Intro to Whitney towers
Part 1A

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Possible outline of Whitney towers presentations

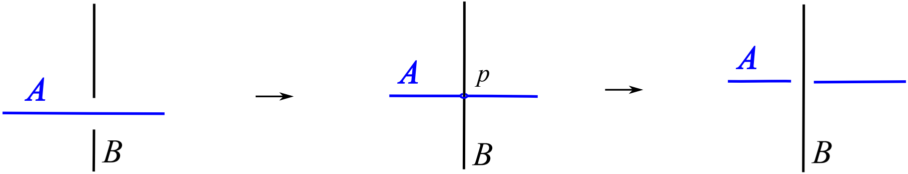
1. Surfaces in 4-space,
   Whitney towers and their trees,
   4-dimensional Jacobi identity

2. Higher-order intersection invariants,
   classification of order n twisted Whitney towers in B^4,
   higher-order Arf invariant conjecture

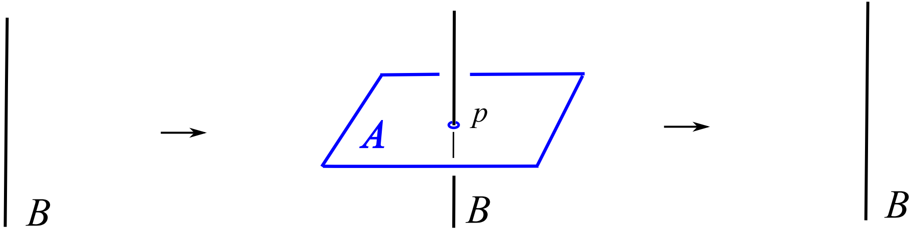
3. Whitney towers on disks in 4-manifolds,
   equivariant Milnor invariants

4. Intersection invariants for 2-spheres in 4-manifolds,
   More details...

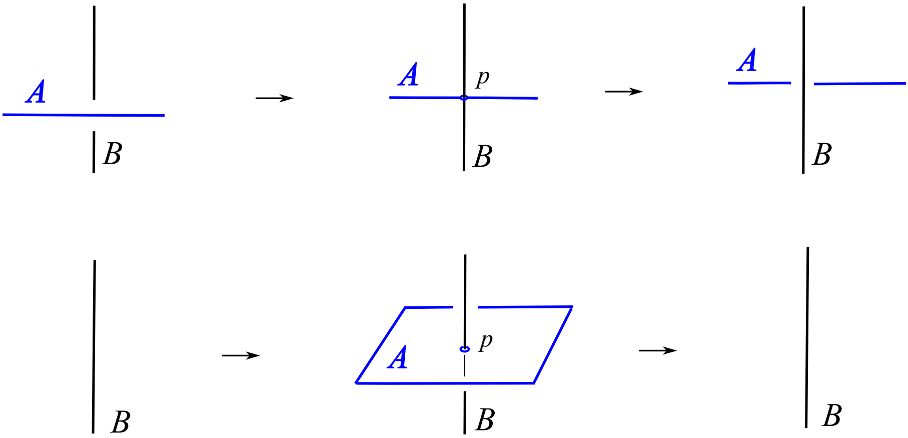
Surface sheets A and B in $B^4 = B^3 \times I$ with $p = A \pitchfork B$



A and B in $B^4 = B^3 \times I$ with $p = A \pitchfork B$ and $A \subset B^3 \times *$

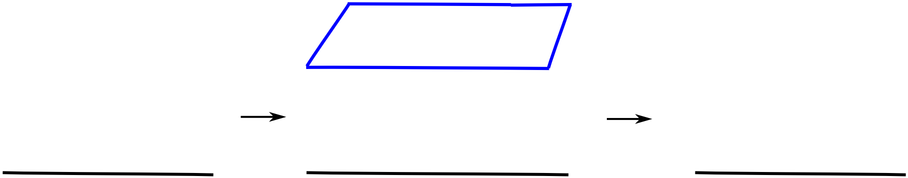


Two views of A and B in $B^4 = B^3 \times I$ with $p = A \pitchfork B$

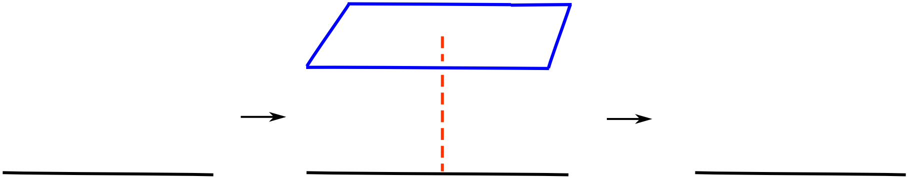


Visualize: Hopf link $= \partial A \cup \partial B \subset S^3 = \partial(B^3 \times I)$

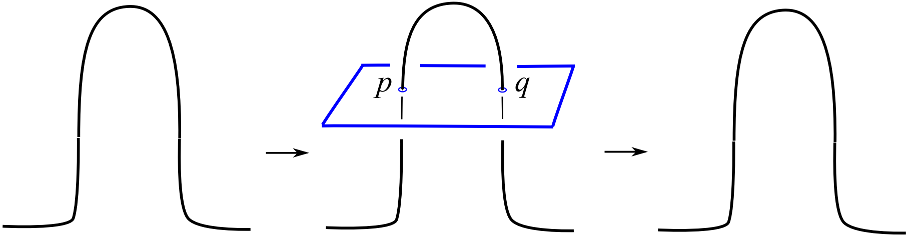
Disjoint surface sheets in $B^4 = B^3 \times I$



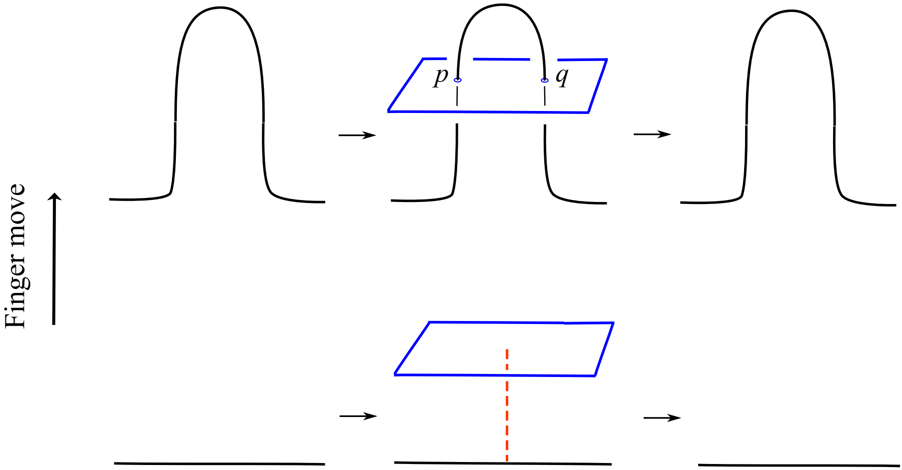
Guiding arc for *Finger Move*



After Finger Move

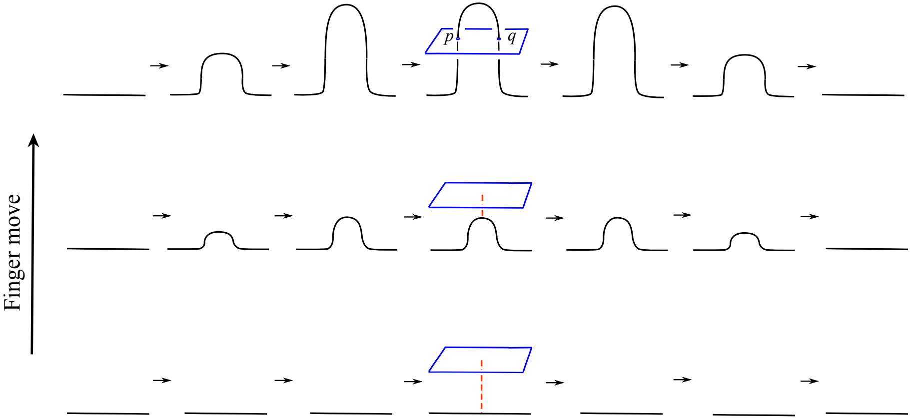


Finger move: Before and after



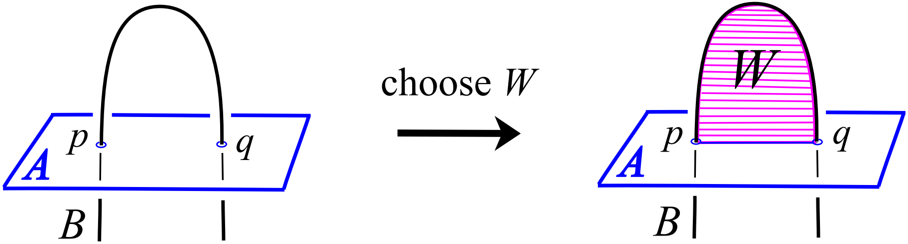
Will usually only show the center pictures.

Larger scale view of finger move

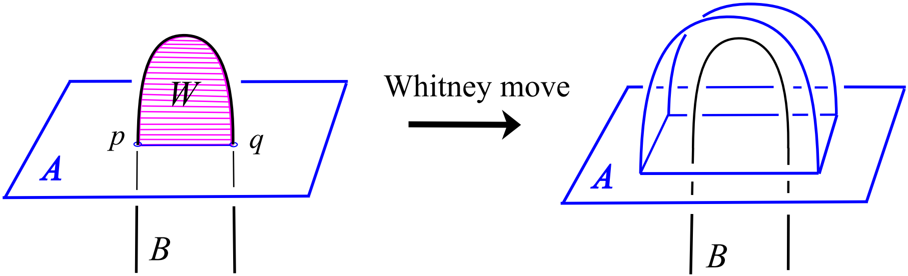


Will usually only show center top and/or center bottom pictures.

Intersections $p, q \in A \pitchfork B$ and a Whitney disk W pairing them

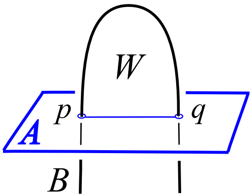


Whitney move: Before and after



Whitney disks in 4-manifolds

Have just seen a *model* Whitney disk W pairing $p, q \in A \pitchfork B$ in B^4:

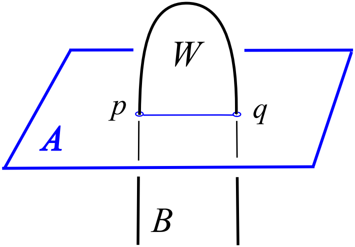


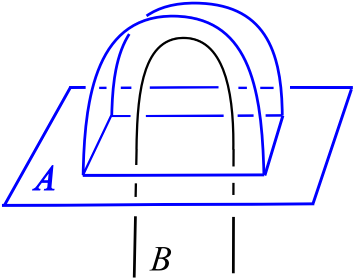
Definition:
A Whitney disk pairing $p, q \in A \pitchfork B$ in a 4-manifold X^4 has a neighborhood obtained by introducing *plumbings* into the model.

So a Whitney disk may have interior self-intersections and intersections with other surfaces.

'Successful' Whitney move: W is 'clean' and 'framed'

Eliminates $p, q \in A \pitchfork B$ without creating new intersections in A or B:





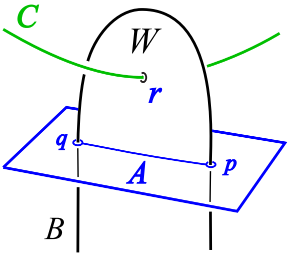
W is *clean* = embedded & interior disjoint from all surfaces.
W is *framed* = W has appropriate parallels.

Fact: 'Up to isotopy, a regular homotopy between surfaces in a 4-manifold is a sequence of finger moves and Whitney moves.'

Want to 'measure' obstructions to successful Whitney moves...

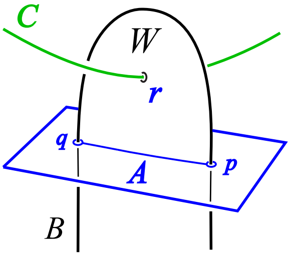
W not clean $\rightsquigarrow$ Whitney move creates new intersections:

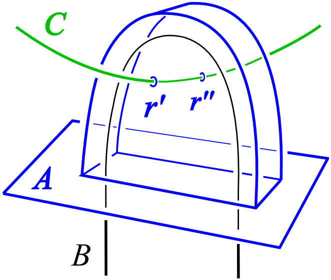
$r \in W \pitchfork C$:



W not clean $\rightsquigarrow$ Whitney move creates new intersections:

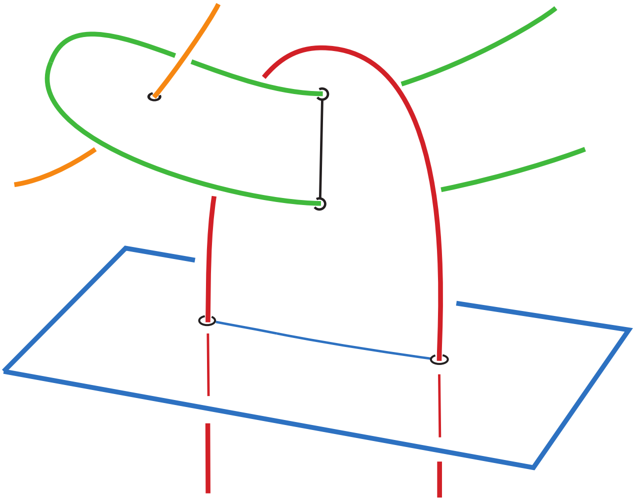
$r \in W \pitchfork C \quad \rightsquigarrow \quad r', r'' \in A \pitchfork C$ after W-move on A:





Visualize: The Borromean Rings $\partial A \cup \partial B \cup \partial C \subset \partial B^4$

Pair 'higher-order intersections' with 'higher-order Whitney disks'...?

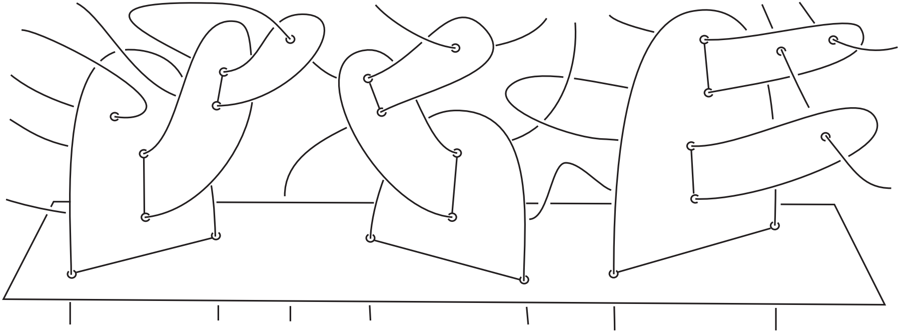


Visualize: The Bing-double of the Hopf link in ∂B^4.

Definition:

A *Whitney tower* on an immersed surface $A^2 \looparrowright X^4$ is defined by:

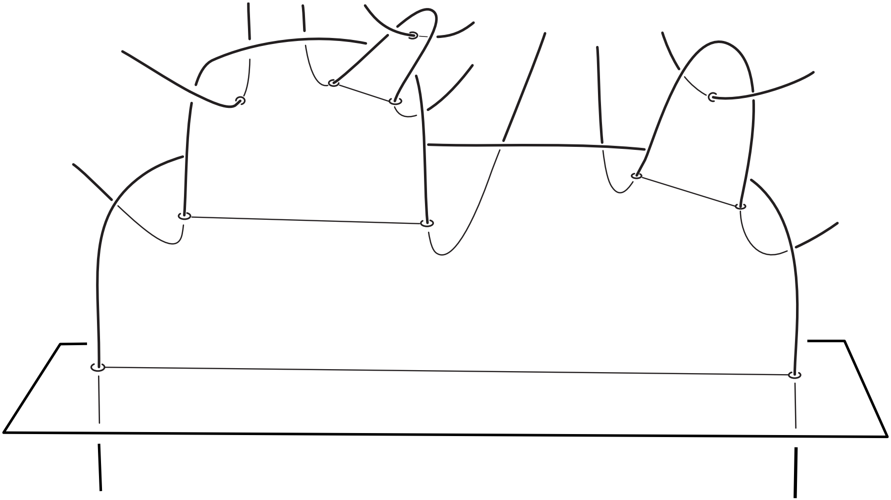
1. A itself is a Whitney tower.
2. If $\mathcal{W}$ is a Whitney tower and W is a Whitney disk pairing intersections in $\mathcal{W}$, then the union $\mathcal{W} \cup W$ is a Whitney tower.



Part of a Whitney tower!

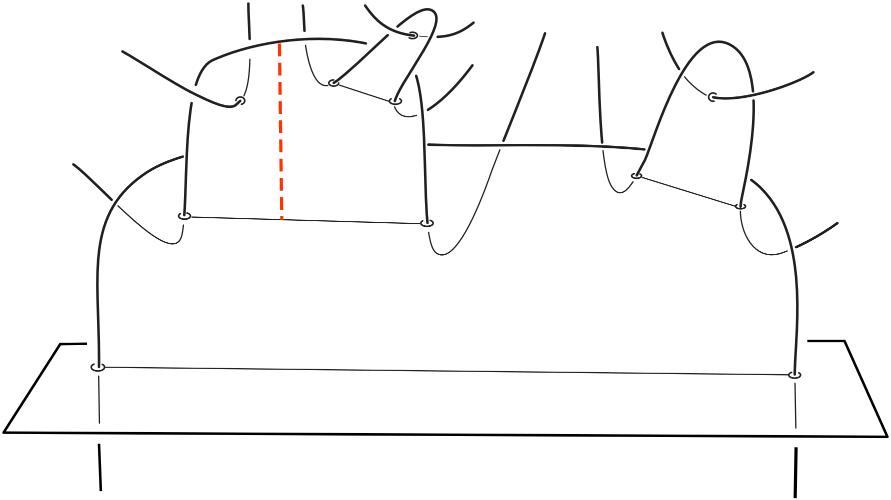
Goal: Study $\mathcal{W}$ to get info about A...

Splitting Whitney towers by finger-moves:



Towards organizing, understanding, controlling Whitney towers...

Splitting Whitney towers by finger-moves:



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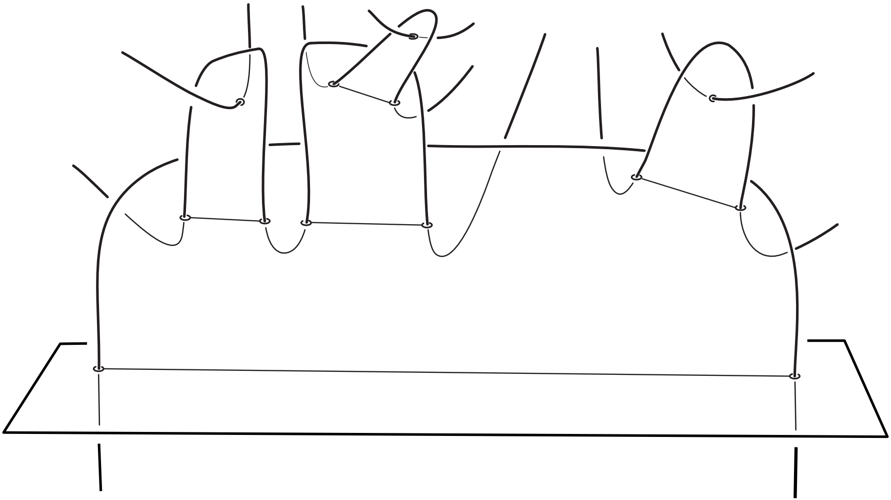
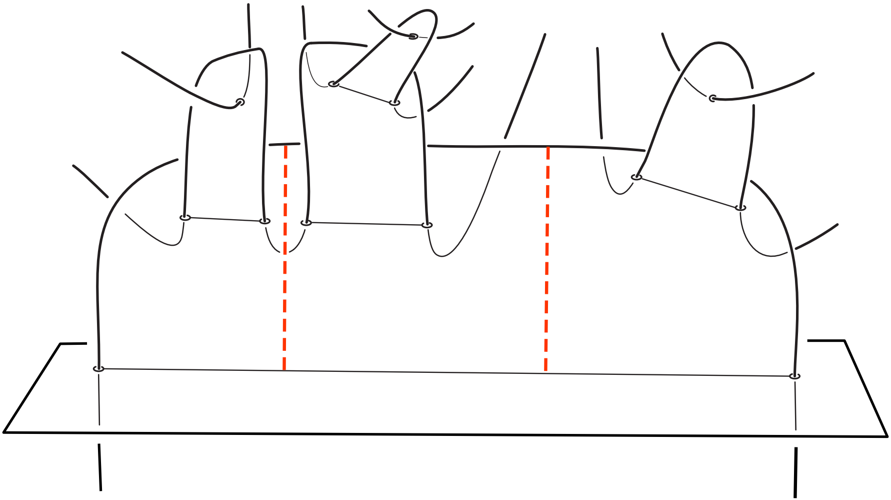
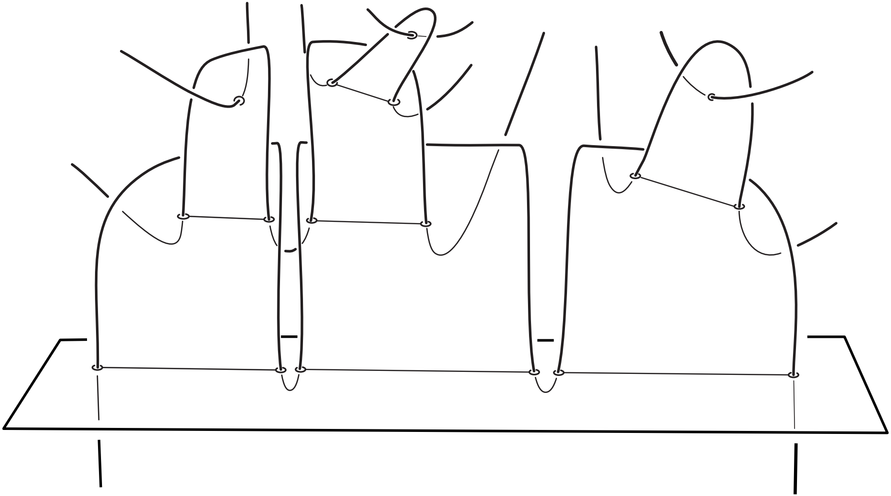


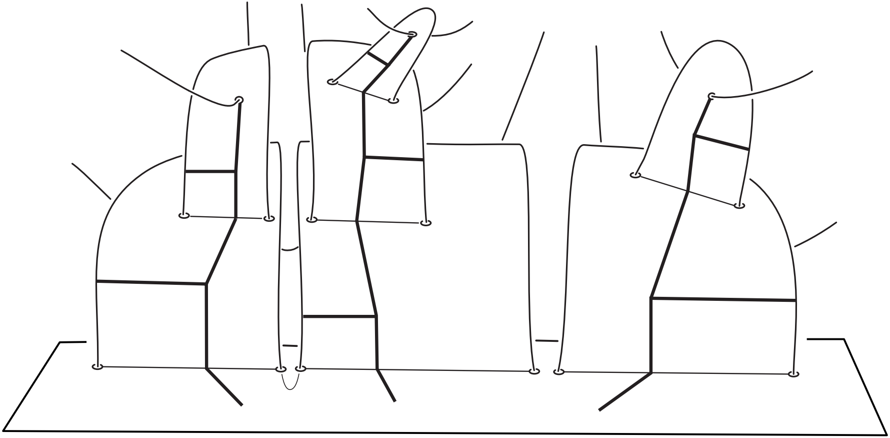
Splitting Whitney towers by finger-moves:



In a *split* Whitney tower each Whitney disk contains only one 'problem' (un-paired intersection or Whitney disk ∂-arc):

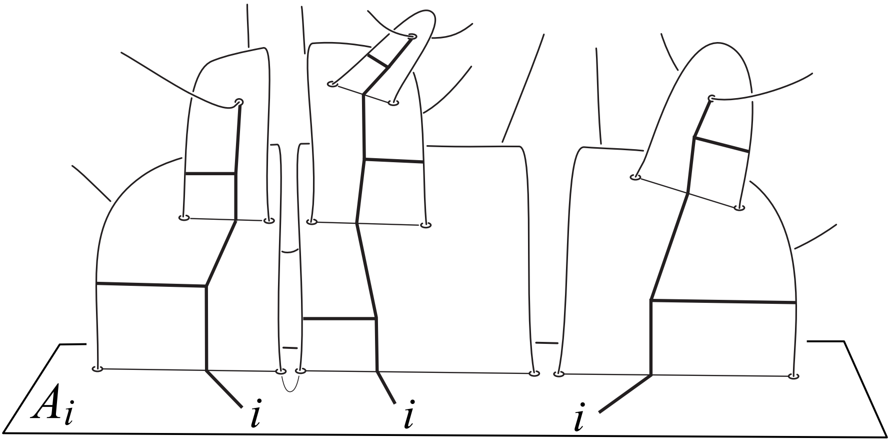


All singularities in split Whitney towers are near **trivalent trees**:



Trees 'bifurcate down' from unpaired intersections.

All singularities in split Whitney towers are near **trivalent trees**:



Trees 'bifurcate down' from unpaired intersections.
Univalent vertices inherit labels from components of the underlying properly immersed surface $A = A_1 \cup A_2 \cup \cdots \cup A_m$.

Preview: Sample Theorems

Will use trees to grade complexity of Whitney towers by *order*...

Theorem: (twisted Whitney towers in B^4)

A link $L \subset S^3$ bounds an order $n+1$ twisted Whitney tower $\mathcal{W} \subset B^4$ if and only if L has vanishing Milnor invariants and higher-order Arf invariants through order n.

Theorem: (Pulling apart surfaces in 4-manifolds)

$A = \cup_{i=1}^m A_i$ bounds an order $m-1$ non-repeating $\mathcal{W} \subset X^4$ if and only if A is homotopic to $A' = \cup_{i=1}^m A'_i$ with $A'_i \cap A'_j = \emptyset$ for all $i \neq j$.