ \kappa"] & H^1(M; \mathcal{K}\Gamma/\mathcal{R}) \arrow[u, "\cong\ \mathrm{P.D}"] \arrow[d, "\cong\ \kappa"] & \\
\beta y \in H_1 W^{\#} \arrow[r, "j^{\#}"] & H_1 M^{\#} &
\end{tikzcd}$$

"$P \subseteq P^\perp$": $\forall p \in P$

$Bl(\partial y) = j^{\#}(\beta y)$

$\Rightarrow B(\partial y, p) = \beta y(j_*(p)) = 0$

"$P^\perp \subseteq P$": $\forall x \in P^\perp$

want $x \longmapsto 0$

$H_1 M/P \overset{j_*}{\rightarrowtail} H_1 W$.

$H_1 W^{\#} \xrightarrow{j^{\#}} (H_1 M/P)^{\#} \subseteq H_1 M^{\#}$ (p.i.d)

$\partial y = x \iff \beta y \longmapsto Bl(x)$ $\blacksquare$

Now apply this for $\Gamma = \mathbb{Z}$, $\mathbb{Z}\Gamma \subseteq \mathcal{R} \subseteq \mathbb{K}\Gamma$

$A_0(K) := H_1(S_0K; \mathcal{R})$ is the $\Leftarrow$ $\mathbb{Z}[t^{\pm 1}] \subseteq \mathbb{Q}[t^{\pm 1}] \subseteq \mathbb{Q}(t)$

rational Alexander module, a fin. dim. $\mathbb{Q}$-vectorspace

$\Uparrow$

Alex. pol. $(K) \neq 1$ fin. gen. $\mathbb{Q}[t^{\pm 1}]$- torsion module

$\Updownarrow$

$A_0(K) \neq 0 \overset{K \text{ slice}}{\Longrightarrow} \exists\, 0 \neq \underset{\substack{\Uparrow \\ x}}{P} = P^\perp \subseteq A_0(K) = \frac{\pi_1 M'}{\pi_1 M''} \otimes_{\mathbb{Z}} \mathbb{Q}$

Bl is non-sing. $H_1 W^\# \overset{j^\#}{\twoheadrightarrow} \left(H_1 S_0 K / P \right)^\#$

$\qquad\qquad\qquad\qquad\qquad\quad \psi \qquad\qquad\qquad\quad \psi$

$\mathrm{Hom}_{\mathbb{Q}[t^{\pm 1}]}\left(H_1(W_{\mathbb{Z}}; \mathbb{Q}), \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \right) \ni y \longmapsto Bl(x) \neq 0 \Longleftrightarrow$

$\Longleftrightarrow H_1 W_{\mathbb{Z}} \rtimes \mathbb{Z} \xrightarrow{\tilde{y}} \frac{\mathbb{Q}(t)}{\mathbb{Q}[t^{\pm 1}]} \rtimes \mathbb{Z} =: \Gamma_1$ $\quad \alpha / \pi_1 M \neq 1$ on $[\pi_1 M, \pi_1 M]$

$\qquad\qquad\quad \shortparallel$

$\pi_1 W \twoheadrightarrow \pi_1 W / \pi_1 W''$

α iso. on H_1

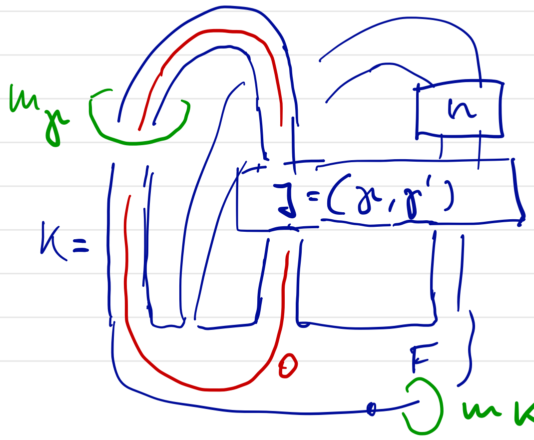
Final steps in the proof of Thm. A :

Relate the Seifert-form S_F and self-linking 0 curve $\gamma \subseteq F$ to $\mathcal{P}$: Genus $F = 1$, K alg. slice

$\Rightarrow A_0(K) \cong \mathbb{Q}[t^{\pm 1}] / A_K \cong \mathbb{Q}^2$ $\quad A_K \neq 1 \Leftrightarrow lk(\gamma, \gamma') \neq 0, 1$ $\quad S_F(\gamma_i, \gamma_i) = 0$

$\exists !$ two $\mathcal{P}_i \underset{1}{\overset{\cup\!\shortmid}{=}} \mathcal{P}_i^\perp$ that are generated by $\gamma_1, \gamma_2 \subseteq F$

[P. Gilmer] W picks one γ ! $\quad \gamma_i$ lift to $(\widetilde{S^3 \setminus K})_{\mathbb{Z}} \Leftarrow lk(\gamma_i, K) = 0$

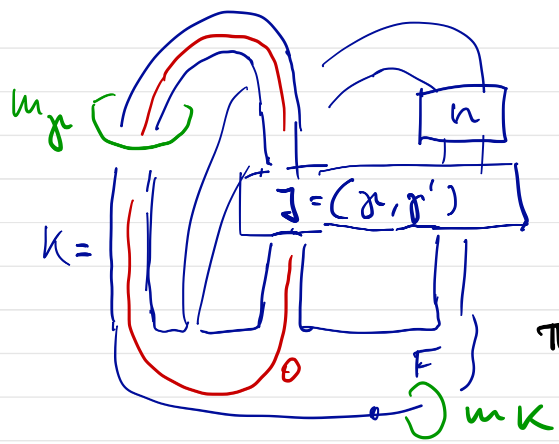


twists

untwisted string link

Moreover, $Bl(\tilde{\gamma}, \tilde{m}_\gamma) \neq 0$

since $A_0(K)$ is always generated by meridians to the 1-handles in F & $\gamma' \simeq lk(\gamma, \gamma') \cdot m_{\gamma'}$ in $S^3 \setminus F$
$\neq 0$



Let's say $W = D^4 \setminus \Delta$ slice complement

picks $\langle \gamma \rangle$ = Lagrangian $\subseteq H_1 F$.

$\tilde{m}_\gamma \mapsto \neq 0$

Then $Bl(\tilde{\gamma}, -): A_0 K \to \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$

$$\begin{tikzcd} \pi_1 S_0 K \arrow[r] \arrow[d] & \Gamma_1 & \Longleftrightarrow & A_0 K \arrow[d] \arrow[r] & \mathbb{Q}(t)/\mathbb{Q}[t^{\pm1}] \\ \pi_1 W \arrow[ur, dashed, "\alpha"] & & & A_0 W \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

By construction, $\alpha(\gamma) = 1 \neq \alpha(m_\gamma)$. This shows $\rho^{(2)}_{\Gamma_1}(K, \alpha) = 0$ but we also have

Lemma: $\rho^{(2)}_{\mathbb{Z}}(\gamma, \text{standard})$ proving Thm. A, see COT2.

Proof: $N^4 := (S_0 K \times I) \cup_{\gamma, 0} h^2 \cup h^3$ $\quad F/\gamma :=$ 2-sphere γ-surgery on F

$\partial N = S_0 K \sqcup S_0 \gamma$

$\pi_1 N \xrightarrow{\alpha} \Gamma_1$ restricts to "standard" on $\pi_1 S_0 \gamma$. Check that $H_2(N; \mathbb{K}\Gamma_1) = 0$ ∎