



# Jacobi identities in low-dimensional topology

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## ABSTRACT

The Jacobi identity is the key relation in the definition of a Lie algebra. In the last decade, it has also appeared at the heart of the theory of finite type invariants of knots, links and 3-manifolds (and is there called the IHX relation). In addition, this relation was recently found to arise naturally in a theory of embedding obstructions for 2-spheres in 4-manifolds in terms of Whitney towers. This paper contains the first proof of the four-dimensional version of the Jacobi identity. We also expose the underlying topological unity between the three- and four-dimensional IHX relations, deriving from a beautiful picture of the Borromean rings embedded on the boundary of an unknotted genus 3 handlebody in 3-space. This picture is most naturally related to knot and 3-manifold invariants via the theory of grope cobordisms.

## 1. Introduction

The only axiom in the definition of a Lie algebra, in addition to the bilinearity and skew symmetry of the Lie bracket, is the *Jacobi identity*

$$[[a, b], c] - [a, [b, c]] + [[c, a], b] = 0.$$

If the Lie algebra arises as the tangent space at the identity element of a Lie group, the Jacobi identity follows from the associativity of the group multiplication. Picturing the Lie bracket as a rooted Y-tree with two inputs (the tips) and one output (the root), the Jacobi identity can be encoded by the diagram in Figure 1.

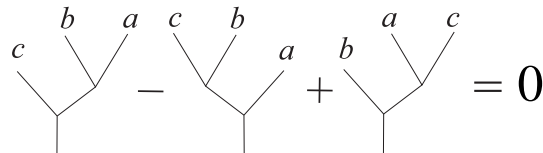


FIGURE 1. The Jacobi identity.

One should read this tree from top to bottom, and note that the planarity of the tree (together with the counterclockwise orientation of the plane) induces an ordering of each trivalent vertex which can thus be used as the Lie bracket. A change of this ordering just introduces a sign due to the skew symmetry of the bracket. This will later correspond to the antisymmetry relation for diagrams. Changing the input letters  $a, b, c$  to  $1, 2, 3$  and labeling the root 4, Figure 1 may be redrawn with the position of the labeled univalent vertices fixed as shown in Figure 2.

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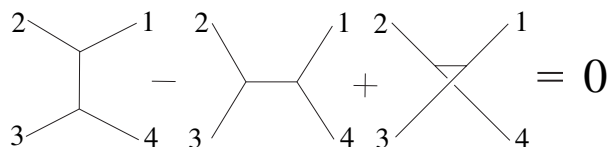


FIGURE 2. The IHX relation.

This (local) relation is an unrooted version of the Jacobi identity, and is well known in the theory of finite type (or Vassiliev) invariants of knots, links and 3-manifolds. Because of its appearance it is called the IHX relation. The precise connection between finite type invariants and Lie algebras is very well explained in many references, see e.g. [BarN95].

Garoufalidis and Ohtsuki [GO98] were the first to prove a version of a three-dimensional IHX relation. It was needed to show that a map from trivalent diagrams to homology 3-spheres was well defined. Habegger [Hab01] improved and conceptualized their construction. Moving to the techniques of claspers (clovers), Garoufalidis, Goussarov and Polyak [GGP01] sketch a proof of Theorem 7 below, a theorem of which Habiro was also aware (private communication). Our proof is completely new, and, we believe, more conceptual. Moreover, it serves as a bridge between the three- and four-dimensional worlds.

### 1.1 A Jacobi identity in four dimensions

In §2 of this paper we will rediscover the Jacobi relation in the context of intersection invariants for Whitney towers in 4-manifolds. It is actually a direct consequence of a beautifully symmetric picture, Figure 3. The expert will see three standard Whitney disks whose Whitney arcs are drawn in an unconventional way (to be explained in §2.3 below). Ultimately, the freedom of choosing the Whitney arcs in this way forces the Jacobi relation upon us.

The reader will recognize the three-component link in the figure as the Borromean rings. Each component consists of a semicircle and a planar arc (solid, dashed, dotted respectively), exhibiting the Borromean rings as embedded on the boundary of an unknotted genus 3 handlebody in 3-space.

The Jacobi relation for Whitney towers plays a key role in the obstruction theory for embedding 2-spheres into 4-manifolds developed in [ST04]. However, it was not proven in that reference and the main purpose of this paper is to give a precise formulation and proof of this Jacobi relation; see Theorem 1 below. In §§2.1, 2.2 and 2.3 of this paper, no background is required of the reader beyond a willingness to try to visualize surfaces in 4-space, and our elementary construction can also serve as an introduction to Whitney towers.

Roughly speaking, a Whitney tower is a 2-complex in a 4-manifold, formed inductively by attaching layers of Whitney disks to pairs of intersection points of previous surface stages; see §2.1. A Whitney tower has an *order* which measures how many layers were used. Moreover, for any unpaired intersection point  $p$  in a Whitney tower  $\mathcal{W}$  of order  $n$ , one can associate a tree  $t(p)$  embedded in  $\mathcal{W}$ ; see Figure 7. The tree  $t(p)$  is a trivalent tree with  $n$  trivalent vertices, each representing a Whitney disk in the tower. Each univalent vertex of  $t(p)$  lies on a bottom stage (immersed) sphere  $A_i$  and is labeled by the index  $i$ .

Orientations of the surface stages in  $\mathcal{W}$  give vertex orientations of  $t(p)$ , i.e. cyclic orderings of the trivalent vertices, and they also give a sign  $\epsilon_p$ . We define the *geometric intersection tree*  $\tilde{\tau}_n(\mathcal{W})$  as the disjoint union of signed vertex-oriented trees, one for each unpaired intersection point  $p$ :

$$\tilde{\tau}_n(\mathcal{W}) := \coprod_p \epsilon_p \cdot t(p).$$

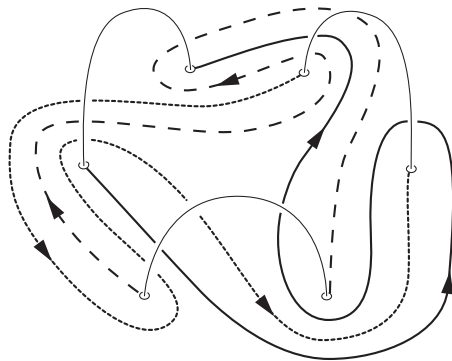


FIGURE 3. The geometric origin of the Jacobi identity in dimension 4.

Properly interpreted, this union represents an obstruction to the existence of an order  $(n + 1)$  Whitney tower extending  $\mathcal{W}$ . (Note that essentially the same geometric intersection tree is denoted by  $t_n(\mathcal{W})$  in [ST04].) The main result of this paper can now be formulated as follows.

**THEOREM 1** (Four-dimensional Jacobi relation). *There exists an order 2 Whitney tower  $\mathcal{W}$  on four immersed 2-spheres in the 4-ball such that  $\tilde{\tau}_2(\mathcal{W}) = (+I) \amalg (-H) \amalg (+X)$ , where  $I$ ,  $H$  and  $X$  are the trees shown in Figure 2.*

This result comes from the fact alluded to before, namely that Whitney towers have the indeterminacy of choosing the Whitney arcs! The local nature of Theorem 1 enables geometric realizations of Jacobi relations via controlled manipulations of Whitney towers, an essential step in the obstruction theory described in [ST04]. It should be mentioned that there is also a four-dimensional geometric Jacobi relation which uses a Whitney move to locally replace an  $I$ -tree by an  $H$ -tree and an  $X$ -tree (see [Sch06]).

In the easiest case  $n = 0$ , a Whitney tower (of order 0) is just a union of immersed 2-spheres  $A_1, \dots, A_\ell : S^2 \looparrowright M^4$ , and its geometric intersection tree  $\tilde{\tau}_0(\bigcup_i A_i)$  is a disjoint union of signed edges, one for each intersection point among the  $A_i$ . The endpoints of the edges are labeled by the 2-spheres, or better by elements of the set  $\{1, \dots, \ell\}$ , organizing the information as to which  $A_i$  are involved in the intersection. Edges with index  $i$  on both ends correspond exactly to self-intersections of  $A_i$ .

In this case we know how to extract an invariant, namely by just summing all the order 0 trees (= edges) in  $\tilde{\tau}_0(\bigcup_i A_i)$ , each signed edge of  $\tilde{\tau}_0$  thought of as an integer  $\pm 1$ , to get exactly the *intersection numbers* among the  $A_i$ . Actually, if  $M$  is not simply connected, these ‘numbers’ should be evaluated in the group ring of  $\pi_1 M$ , rather than in  $\mathbb{Z}$ , leading to Wall’s intersection invariants [Wal70]. This corresponds to putting orientations and group elements on the edges of each  $t(p)$ , and has been worked out in [ST04] for higher order Whitney towers. Note that for identical indices at the ends of an edge, the two possible orientations on the edge give isomorphic pictures, leading to the usual relations in the group ring when measuring self-intersections:

$$w_1(g) \cdot g = g^{-1}, \quad \text{for all } g \in \pi_1 M.$$

In the present paper our constructions are local so that we may safely ignore these group elements.

If  $\tilde{\tau}_0(\bigcup_i A_i)$  sums to zero then all the intersections can be paired up by Whitney disks, i.e. there is a Whitney tower  $\mathcal{W}$  of order 1 with the  $A_i$  as bottom stages. Then  $\tilde{\tau}_1(\mathcal{W})$  is a disjoint union of signed (vertex-oriented)  $Y$ -trees, and again the univalent vertices have labels from  $\{1, \dots, \ell\}$ . It was shown in [ST01] (and in [Mat78, Yam79] for simply connected 4-manifolds) that a summation as above leads to an invariant  $\tau_1(\mathcal{W})$  which vanishes if and only if there is a Whitney tower  $\mathcal{W}$  of

order 2 with the  $A_i$  as bottom stages. In fact, if defined in the correct target group,  $\tau_1(\mathcal{W})$  only depends on the regular homotopy classes of the  $A_i$  and hence is a well-defined higher obstruction for representing these classes by disjoint embeddings.

Theorem 1 only becomes relevant for  $\tilde{\tau}_2$  and higher, and we next give a proper formulation of some necessary notation and terminology for intersection trees.

DEFINITION 2. We define the *order* of a trivalent tree to be the number of trivalent vertices, and the *degree* to be one more than that number. The degree is also one-half of the total number of vertices, or one less than the number of univalent vertices. This definition is consistent with the theory of finite type invariants, where the degree goes back to Vassiliev.

DEFINITION 3. Consider pairs  $(\epsilon, t)$  where  $\epsilon \in \{\pm\}$  and  $t$  is a vertex-oriented trivalent tree of degree  $n$ , with univalent vertices labeled from  $\{1, \dots, \ell\}$ .

- (i) An AS (antisymmetry) relation is of the form

$$(\epsilon, t) = (-\epsilon, t'),$$

where  $t'$  is isomorphic to  $t$  and its orientation differs from that of  $t$  by changing the cyclic orientation at a single vertex. All AS relations generate an equivalence relation, and we let  $\tilde{\mathcal{B}}_n^t(\ell)$  be the commutative monoid with unit generated by the set of equivalence classes of such pairs  $(\epsilon, t)$ . We think of the monoid operation as disjoint union,  $\amalg$ , and we write  $\epsilon \cdot t$  for the equivalence class of  $(\epsilon, t)$ .

- (ii) The abelian group  $\hat{\mathcal{B}}_n^t(\ell)$  is obtained by dividing the monoid  $\tilde{\mathcal{B}}_n^t(\ell)$  by all relations of the form

$$(\epsilon \cdot t) \amalg (-\epsilon \cdot t) = 0,$$

where 0 is the unit of the monoid  $\tilde{\mathcal{B}}_n^t(\ell)$ . This clearly introduces inverses and the monoid operation  $\amalg$  becomes a group addition which we write as ‘+’.

- (iii) The abelian group  $\mathcal{B}_n^t(\ell)$  is obtained from  $\hat{\mathcal{B}}_n^t(\ell)$  by dividing out all Jacobi (IHX) relations.

Remark 4. Definition 3(i) can be spelled out more concretely at two points: the equivalence relation generated by AS relations as above is just given by relations of the form

$$(\epsilon, t) = ((-1)^k \epsilon, t'),$$

where  $t'$  is isomorphic to  $t$  and its orientation differs from that of  $t$  by changing the cyclic orientation at exactly  $k$  vertices. Moreover, the commutative monoid  $\tilde{\mathcal{B}}_n^t(\ell)$  generated by such equivalence classes can be described by working with (equivalence classes of) *finite unions* of vertex-oriented trees, with each connected component labeled by a sign  $\epsilon$ . Then the disjoint union really gives a monoid structure on this set which is clearly commutative and generated by trees. Its unit is given by the empty graph.

Let  $\mathbb{W}_{(n-1)}(\ell)$  denote the set of Whitney towers of order  $(n - 1)$  on bottom stages  $A_1, \dots, A_\ell$ . We have been discussing a map  $\tilde{\tau}_{n-1}$  which we can now write as

$$\tilde{\tau}_{n-1}: \mathbb{W}_{(n-1)}(\ell) \rightarrow \tilde{\mathcal{B}}_n^t(\ell).$$

Working modulo the relations in definition (ii) above, we get a summation map

$$\hat{\tau}_{(n-1)}: \mathbb{W}_{(n-1)}(\ell) \longrightarrow \hat{\mathcal{B}}_n^t(\ell).$$

More explicitly, if  $\tilde{\tau}_{n-1}(\mathcal{W}) = \coprod_p \epsilon_p \cdot t(p)$  is the geometric intersection tree of an order  $(n - 1)$  Whitney tower  $\mathcal{W}$  we set

$$\hat{\tau}_{(n-1)}(\mathcal{W}) := \sum_p \epsilon_p \cdot t(p) \in \hat{\mathcal{B}}_n^t(\ell).$$

It is a consequence of the AS relation that only orientations of the *bottom stages*  $A_i$  are relevant for the definition of  $\tilde{\tau}_{(n-1)}(\mathcal{W})$ ; see Lemma 14. From our geometric point of view, this is the main reason for introducing AS relations.

The question arises as to whether  $\hat{\tau}_{(n-1)}(\mathcal{W})$  can be made into an obstruction for representing the bottom stages  $A_i$ , up to homotopy, by disjoint embeddings. The punchline of the first part of this paper is that this can only be possible if we quotient the groups  $\hat{\mathcal{B}}_n^t(\ell)$  by all Jacobi relations. This gives the above-mentioned groups  $\mathcal{B}_n^t(\ell)$ , containing elements  $\tau_{(n-1)}(\mathcal{W})$ , which are more customary in the theory of finite type invariants. In fact, in the general finite type theory the superscript ‘ $t$ ’, for tree, does not appear because one uses all trivalent graphs instead of just unions of trees.

Theorem 1 implies that one needs to study these quotients if one wants to obtain invariants of the bottom spheres  $A_i$  from the intersection trees associated to Whitney towers. As shown in [ST04], the vanishing of  $\tau_{(n-1)}(\mathcal{W})$  in  $\mathcal{B}_n^t(\ell)$  is sufficient for finding a next order  $n$  Whitney tower on the  $A_i$  (up to homotopy). However, it is an open problem what precise further quotient of  $\mathcal{B}_n^t(\ell)$  is necessary to get a well-defined invariant which only depends on the homotopy classes of the  $A_i$  (and *not* on the order  $(n - 1)$  Whitney tower) and gives the complete obstruction to the existence of an order  $n$  Whitney tower.

### 1.2 From four- to three-dimensional Jacobi relations

In § 3 we connect the geometric Jacobi relation explained above to a three-dimensional setting via a correspondence between capped grope concordances and Whitney towers. This translation becomes important because, to date, there is no useful definition of a Whitney tower in three dimensions. On the other hand, two of us have introduced in [CT04a] a theory of (capped) grope cobordisms between knots in 3-space, and the third member in our group [Sch06] has worked out a four-dimensional correspondence between capped grope concordances and Whitney towers.

A grope is a certain 2-complex, built out of layers of surfaces. The number of these layers is measured by the *class* of the grope, later corresponding to the degree of a tree. A grope contains a specified *bottom stage surface*, usually with one or two boundary circles, depending on whether it is used to relate string links or links. This is explained in detail in § 3.1 and we shall introduce the notation that:

- (a) a grope *cobordism* is an embedding of a grope into 3-space (see § 3.2);
- (b) a grope *concordance* is an embedding of a grope into 4-space (more precisely, the embedding is into  $B^3 \times [0, 1]$ ; see § 3.5).

We shall also explain the notions of *capped* grope cobordism and concordance. The *caps* are embedded disks whose boundaries lie on the top stages of the grope. The (interiors of the) caps are allowed to intersect the grope only in the bottom stage surface. The punchline is that these intersections are going to be:

- (a) arcs, from one part of the boundary to another, in the bottom stage of a grope cobordism;
- (b) points in the bottom stage of a grope concordance.

These statements are the generic case in dimension 4 and need certain cleaning up operations in dimension 3. In any case, when pushing a grope cobordism into 4-space, the arcs become points, and one loses the information of the order in which the arcs hit the boundary. More precisely, in § 3.5 we shall explain in full detail the following commutative diagram.

$$\begin{array}{ccc}
 \mathbb{G}_n^c(\ell) & \xrightarrow{\text{push-in}} & \overline{\mathbb{W}}_{(n-1)}(\ell) \\
 \tilde{\tau}_n^c \downarrow & & \tilde{\tau}_{(n-1)} \downarrow \\
 \tilde{\mathcal{A}}_n^t(\ell) & \xrightarrow{\text{pull-off}} & \tilde{\mathcal{B}}_n^t(\ell)
 \end{array} \tag{1}$$

Here  $\mathbb{G}_n^c(\ell)$  is the set of class  $n$  capped grope cobordisms of  $\ell$ -string links. The set  $\overline{\mathbb{W}}_{(n-1)}(\ell)$  is a quotient of  $\mathbb{W}_{(n-1)}(\ell)$  by the relation equating Whitney towers which are assigned the same element by  $\tilde{\tau}_{(n-1)}$ . The map *push-in* takes a capped grope, pushes it slightly into the 4-ball, and then surgers the resulting grope concordance into a Whitney tower (of order  $(n - 1)$ ). This procedure has some non-uniqueness which is why we need the space  $\overline{\mathbb{W}}_{(n-1)}(\ell)$  as opposed to  $\mathbb{W}_{(n-1)}(\ell)$ . The monoid  $\tilde{\mathcal{A}}_n^t(\ell)$  is just like its  $\mathcal{B}$  analogue, except that the univalent vertices of the trees are attached to  $\ell$  numbered strands (which form a trivial string link). More precisely, we have the following definition.

DEFINITION 5. Consider pairs  $(\epsilon, t)$  where  $\epsilon \in \{\pm\}$ , and  $t$  is a vertex-oriented trivalent tree of degree  $n$  whose tips are attached to the trivial  $\ell$ -string link.

- (i) As in Definition 3, AS (antisymmetry) relations of these pairs generate an equivalence relation, and we let  $\tilde{\mathcal{A}}_n^t(\ell)$  be the abelian monoid generated by the equivalence classes. As before, the monoid operation is given by disjoint union and we write  $\epsilon \cdot t$  for  $(\epsilon, t)$ .
- (ii) The abelian group  $\hat{\mathcal{A}}_n^t(\ell)$  is obtained from  $\tilde{\mathcal{A}}_n^t(\ell)$  by dividing by all relations of the form

$$(\epsilon \cdot t) \amalg (-\epsilon \cdot t) = 0,$$

where 0 is the trivial  $\ell$ -string link with the ‘empty graph’ attached. The monoid operation  $\amalg$  becomes the group addition ‘+’.

- (iii) The abelian group  $\mathcal{A}_n^t(\ell)$  is obtained from  $\hat{\mathcal{A}}_n^t(\ell)$  by dividing out all Jacobi (IHX) relations.

The homomorphism *pull-off* in diagram (1) pulls each tree off the strands of the trivial  $\ell$ -string link, just remembering their indices in  $\{1, \dots, \ell\}$ . Thus the diagram above says exactly what information is lost when one moves from three to four dimensions, namely the orders in which the caps hit the bottom stages.

The map  $\tilde{\tau}_n^c$  is defined precisely in Definition 24 using a notion of geometric intersection trees for gropes (Definition 16), and, just as in the case of Whitney towers, leads to maps  $\hat{\tau}_n^c$  and  $\tau_n^c$ .

By re-interpreting our central picture, Figure 3, in terms of capped gropes in 3-space,  $\tilde{\tau}_n^c$  will be used to show that the four-dimensional Jacobi relation from Theorem 1 can be lifted to a three-dimensional version, as follows.

THEOREM 6 (Three-dimensional Jacobi relation). *Suppose  $t_I, t_H, t_X$  are the three terms in any IHX relation in  $\tilde{\mathcal{A}}_3^t(\ell)$ . Then there is a class 3 simple grope cobordism  $G^c$ , which takes the  $\ell$ -component trivial string link to itself, such that  $\tilde{\tau}_3^c(G^c) = (+t_I) \amalg (-t_H) \amalg (+t_X)$ .*

We should remark that, by a main theorem of [CT04a], we can think of  $\mathbb{G}_n^c(\ell)$  as being the set of degree  $n$  capped (or simple) claspers in the complement of some  $\ell$ -component (string) link. The map  $\tilde{\tau}_n^c$  is then the obvious map which sends a clasper to its tree type, with univalent vertices attached to the link components which link with the corresponding tips. However,  $\tilde{\tau}_n^c$  can also be directly defined for capped gropes, as we explain in Definition 24. One consequence of our work is the following theorem.

THEOREM 7 (Three-dimensional Jacobi relation for claspers). *Suppose three tree claspers  $C_i$  differ locally by the three terms in the Jacobi relation. Given an embedding of  $C_1$  into a 3-manifold, there are embeddings of  $C_2$  and  $C_3$  inside a regular neighborhood of  $C_1$ , such that the leaves are parallel copies of the leaves of  $C_1$ , and the edges avoid any caps that  $C_1$  may have. Moreover, surgery on  $C_1 \cup C_2 \cup C_3$  is diffeomorphic (rel boundary of the handlebody neighborhood) to doing no surgery at all.*

This theorem was stated and utilized in [CT04b], although the fact that the claspers must be tree claspers was accidentally omitted. The theorem was needed to prove Lemma 3.11(c) in [CT04b].

We re-prove this lemma as Lemma 40 of the current paper, as a simple consequence of our general machinery. The map  $\tau_n^c$  is relevant to the theory of Vassiliev invariants. Given a simple grope cobordism between two links, it records the difference in the Vassiliev invariants of degree  $n$  between the two links. Thus, similar in spirit to  $\tau_n$  for Whitney towers,  $\tau_n^c$  could represent an obstruction to two links being isotopic. However, again the question remains on how it depends on the particular choice of the given grope cobordism.

### 1.3 Grope cobordism of string links

In § 5, we shall use the techniques developed in this paper to obtain new information about string links. Let  $\mathcal{L}(\ell)$  be the set of isotopy classes of string links in  $D^3$  with  $\ell$  components (which is a monoid with respect to the usual ‘stacking’ operation). Its quotient by the relation of grope cobordism of class  $n$  is denoted  $\mathcal{L}(\ell)/G_n$ ; compare Definition 18. The quotient by the relation of *capped* grope cobordism of class  $n$  is denoted by  $\mathcal{L}(\ell)/G_n^c$ . The submonoid of  $\mathcal{L}(\ell)$  consisting of those string links which cobound a class  $n$  grope with the trivial string link is denoted by  $G_n(\ell)$ , and similarly the submonoid consisting of those string links which cobound a class  $n$  capped grope with the trivial string link is denoted by  $G_n^c(\ell)$ . The relation of capped (respectively not capped) grope cobordism of class  $n$  coincides with the relation that two string links differ by a sequence of simple (respectively rooted) clasper surgeries of degree  $n$ . Using this connection and results of Habiro [Hab00] we show the following theorem.

**THEOREM 8.** *The quotients  $\mathcal{L}(\ell)/G_{n+1}$  and  $\mathcal{L}(\ell)/G_{n+1}^c$  are finitely generated groups and the iterated quotients  $G_n(\ell)/G_{n+1}$  and  $G_n^c(\ell)/G_{n+1}^c$  are central subgroups. As a consequence,  $\mathcal{L}(\ell)/G_{n+1}$  and  $\mathcal{L}(\ell)/G_{n+1}^c$  are nilpotent.*

In the case of knots,  $\ell = 1$ , results of [Hab00] and also [CT04b] imply that  $G_n^c(1)/G_{n+1}^c$  is rationally isomorphic to the space  $\mathcal{B}_n \otimes \mathbb{Q}$  appearing in the theory of Vassiliev invariants. (Indeed, we alluded to  $\mathcal{B}_n = \mathcal{B}_n(1)$  a few paragraphs after Definition 3.)

For the case of  $\ell \geq 2$  no such theorem is known, but we show that, if one relaxes the requirement that the gropes be capped (which is the same as relaxing the requirement that all leaves of the clasper bound disjoint disks to the requirement that only one leaf does), then one does get such a statement. Using our geometric IHX relations, we will construct a surjective homomorphism from diagrams to string links modulo grope cobordism,

$$\Phi_n: \mathcal{B}_n^g(\ell) \twoheadrightarrow G_n(\ell)/G_{n+1},$$

where  $\mathcal{B}_n^g$  denotes the usual abelian group of trivalent graphs, modulo IHX and AS relations, but graded by the *grope degree* (which is the Vassiliev degree plus the first Betti number of the graph); compare § 4.2.

**THEOREM 9.** *The map  $\Phi_n \otimes \mathbb{Q}: \mathcal{B}_n^g(\ell) \otimes \mathbb{Q} \xrightarrow{\cong} G_n(\ell)/G_{n+1} \otimes \mathbb{Q}$  is an isomorphism.*

This extends a result in [CT04b] from knots to string links and it relies on the existence of the Kontsevich integral for string links, which serves as an inverse to the above map. Although Theorem 9 is an elementary modification of the argument in [CT04b], we found it to be quite surprising in light of the fact that the corresponding statement for capped gropes and simple clasps is unknown.

The map  $\Phi_n$  comes from a map  $\widehat{\tau}_n^g$  defined in § 4.2, which assigns a linear combination of vertex-oriented unitrivalent graphs of grope degree  $n$  to any grope cobordism of class  $n$ . This map is a technical improvement of our methods in [CT04b], and is necessary for us to realize the IHX relation in the uncapped case. To define the map in that paper, we first turned a grope cobordism into a sequence of simple clasper surgeries, and then read off the unitrivalent graphs from the graph types

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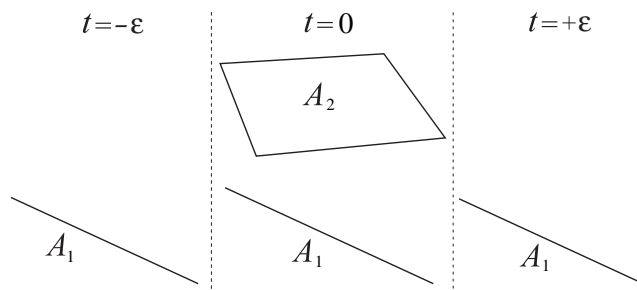


FIGURE 4.

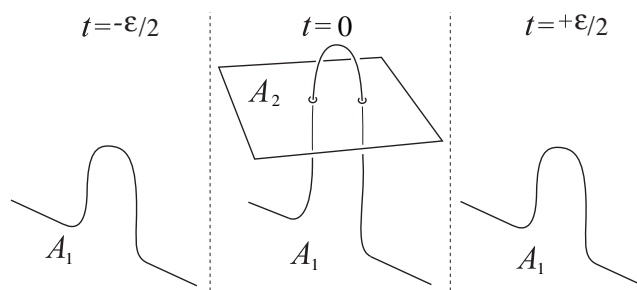


FIGURE 5.

of the claspers. In this paper, we read off the graphs directly from the (genus one) grope itself. The proof that this map induces an isomorphism still requires the techniques of [CT04b], and, in particular, still requires the passage to claspers, since the Kontsevich integral’s behavior with respect to claspers is well understood.

In Appendix A we define the map  $\widehat{\tau}_n^g$  for arbitrary grope cobordisms, which is more general than the genus one gropes used in the body of the paper. This is logically not necessary but included for completeness and possibly for future use.

## 2. A Jacobi identity in dimension 4

In this section we prove Theorem 1, but we first explain some background material and state an important corollary which is used in [ST04]. For more details on immersed surfaces in 4-manifolds we refer to [FQ90]; for more details on Whitney towers compare [Sch06], [Sch05] and [ST04].

### 2.1 Whitney towers

Using local coordinates  $\mathbb{R}^3 \times (-\epsilon, +\epsilon)$ , Figure 4 shows a pair of disjoint local sheets of oriented surfaces  $A_1$  and  $A_2$  in 4-space. We think of the fourth coordinate as ‘time’, so the sheet  $A_2$  lies completely in the *present*  $t = 0$ , whereas  $A_1$  moves through time and thus also forms a two-dimensional sheet represented by an arc which extends from *past* into *future*. Figure 5 shows the result of applying a (Casson) *finger move* to the sheets of Figure 4, with  $A_1$  having been changed by an isotopy supported near an arc from  $A_1$  to  $A_2$ , creating a pair of transverse intersection points in  $A_1 \cap A_2 \subset \mathbb{R}^3 \times \{0\}$ . Such a pair of intersection points is called a *canceling pair* since their signs differ and they can be paired by a *Whitney disk* as illustrated in Figure 6. Note that the boundary of the Whitney disk is a pair of arcs, one in each sheet, connecting the canceling pair of intersections. A Whitney disk guides a motion (of either sheet) called a *Whitney move* that eliminates the pair



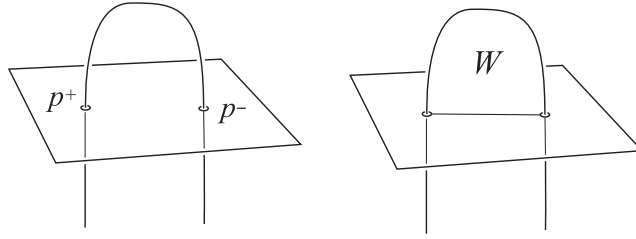


FIGURE 6. Left: A canceling pair of intersections  $p^\pm$ . Right: A Whitney disk pairing  $p^\pm$ .

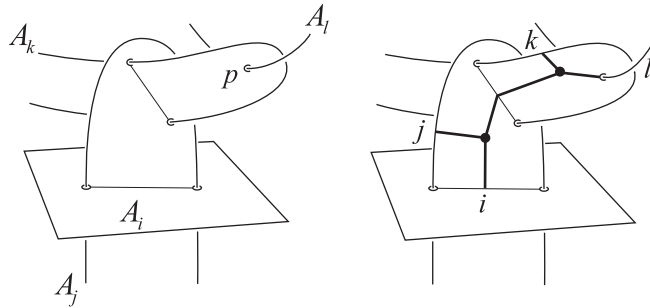


FIGURE 7. Part of an order 2 Whitney tower on order 0 surfaces  $A_i, A_j, A_k$ , and  $A_l$ , and the labeled tree  $t(p)$  of order  $2 = \text{Vassiliev degree } 3$ , associated to the order 2 intersection point  $p$ .

of intersection points [FQ90]. A Whitney move guided by a Whitney disk whose interior is free of singularities can be thought of as an ‘inverse’ to the finger move since it eliminates a canceling pair without creating any new intersections. In general, Whitney disks may have interior self-intersections and intersections with other surfaces so that eliminating a canceling pair via a Whitney move may also create new singularities. Pairing up ‘higher order’ interior intersections in a Whitney disk by ‘higher order’ Whitney disks leads to the notion of a Whitney tower, as follows.

DEFINITION 10 (Compare [Sch06], [Sch05] and [ST04]).

- (i) A *surface of order 0* in a 4-manifold  $M$  is an oriented surface in  $M$  with boundary embedded in the boundary and interior *immersed* in the interior of  $M$ . A *Whitney tower of order 0* is a collection of order 0 surfaces. These are usually referred to as the *bottom stage surfaces* or *underlying surfaces*, and a (higher order) Whitney tower is built *on* these surfaces.
- (ii) The *order of a (transverse) intersection point* between a surface of order  $n$  and a surface of order  $m$  is  $n + m$ .
- (iii) The *order of a Whitney disk* is  $(n + 1)$  if it pairs intersection points of order  $n$ .
- (iv) For  $n \geq 1$ , a *Whitney tower of order  $n$*  is a Whitney tower  $\mathcal{W}$  of order  $(n - 1)$  together with order  $n$  Whitney disks pairing all order  $(n - 1)$  intersection points of  $\mathcal{W}$  (see Figure 7). These order  $n$  Whitney disks are allowed to self-intersect, and/or intersect each other, as well as lower order surfaces.

The boundaries of the Whitney disks in a Whitney tower are required to be disjointly embedded and the Whitney disks themselves are required to be *framed*.

Framings of Whitney disks will not be discussed here (see e.g. [FQ90]). In the construction of an order 2 Whitney tower (proof of Theorem 1) the reader familiar with framings can check that the Whitney disks are correctly framed.

## 2.2 Intersection trees for Whitney towers

For each order  $n$  intersection point  $p$  in a Whitney tower  $\mathcal{W}$  there is an associated labeled trivalent tree  $t(p)$  of order  $n$  (Figure 7). The order of a tree is the number of trivalent vertices (which is one less than the Vassiliev degree). This tree  $t(p)$  is most easily described as a subset of  $\mathcal{W}$  which ‘branches down’ from  $p$  to the order 0 surfaces, bifurcating in each Whitney disk: the trivalent vertices of  $t(p)$  correspond to Whitney disks in  $\mathcal{W}$ , the labeled univalent vertices of  $t(p)$  correspond to the labeled order 0 surfaces of  $\mathcal{W}$ , and the edges of  $t(p)$  correspond to sheet-changing paths between adjacent surfaces in  $\mathcal{W}$ .

Fixing orientations on the surfaces in  $\mathcal{W}$  (including Whitney disks) endows each intersection point  $p$  with a sign  $\epsilon_p \in \{\pm\}$ , determined as usual by comparing the orientations of the intersecting sheets at  $p$  with that of the ambient manifold. These orientations also determine a cyclic orientation for each of the trivalent vertices of  $t(p)$  via a bracketing convention which will be illustrated explicitly during the proof of Theorem 1 below. We shall henceforth assume that our Whitney towers come equipped with such orientations.

The order  $n$  intersection points are the ‘interesting’ intersection points in an order  $n$  Whitney tower  $\mathcal{W}$ , since these points may represent an obstruction to the existence of an order  $(n + 1)$  Whitney tower on the order 0 surfaces of  $\mathcal{W}$ . (In fact, all intersections of order greater than  $n$  can be eliminated by finger moves on the Whitney disks.)

Recall  $\tilde{\mathcal{B}}_{n+1}^t$  from Definition 3.

DEFINITION 11. For an oriented order  $n$  Whitney tower  $\mathcal{W}$ , define  $\tilde{\tau}_n(\mathcal{W}) \in \tilde{\mathcal{B}}_{n+1}^t(\ell)$ , the order  $n$  *geometric intersection tree* of  $\mathcal{W}$ , to be the disjoint union of signed labeled vertex-oriented trivalent trees

$$\tilde{\tau}_n(\mathcal{W}) := \coprod_p \epsilon_p \cdot t(p)$$

over all order  $n$  intersection points  $p \in \mathcal{W}$ .

We emphasize that  $\tilde{\tau}_n(\mathcal{W})$  is a collection of signed trees of order  $n$ , possibly with repetitions, *without* cancellation of terms. (The geometric intersection tree is denoted by  $t_n(\mathcal{W})$  in [ST04], as is an unoriented version in [Sch06].)

Note that there is a natural map  $\pi: \tilde{\mathcal{B}}_{n+1}^t(\ell) \rightarrow \hat{\mathcal{B}}_{n+1}^t(\ell)$  given by sending the monoid operation to the group addition.

DEFINITION 12. Given an oriented order  $n$  Whitney tower  $\mathcal{W}$ , define  $\hat{\tau}_n(\mathcal{W}) = \pi(\tilde{\tau}_n(\mathcal{W}))$ .

It turns out (see Lemma 14 below) that, for any fixed Whitney tower  $\mathcal{W}$ , the AS antisymmetry relations correspond *exactly* to the indeterminacies coming from orientation choices on the Whitney disks in  $\mathcal{W}$ , so that the element  $\tilde{\tau}_n(\mathcal{W}) \in \tilde{\mathcal{B}}_{n+1}^t$  only depends on the orientations of the bottom stage surfaces. On the other hand, by fixing the bottom stage surfaces and varying the choices of Whitney disks, we are led to the IHX relations, as we describe in the next section.

Since the ultimate goal of studying Whitney towers is to extract homotopy invariants  $\tau_n$  of the underlying order 0 surfaces from the geometric intersection tree, such an element should vanish for any Whitney tower  $\mathcal{W}$  on immersed 2-spheres into 4-space since all such spheres are null-homotopic. Theorem 1 from the Introduction (proven below) and its corollary (Corollary 13) illustrate the necessity of the IHX relation in the target of  $\tau_n$ . Since Theorem 1 is a local statement (taking place in a 4-ball) it can be used to ‘geometrically realize’ all higher degree IHX relations for Whitney towers in arbitrary 4-manifolds, a key part of the obstruction theory described in [ST04]. The following corollary of Theorem 1 is proved in [ST04].

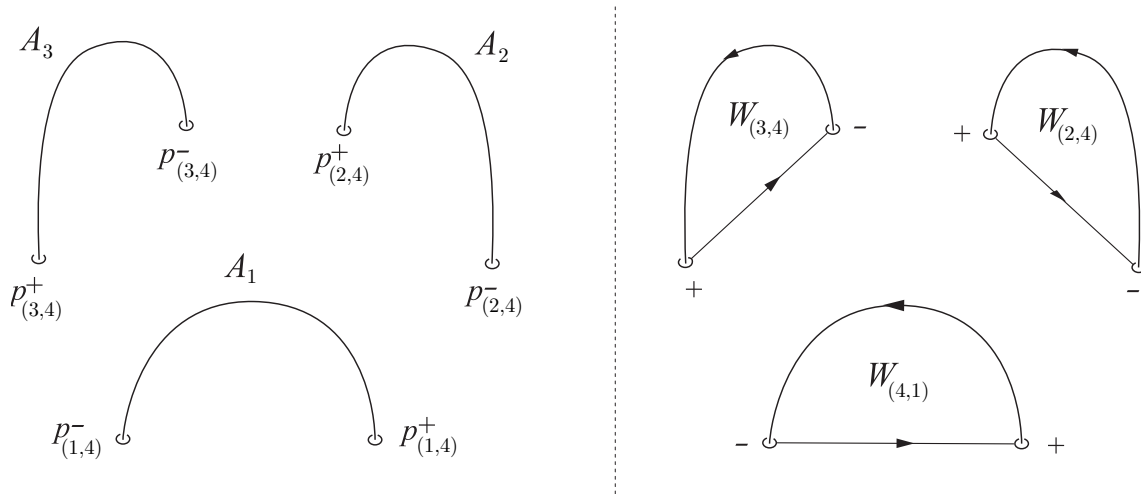


FIGURE 8. The clean order 1 Whitney tower  $\mathcal{W}^0$  is shown on the right.

COROLLARY 13. *Let  $\mathcal{W}$  be an order  $n$  Whitney tower on surfaces  $A_i$ . Then, given any order  $n$  trivalent trees  $t_I, t_H$  and  $t_X$  differing only by a local IHX relation, there exists an order  $n$  Whitney tower  $\mathcal{W}'$  on  $A'_i$  homotopic (rel boundary) to the  $A_i$  such that*

$$\tilde{\tau}_n(\mathcal{W}') = \tilde{\tau}_n(\mathcal{W}) \amalg (+t_I) \amalg (-t_H) \amalg (+t_X).$$

The idea of the proof of Corollary 13 is that by applying finger moves to surfaces in a Whitney tower one can create clean Whitney disks which are then tubed into the spheres in Theorem 1. This construction can be done without creating extra intersections since finger moves are supported near arcs and the construction of Theorem 1 is contained in a 4-ball.

### 2.3 Proof of the main result (Theorem 1)

The four-dimensional IHX construction starts with any four disjointly embedded oriented 2-spheres  $A_1, A_2, A_3$  and  $A_4$  in 4-space. Perform finger moves on each  $A_i$ , for  $i = 1, 2, 3$ , to create a canceling pair of order 0 intersection points  $p_{(i,4)}^{\pm}$  between each of the first three 2-spheres (still denoted  $A_i$ ) and  $A_4$  as pictured on the left-hand side of Figure 8, where  $A_4$  appears as the ‘plane of the paper’ with the standard counterclockwise orientation, sitting in the ‘present’ slice  $\mathbb{R}^3 \times \{0\}$  of local coordinates  $\mathbb{R}^3 \times (-\epsilon, +\epsilon)$  in 4-space. Choose disjointly embedded oriented order 1 Whitney disks  $W_{(3,4)}, W_{(2,4)}$  and  $W_{(4,1)}$  for the canceling pairs  $p_{(i,4)}^{\pm}$  as on the right-hand side of Figure 8. Here the bracket subscript notation corresponds to the following *orientation convention*: the bracket subscript  $(i, j)$  on a Whitney disk indicates that the boundary  $\partial W_{(i,j)}$  of the Whitney disk is oriented from the negative intersection point to the positive intersection point along  $A_i$  and from the positive to the negative intersection point along  $A_j$ . This orientation of  $\partial W_{(i,j)}$  together with a second ‘inward pointing’ tangent vector induces the orientation of  $W_{(i,j)}$ . We have constructed an order 1 Whitney tower  $\mathcal{W}^0$  which is *clean*, meaning that  $\mathcal{W}^0$  has no unpaired intersection points and hence is in fact a Whitney tower of order  $n$  for all  $n$ . As illustrated in Figure 8, the three order 1 Whitney disks of  $\mathcal{W}^0$  all lie in the present slice of local coordinates. In the following construction, these three Whitney disks will be modified to create the three terms in the IHX relation. The modified  $W_{(3,4)}$  will remain entirely in the present, while most of  $W_{(2,4)}$  will be perturbed slightly into the future, and most of  $W_{(4,1)}$  will be perturbed slightly into the past. These perturbations are essential for keeping the Whitney disks disjoint!

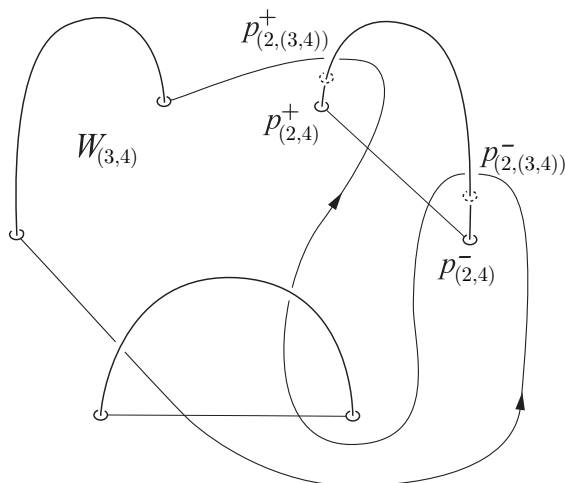


FIGURE 9.

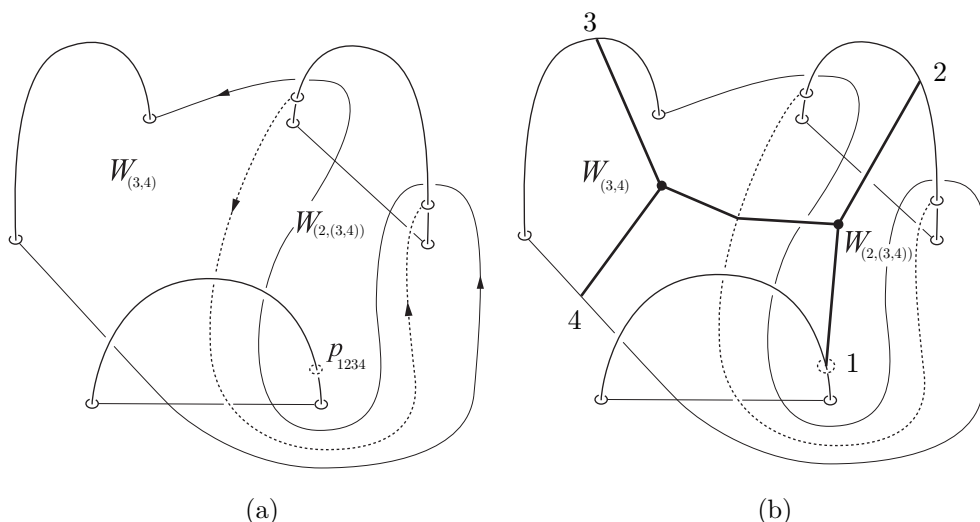


FIGURE 10. The construction of the  $I$ -tree. Both sides (a) and (b) of this figure show the same present slice of local coordinates.

Continuing with the construction, change  $W_{(3,4)}$  by isotoping its boundary  $\partial W_{(3,4)}$  along  $A_4$  and across  $p_{(2,4)}^+$  and  $p_{(2,4)}^-$  as indicated in Figure 9 and extending this isotopy to a collar of  $\partial W_{(3,4)}$ . Note that a canceling pair of order 1 intersection points  $p_{(2,3,4)}^\pm$  has been created between  $A_2$  and the interior of the ‘new’  $W_{(3,4)}$  (still denoted by  $W_{(3,4)}$ ). The pair  $p_{(2,3,4)}^\pm$  is indicated in Figure 9 by the small dashed circles near  $p_{(2,4)}^\pm$  and, since the orientation of  $A_4$  is the standard counterclockwise orientation of the plane, the sign of  $p_{(2,3,4)}^+$  (respectively  $p_{(2,3,4)}^-$ ) agrees with the sign of  $p_{(2,4)}^+$  (respectively  $p_{(2,4)}^-$ ). By perturbing most of  $W_{(2,4)}$  into the future, we may assume that  $p_{(2,3,4)}^\pm$  lie near, but not on,  $\partial W_{(2,4)}$ . Specifically, the only part of  $W_{(2,4)}$  that we do not push into the future is a small collar of the arc of  $\partial W_{(2,4)}$  which lies on  $A_4$ . For now,  $W_{(3,4)}$  has intersections with the other first order Whitney disks in and near its boundary on  $A_4$ , but these will be removed later in the construction.

A Whitney disk  $W_{(2,(3,4))}$  (of order 2) for the canceling pair  $p_{(2,(3,4))}^\pm$  can be constructed by altering a parallel copy of  $W_{(2,4)}$  in a collar of its boundary as indicated in Figure 10(a). Note that

$W_{(2,(3,4))}$  sits entirely in the present. The part of the boundary of  $W_{(2,(3,4))}$  that lies on  $W_{(3,4)}$  is indicated by a dashed line in Figure 10(a). The other arc of  $\partial W_{(2,(3,4))}$  runs along  $A_2$  where there used to be an arc of  $\partial W_{(2,4)}$  before most of  $W_{(2,4)}$  was pushed into the future.

Take the orientation of  $W_{(2,(3,4))}$  that corresponds to its bracket subscript via the above convention, i.e. that induced by orienting  $\partial W_{(2,(3,4))}$  from  $p_{(2,(3,4))}^-$  to  $p_{(2,(3,4))}^+$  along  $A_2$  and from  $p_{(2,(3,4))}^+$  to  $p_{(2,(3,4))}^-$  along  $W_{(3,4)}$  together with a second inward pointing vector.

Note that  $W_{(2,(3,4))}$  has a single *positive* intersection point  $p_{1234}$  (of order 2) with  $A_1$  (in the present). By pushing most of  $W_{(4,1)}$  into the past, we can arrange that  $W_{(2,(3,4))}$  (which sits entirely in the present) is disjoint from  $W_{(4,1)}$ . Specifically, the only part of  $W_{(4,1)}$  that is not pushed into the past is a small collar on the arc of  $\partial W_{(4,1)}$  which lies in  $A_4$ . To the point  $p_{1234}$  we associate the positively signed labeled  $I$ -tree (of order 2) as illustrated in Figure 10(b). This  $I$ -tree,  $t(p_{1234})$ , is embedded in the construction with the trivalent vertices lying in the interiors of the Whitney disks,  $W_{(2,4)}$  and  $W_{(2,(3,4))}$ , and each  $i$ -labeled univalent vertex lying on  $A_i$ . Each trivalent vertex of  $t(p_{1234})$  inherits a cyclic orientation from the ordering of the components in the bracket associated to the corresponding oriented Whitney disk. Note that the pair of edges which pass from a trivalent vertex down into the lower order surfaces paired by a Whitney disk determine a ‘corner’ of the Whitney disk which does not contain the other edge of the trivalent vertex. If this corner contains the *positive* intersection point paired by the Whitney disk, then the vertex orientation and the Whitney disk orientation agree [ST04]. Our figures are all drawn to satisfy this convention.

We have described how to construct (from the original  $W_{(3,4)}$  of  $\mathcal{W}^0$ ) Whitney disks  $W_{(2,(3,4))}$  and  $W_{(3,4)}$  (both lying entirely in the present) such that  $W_{(2,(3,4))}$  pairs  $A_2 \cap W_{(3,4)}$  and such that  $A_1 \cap W_{(2,(3,4))}$  consists of a single point  $p_{1234}$  whose associated tree is the  $I$  term in the IHX relation. In fact, a parallel version of this construction can be carried out simultaneously on all of the original Whitney disks in  $\mathcal{W}^0$  yielding additional order 2 intersection points  $p_{2341} \in A_2 \cap W_{(3,(4,1))}$  (with negative sign and associated labeled trivalent tree  $H$ ) and  $p_{3124} \in A_3 \cap W_{(1,(2,4))}$  (with positive sign and associated labeled trivalent tree  $X$ ). Here  $W_{(3,(4,1))}$  pairs  $A_3 \cap W_{(4,1)}$  and  $W_{(1,(2,4))}$  pairs  $A_1 \cap W_{(2,4)}$  and it can be arranged that all the Whitney disks have pairwise disjointly embedded interiors and pairwise disjointly embedded boundaries. To see this, first observe that the boundaries of the first order Whitney disks  $W_{(3,4)}$ ,  $W_{(4,1)}$  and  $W_{(2,4)}$  can be disjointly embedded in the present, as pictured in Figure 3, which shows how collars on the parts of the Whitney disk boundaries that lie on  $A_4$  can be simultaneously changed in the same way that we previously changed  $W_{(3,4)}$ . Recall that, in the above construction, the part of  $W_{(4,1)}$  that was pushed into the past was exactly the complement of a collar on the boundary arc of  $\partial W_{(4,1)}$  which lies on  $A_4$ . Thus, (a collar on) the boundary arc of  $\partial W_{(4,1)}$  which lies on  $A_4$  as pictured in Figure 3 can be extended (without creating any new intersections) to connect to the rest of  $W_{(4,1)}$  which has been perturbed into the past, and the  $-H$  term can be created by a parallel construction to the construction of the  $I$  term, as illustrated in Figure 11, which shows the relevant past slice of local coordinates. Specifically, the second order Whitney disk  $W_{(3,(4,1))}$  sits entirely in the past, and is made from a parallel copy of  $W_{(3,4)}$  by pushing a collar to create the intersection  $p_{2341}$  with  $A_2$ . Note that since  $A_4$  sits entirely in the present, it does not appear in Figure 11, which shows exclusively the past. The signs of all intersection points can be determined from the signs of the original intersections in Figure 8 using our orientation conventions: the vertex orientations of the embedded  $H$ -tree in Figure 11(b) agree with the orientations of the Whitney disks, and the sign of the intersection point  $p_{2341}$  is  $-1$ , as desired.

The  $X$ -tree term is created similarly by extending a collar of the boundary arc of  $\partial W_{(2,4)}$  as pictured in Figure 3 into the future and performing a parallel construction as illustrated in Figure 12. The resulting order 2 Whitney tower  $\mathcal{W}$  has exactly three order 2 intersection points with  $\tilde{\tau}_2(\mathcal{W}) = (+I) \amalg (-H) \amalg (+X)$ . The correspondence between the Whitney disks in this construction and the trivalent vertices in the IHX relation is indicated in Figure 13.

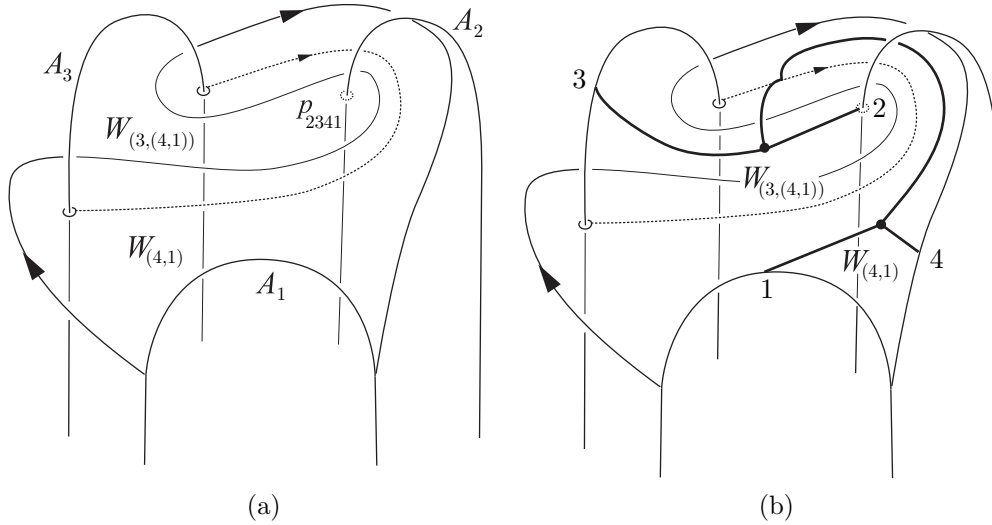


FIGURE 11. The construction of the  $H$ -tree. Both sides (a) and (b) of this figure show the same slice of local coordinates, just in the past of Figure 10.

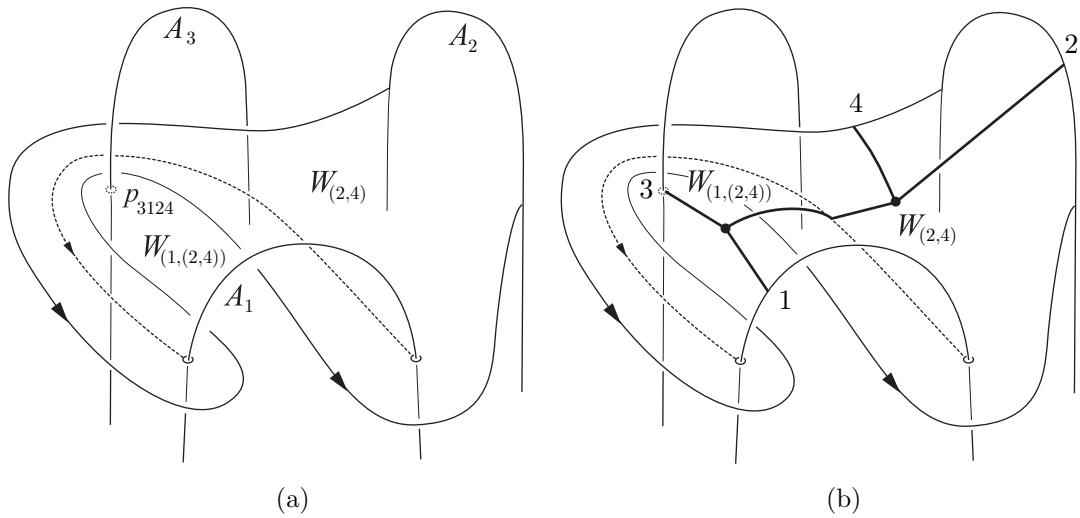


FIGURE 12. The construction of the  $X$ -tree. Both sides (a) and (b) of this figure show the same slice of local coordinates, just in the future of Figure 10.

The proof of Theorem 1 is now complete, but before moving on to connections with the three-dimensional Jacobi relations we note here a lemma which can now be appreciated by the reader who has carefully kept track of the orientations in the above constructions.

LEMMA 14. For a fixed order  $n$  Whitney tower  $\mathcal{W}$ , the geometric intersection tree  $\tilde{\tau}_n(\mathcal{W}) \in \tilde{\mathcal{B}}_{n+1}^t(\ell)$  only depends on the orientations of the order 0 surfaces.

*Proof.* Recall that  $\tilde{\tau}_n(\mathcal{W})$  is a disjoint union of signed vertex-oriented trees associated to the order  $n$  intersection points in  $\mathcal{W}$ , and the AS relations change the sign of a tree whenever a vertex orientation is changed. Each tree  $t(p)$  is most easily defined as a subset of  $\mathcal{W}$  which bifurcates down through the

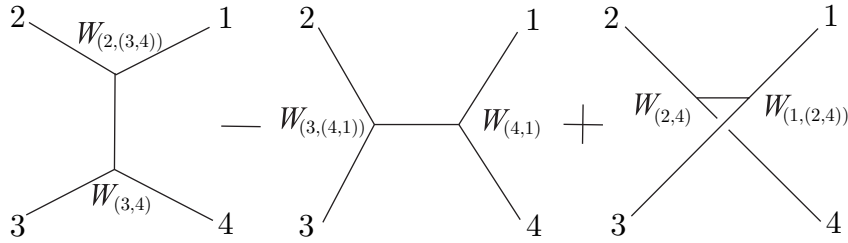


FIGURE 13. The correspondence between the trivalent vertices in the IHX relation and the (oriented) Whitney disks in the construction. (The trivalent orientations are all counterclockwise.)

Whitney disks, with each trivalent vertex of  $t(p)$  lying in a Whitney disk. Each trivalent vertex has two *descending* edges which pass into the lower order sheets paired by the Whitney disk, and one *ascending* edge which either passes through the intersection point  $p$  or passes into a higher order Whitney disk. Assuming fixed orientations on all the surfaces in  $\mathcal{W}$  (including Whitney disks), our orientation convention for  $t(p)$  can be summarized as follows: the descending edges of a trivalent vertex determine a corner of the corresponding Whitney disk which does not contain the ascending edge. If this corner encloses the positive intersection point (of the intersections paired by the Whitney disk), then the vertex orientation is the same as that induced by the orientation of the Whitney disk. If this corner encloses the negative intersection point, then the vertex orientation is the opposite of the orientation induced by the Whitney disk.

We remark here that in practice the geometric intersection tree  $\tilde{\tau}_n(\mathcal{W})$  usually sits as an *embedded* subset of  $\mathcal{W}$ , as can be arranged easily by ‘splitting’ the Whitney tower [Sch06, ST04]. However, in general  $\tilde{\tau}_n(\mathcal{W})$  will not be embedded if any Whitney disks contain self-intersections and/or multiple (pairs of) intersections.

To check that each signed tree  $\epsilon_p \cdot t(p)$  in  $\tilde{\tau}_n(\mathcal{W})$  only depends, modulo antisymmetry, on the orientations of the underlying order 0 surfaces, it is enough to consider the effect of changing any single Whitney disk orientation. There are two cases to consider.

First consider a Whitney disk  $W$  containing a trivalent vertex  $v$  of a signed tree  $\epsilon_p \cdot t(p)$ , where the ascending edge of  $v$  passes into a higher order Whitney disk  $W'$  containing an adjacent trivalent vertex  $v'$  of  $t(p)$ . Changing the orientation of  $W$  changes the vertex orientation of  $v$ , and also changes the vertex orientation at  $v'$  because the signs of the intersection points (in  $W$ ) which are paired by  $W'$  are reversed. Thus, the signed tree  $\epsilon_p \cdot t(p)$  does not change.

Now consider a Whitney disk  $W$  containing a trivalent vertex  $v$  of a signed tree  $\epsilon_p \cdot t(p)$ , where the ascending edge of  $v$  passes through the intersection point  $p$ . In this case, changing the orientation of  $W$  changes the vertex orientation of  $v$  and changes the sign of the intersection point  $p$ , provided  $p$  is not a self-intersection point of  $W$ , so that  $\epsilon_p \cdot t(p)$  is changed exactly by an antisymmetry relation at  $v$ . In the case that  $p$  is a self-intersection point of  $W$ , then changing the orientation of  $W$  changes both trivalent vertices adjacent to  $p$ , namely  $v$  and another trivalent vertex of  $t(p)$  which also sits in  $W$ . This completes the proof of Lemma 14.  $\square$

### 3. Connecting four- and three-dimensional Jacobi relations

In this section we explain in detail the commutative diagram (1) in §1.2 of the Introduction. But first we need to introduce some background material.

### 3.1 Gropes and their associated trees

For technical simplicity, we will use only genus 1 gropes, which are sufficient for our purposes. We will not specify the genus 1 assumption in the body of this paper but we note that there is a *grobe refinement* procedure [CT04a, §2.3] that allows one to replace an arbitrary grope by a genus 1 grope. In fact, we allow here the bottom stage surface to have arbitrary genus, which is why we do not need *sequences of* genus 1 gropes as in [CT04a]. In Appendix A we deal with general gropes, and the reader is referred to [CT04a] for their definition.

DEFINITION 15. A (genus 1) *grobe*  $g$  is constructed by the following method.

- (i) Start with a compact oriented connected surface of any genus, the *bottom stage* of  $g$ , and choose a symplectic basis of circles on this bottom stage surface.
- (ii) Attach punctured tori to any number of the basis circles and choose hyperbolic pairs of circles on each attached torus.

Iterating the second step a finite number of times yields the grope  $g$ . The attached tori are the *higher stages* of  $g$ . The basis circles in all stages of  $g$  that do not have a torus attached to them are called the *tips* of  $g$ . Attaching 2-disks along all the tips of  $g$  yields a *capped* grope (of genus 1), denoted  $g^c$ . In the case of an (uncapped) grope, it is often convenient to attach an annulus along one of its boundary components to each tip. These annuli are called *pushing annuli*, and every tame embedding of a grope in a 3-manifold can be extended to include the pushing annuli.

Let  $g^c$  be a capped grope. We define a rooted trivalent tree  $t(g^c)$  as follows.

DEFINITION 16. Assume first that the bottom stage of  $g^c$  is a genus 1 surface with boundary. Then define  $t(g^c)$  to be the rooted trivalent tree which is dual to the 2-complex  $g^c$ ; specifically,  $t(g^c)$  sits as an embedded subset of  $g^c$  in the following way. The root univalent vertex of  $t(g^c)$  is a point in the boundary of the bottom stage of  $g^c$ ; each of the other univalent vertices are points in the interior of a cap of  $g^c$ ; each higher stage of  $g^c$  contains a single trivalent vertex of  $t(g^c)$ ; and each edge of  $t(g^c)$  is a sheet-changing path between vertices in adjacent stages or caps (here ‘adjacent’ means ‘intersecting in a circle’), see Figure 14(b).

In the case where the bottom stage of  $g^c$  has genus greater than 1, then  $t(g^c)$  is defined by cutting the bottom stage into genus 1 pieces and taking the disjoint union of the trees just described. In the case of genus 0,  $t(g^c)$  is the empty tree.

We can now define the relevant complexity of a grope.

DEFINITION 17. The *class* of  $g^c$  is the minimum of the Vassiliev degrees of the connected trees in  $t(g^c)$ . The underlying uncapped grope  $g$  (the *body* of  $g^c$ ) inherits the same tree,  $t(g) = t(g^c)$ , and the same notion of class. If the grope consists of a surface of genus 0, we regard it as a grope of class  $n$  for all  $n$ . The non-root univalent vertices of  $t(g)$  are called *tips* and each tip of  $t(g)$  corresponds to a tip of  $g$ .

We will assume throughout this paper that all surface stages in our gropes contribute to the class of the grope, i.e. we ignore surface stages that can be deleted without changing the class.

### 3.2 Grope cobordism

DEFINITION 18. A *class  $n$  grope cobordism* between  $\ell$ -component string links  $\sigma$  and  $\sigma'$  is defined as follows. For each  $1 \leq i \leq \ell$ , let  $\sigma_i$ , respectively  $\sigma'_i$ , be the  $i$ th string link component of  $\sigma$ , respectively  $\sigma'$ . Suppose that, for each  $i$ , there is an embedding of a class  $n$  grope  $g_i$  into the 3-ball whose (oriented) boundary is decomposed into two arcs representing the (oriented) isotopy classes of  $\sigma_i$  and  $-\sigma'_i$ . This collection of gropes is called a *grobe cobordism*  $G$  from  $\sigma$  to  $\sigma'$  if the gropes



$g_i$  are embedded disjointly. We sometimes also say that  $G$  is a grope cobordism of  $\sigma$  and note the asymmetry coming from the above orientation convention.

If all the tips of each  $g_i$  bound embedded caps whose interiors are disjoint from each other and disjoint from all but the bottom stages of the  $g_i$ , then  $G$  together with these caps form a (class  $n$ ) capped grope cobordism  $G^c$  from  $\sigma$  to  $\sigma'$  (or of  $\sigma$ ).

Note that this definition does not specify the relative embedding of  $\sigma$  and  $\sigma'$ .

*Remark 19.* The above definition is a generalization of the one given in [CT04a] for knots. By considering disjointly embedded gropes in 3-space, each with two boundary circles, one also gets a notion of grope cobordism of links. The arguments of [CT04a] adapt to show that grope cobordism (of links or string links) is an equivalence relation.

Let  $G^c$  be a capped grope cobordism from  $\sigma$  to  $\sigma'$ . It turns out that one can assume that the intersections of the caps with the bottom stages are arcs from  $\sigma$  to  $\sigma'$ . This can be accomplished by finger moves of the caps across the boundary of the bottom stages. Also, by applying Krushkal's splitting technique (as adapted to three dimensions in [CT04a]) it can be assumed that each cap contains just a single intersection arc.

DEFINITION 20. The following notions will be used for capped grope cobordisms.

- (i) A capped grope cobordism which has been simplified as described in the previous paragraph will be referred to as a *simple grope cobordism*.
- (ii) Denote by  $\mathbb{G}_n^c(\ell)$  the set of class  $n$  simple grope cobordisms of  $\ell$ -component string links.
- (iii) Denote by  $\mathbb{G}_n(\ell)$  the set of class  $n$  grope cobordisms (that is, grope cobordisms which are not required to have caps).

### 3.3 Claspers and gropes

DEFINITION 21. The following definitions can be found in [Hab00] and/or [CT04a].

- (i) A *clasper* is a surface embedded in the complement of a link or string link in a 3-manifold, formed by gluing together edges, nodes and leaves. An edge is homeomorphic to  $I \times I$ , and each end  $I \times \{0\}$  or  $I \times \{1\}$  is glued to a node or a leaf. A node is homeomorphic to  $D^2$  and must have three edges glued to its boundary. A leaf is homeomorphic to  $S^1 \times I$  and must have a single edge glued to one of its boundary components.
- (ii) A clasper is said to be *capped* if all of (the cores of) its leaves bound disjoint disks (called caps) which may hit the link or string link, but only intersect the clasper along their boundaries.
- (iii) A clasper is said to be *simple* if it is capped and if the caps each only hit the link or string link in a single transverse intersection.
- (iv) Given a clasper  $C$ , we can form an oriented graph by collapsing each edge to a one-dimensional edge, each node to a trivalent vertex, and each leaf to a univalent vertex. The vertex orientation of the graph is somewhat subtle, especially when the resulting graph is not a tree, and we refer the reader to [CT04b] for details.
- (v) A *tree clasper* is a clasper whose associated graph is a tree.
- (vi) A tree clasper is said to be *rooted* if there is at least one leaf which has a cap that hits the link or string link in a single transverse intersection.
- (vii) Given a clasper, there is a way of producing an embedded framed link, and surgery on the clasper is defined to be surgery on this framed link. If the clasper is rooted (which is implied by 'simple' and 'capped') then the surgery does not change the ambient manifold and can instead be regarded as changing the link or string link.

DEFINITION 22. The (Vassiliev) *degree* of a clasper is half the total number of vertices of the associated graph. The *grope degree* of a clasper is the (Vassiliev) degree plus the first Betti number of the associated graph.

Claspers and gropes are closely related, as discussed in detail in [CT04a]. Here are some important results, which were stated for knots, but hold true for links and string links as well.

THEOREM 23. *The following statements can be proven by the techniques of [CT04a].*

- (i) *Two links or string links in a 3-manifold differ by a sequence of simple clasper surgeries of Vassiliev degree  $n$  if and only if they are related by a simple grope cobordism of class  $n$ .*
- (ii) *Two links or string links in a 3-manifold differ by a sequence of rooted tree clasper surgeries of Vassiliev degree  $n$  if and only if they are related by a grope cobordism of class  $n$ .*
- (iii) *Two links or string links in a 3-ball differ by a sequence of simple clasper surgeries of grope degree  $n$  if and only if they are related by a grope cobordism of class  $n$ .*

Habiro [Hab00] has shown that two knots share the same Vassiliev invariants up to degree  $n$  if and only if they differ by a sequence of simple clasper surgeries of degree  $(n + 1)$ . Together with the above theorem, this implies that two knots have the same Vassiliev invariants up to degree  $n$  if and only if they cobound a simple grope cobordism of class  $(n + 1)$ . The corresponding statements for string links are not known, but see §5.

### 3.4 Geometric intersection trees for grope cobordisms

Let  $G^c \in \mathbb{G}_n^c(\ell)$  be a class  $n$  simple grope cobordism of a string link  $\sigma$ , and let  $g_i^c$  be a capped grope component of  $G^c$ . Each cap of  $g_i^c$  contains only a single arc of intersections, which can be with any bottom stage surface in  $g_j^c \subset G^c$ . The bottom stage surface of  $g_i^c$  inherits an orientation from its boundary, and we now describe how to orient the higher stages of the grope cobordism, up to a certain indeterminacy.

Each surface stage or cap is attached to a previous stage along a circle, which hits the attaching region for one other surface stage or cap in a point. Near this point, the 2-complex is modeled by the following subset of  $\mathbb{R}^3$ :

$$\{(x, y, z) : z = 0\} \cup \{(x, y, z) : x = 0, z \geq 0\} \cup \{(x, y, z) : y = 0, z \leq 0\}.$$

Distinguish two of the quadrants as *positive*, namely the quadrants where both  $x, y > 0$ , respectively where both  $x, y < 0$ ; see Figure 14(a), where the two positive quadrants are indicated. Now suppose that the lower stage ( $z = 0$ ) has an orientation and choose one of the two positive quadrants. The orientation of the surface induces an orientation of a small triangle in the positive quadrant which has a vertex at the origin and two edges contained in the axes. This then induces an orientation of the boundaries of the two higher surface stages, and hence induces an orientation of the higher surface stages. If we use the other positive quadrant instead, this has the effect of flipping the orientation of both higher surface stages, and this is the indeterminacy that we allow.

The above orientations of the surface stages in a capped grope  $g^c$  induce vertex orientations of the trivalent vertices of  $t(g^c)$  by taking each trivalent vertex of  $t(g^c)$  to lie in a chosen positive quadrant; see Figure 14(b). Here, also, the pairs of edges that cross into the next stages are required to do so through that positive quadrant.

Recall that  $t(g_i^c)$  is a disjoint union  $\coprod_r t_i^r$  of trees  $t_i^r$ , each of which sits as an embedded subset of  $g_i^c$ , with the root of  $t_i^r$  lying on the  $i$ th strand of  $\sigma$  (in  $\partial g_i^c$ ) and each tip of  $t_i^r$  lying inside a cap. The interior of each cap intersects the cobordism in a single intersection arc which corresponds to some strand of  $\sigma$ . Hence we can regard these tips as actually lying on a  $j$ th strand of  $\sigma$  at an intersection point between a cap of  $g_i^c$  and that  $j$ th strand (see left-hand side of Figure 15). Associate to each

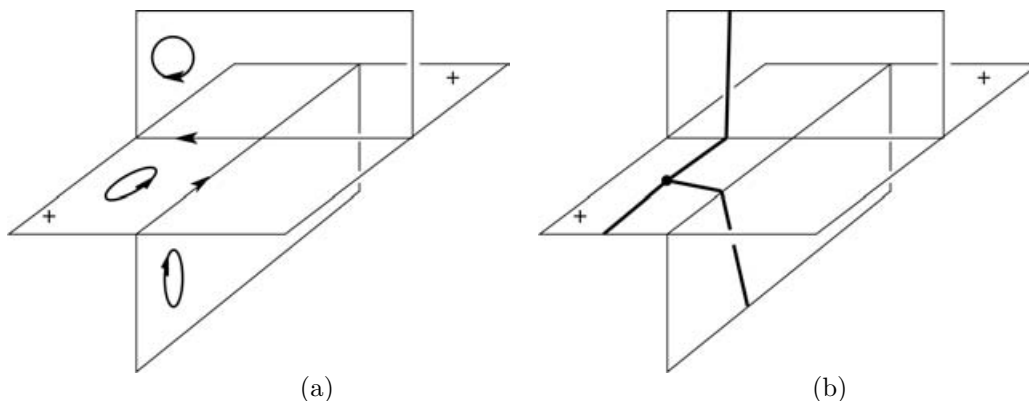


FIGURE 14. (a) Positive quadrants and orientation. (b) A trivalent vertex of  $t(g^c)$ .

tip of  $t_i^r$  the sign of the corresponding intersection point (between the cap and the  $j$ th strand) and denote by  $\epsilon_i^r \in \{+, -\}$  the product of these signs.

The vertices of  $t(g_i^c)$  can be oriented by regarding the tree as a subset of  $g_i^c$  where the two edges emanating from a trivalent vertex must pass to the higher stages in a positive quadrant, as depicted in Figure 14.

Recall  $\tilde{\mathcal{A}}_n^t(\ell)$  from Definition 5.

DEFINITION 24. Let  $G^c$  be a capped class  $n$  grope cobordism of  $\ell$ -string links. The *geometric intersection tree*  $\tilde{\tau}_n^c(G^c) \in \tilde{\mathcal{A}}_n^t(\ell)$  is defined to be the disjoint union  $\coprod_i \coprod_r \epsilon_i^r \cdot t_i^r$  of all the vertex-oriented signed trees associated to all the  $g_i^c$ . Note that each tree should avoid the intersections between caps and the bottom stage, and this forces the roots to attach to the strands of  $\sigma$  in a specific ordering.

LEMMA 25. *The geometric intersection tree  $\tilde{\tau}_n^c(G^c)$  is well defined.*

*Proof.* The issue is whether the choice of positive quadrants can affect  $\tilde{\tau}_n^c(G^c)$ . Choosing a different positive quadrant does not change the cyclic order of the corresponding vertex, but it does change the orientations of all of the higher stages, including the caps. This has the effect of switching the cyclic orders at each of the vertices representing these higher stages, as well as switching the sign of all of the tips representing these caps. In other words a sign is introduced for every vertex (both uni- and trivalent) lying above the vertex we started with. A simple induction shows that there must be an even number of these. Hence, we arrive at the same signed tree, modulo AS relations.  $\square$

DEFINITION 26.

- (i) Let  $\rho: \tilde{\mathcal{A}}_n^t(\ell) \rightarrow \hat{\mathcal{A}}_n^t(\ell)$  be the natural map sending the monoid operation to the group addition.
- (ii) Let  $\hat{\tau}_n^c: \mathbb{G}_n^c(\ell) \rightarrow \hat{\mathcal{A}}_n^t(\ell)$  be defined by  $\hat{\tau}_n^c(G^c) = \rho(\tilde{\tau}_n^c(G^c))$ .

Remark 27. If one translates a simple grope into a union of simple tree claspers, the map  $\hat{\tau}_n^c$  can be regarded as the map which sums over the set of claspers, collapsing each clasper to its underlying tree, with univalent vertices attaching to the  $\ell$  strands according to where the caps of the clasper meet the string link. This was the point of view taken in [CT04b].

### 3.5 From grope cobordism to Whitney concordance

DEFINITION 28. A *singular concordance* between string links  $\sigma$  and  $\sigma'$  is a collection of properly immersed 2-disks  $D_i$  in the product  $B^3 \times I$  of the 3-ball with the unit interval  $I = [0, 1]$ , with

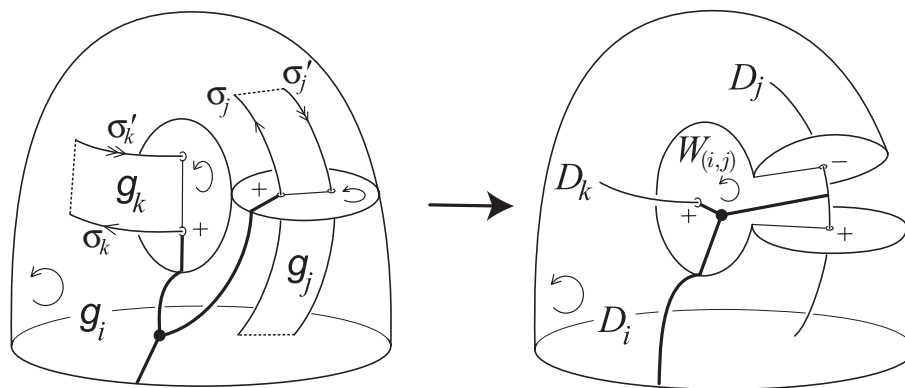


FIGURE 15. Left: A top stage of a capped grope cobordism. Right: The corresponding part of a Whitney concordance after pushing into 4-space and surgering a cap.

$\partial D_i$  equal to the union of the  $i$ th strands  $\sigma_i \subset B^3 \times \{0\}$  and  $\sigma'_i \subset B^3 \times \{1\}$  together with their endpoints crossed with  $I$ . For instance, any generic homotopy between  $\sigma$  and  $\sigma'$  defines such a singular concordance. A singular concordance of  $\sigma$  induces the orientation of  $\sigma$ .

An (order  $n$ ) Whitney tower whose bottom stages form a singular concordance is called an (order  $n$ ) *Whitney concordance*. Denote by  $\mathbb{W}_n(\ell)$  the set of order  $n$  Whitney concordances of  $\ell$ -component string links.

Let  $G^c$  be a simple grope cobordism (from  $\sigma$  to  $\sigma'$ ) in  $\mathbb{G}_n^c(\ell)$ . Think of  $G^c$  as sitting in the middle slice  $B^3 \times \{1/2\}$  of  $B^3 \times I$ . Extending  $\sigma \subset G^c$  to  $B^3 \times \{0\}$ , via the product with  $[0, 1/2]$ , and extending  $\sigma' \subset G^c$  to  $B^3 \times \{1\}$ , via the product with  $[1/2, 1]$ , yields a collection of class  $n$  capped gropes properly embedded in  $B^4 = B^3 \times I$ , i.e. a *grope concordance*, from  $\sigma$  to  $\sigma'$ . After perturbing the interiors of the caps slightly, we may assume that all caps are still disjointly embedded and that a cap which intersected the  $j$ th string link component in the grope cobordism now has a single transverse intersection point with the interior of a bottom stage of the  $j$ th grope in the grope concordance. By fixing the appropriate orientation conventions, this construction preserves the signs of these intersection points.

Consider the effect of the construction on the (degree  $n$ ) trees  $t(g_i^c)$  which were embedded in the original  $G^c$  and are now sitting in the class  $n$  capped gropes in the 4-ball. Any root vertex that was lying on an  $i$ th string link strand is now in the interior of the  $i$ th bottom stage, and any tip that corresponded to an intersection between a cap and a  $j$ th strand now corresponds to an intersection between a cap and a  $j$ th bottom stage. These are exactly the labeled trees associated to gropes in 4-manifolds as described in [Sch06], and Theorem 6 of [Sch06] describes how to surger such gropes to an order  $(n - 1)$  Whitney concordance  $\mathcal{W}$  while preserving trees, meaning that the labeled trees associated to the gropes become the order  $(n - 1)$  geometric intersection tree  $\tilde{\tau}_{n-1}(\mathcal{W})$ . Although signs and orientations are not discussed in [Sch06], the notation there is compatible with the sign conventions of this paper and a basic case of the compatibility is illustrated in Figure 15 which shows a ‘push and surger’ step in the modification of a three-dimensional grope cobordism to a Whitney concordance applied to a top stage. The modification in general involves ‘hybrid’ grope towers but reduces essentially to this case as explained in [Sch06].

DEFINITION 29. The commutative diagram (1) in the Introduction is explained as follows.

- (i) Let  $\overline{\mathbb{W}}_{(n-1)}$  be the set of order  $(n - 1)$  Whitney concordances modulo the relation that two Whitney towers with the same geometric intersection tree are the same.

- (ii) The above constructions define the map  $push-in : \mathbb{G}_n^c(\ell) \longrightarrow \overline{\mathbb{W}}_{(n-1)}(\ell)$  which pushes a grope into  $B^3 \times I$  and surgers it into a Whitney tower. It is used in our main diagram (1) in the Introduction. The Whitney tower produced from a grope is not unique, as it depends on the choice of caps one uses to surger, which is why we need to divide  $\mathbb{W}_{(n-1)}(\ell)$  by an appropriate equivalence relation.

*Remark 30.* The only information contained in the original geometric intersection tree  $\tilde{\tau}_n^c(G^c)$  that is *lost* by the map (induced by)  $push-in$  is the ordering in which the univalent vertices of the trees in  $\tilde{\tau}_n^c(G^c)$  were attached to the string link components. Thus, pushing a class  $n$  grope cobordism into four dimensions, surgering to an order  $(n - 1)$  Whitney concordance and applying the map  $\tilde{\tau}_{(n-1)}$  is the same as the composition of the map  $\tilde{\tau}_n^c$  with the homomorphism

$$pull-off : \hat{\mathcal{A}}_n^t(\ell) \longrightarrow \hat{\mathcal{B}}_n^t(\ell)$$

that pulls the trees off the string link components and labels the univalent vertices accordingly.

Notice that the map  $pull-off$  is very different from the rational PBW-type isomorphism  $\sigma : \mathcal{A} \otimes \mathbb{Q} \rightarrow \mathcal{B} \otimes \mathbb{Q}$  as defined in [BarN95].

### 4. Jacobi identities in dimension 3

As a consequence of our work so far, IHX relations appear in  $\tilde{\mathcal{B}}_n^t$ , and hence  $\hat{\mathcal{B}}_n^t$ , as the image under  $\tilde{\tau}_{(n-1)}$  (respectively  $\hat{\tau}_{(n-1)}$ ) of Whitney concordances from any string link to itself (e.g. tube the 2-spheres in Theorem 1 into a product concordance). In §§ 4.1 and 4.2 we show that this phenomenon pulls back to the three-dimensional world: there are capped grope cobordisms from any string link to itself whose images under  $\tilde{\tau}_n^c$  (and  $\hat{\tau}_n^c$ ) give all IHX relations. We will also realize all IHX relations in a group generated by univalent *graphs* by defining a more general map  $\tilde{\tau}_n^g$  on *uncapped* class  $n$  grope cobordisms. In Appendix A, we will show how to interpret this map for grope cobordisms where genus is allowed at all stages.

#### 4.1 The IHX relation for string links

The geometric IHX construction for string links contained in Theorem 6 will play a key role in all subsequent IHX constructions. At the heart of the proof of Theorem 6 is a three-dimensional interpretation of Figure 3 which leads to the following construction of a capped grope cobordism that is (slightly) singular – these singularities will be removed in subsequent constructions.

CONSTRUCTION 31. Consider a trivial three-component string link in the 3-ball. We will construct a singular capped grope  $\bar{g}^c$  of class 3 with an unknotted boundary component on the surface of the ball. Its bottom stage is of genus 3 and embedded. The second stage surfaces of  $\bar{g}^c$  are of genus 1 and are each embedded. The interiors of the second stage surfaces intersect each other but are disjoint from the bottom stage of  $\bar{g}^c$ . Only the caps of  $\bar{g}^c$  intersect the three trivial string link strands. Denote by  $\bar{G}^c$  the union of  $\bar{g}^c$  together with trivial cobordisms of the strands of the string link (embedded 2-disks traced out by perturbations of the interiors of the strands). Then the key property of  $\bar{g}^c$  is that  $\tilde{\tau}_3^c(\bar{G}^c) \in \tilde{\mathcal{A}}_3^t(4)$  is equal to the three terms of the IHX relation in Figure 2. Here the strands of the trivial string link are labeled by 1, 2, 3 and  $\bar{g}^c$  is interpreted as a *null* bordism of its unknotted boundary which is labeled 4. Note that  $\tilde{\tau}_3^c$  still makes sense as a disjoint union of subtrees of  $\bar{g}^c$  whose tips are attached to intersections with caps, even though  $\bar{g}^c$  is singular.

To begin the construction of  $\bar{g}^c$ , consider Figure 3 again. Think of it as taking place inside a 3-ball  $B$ , so that the horizontal plane has an unknotted boundary on  $\partial B$ . The arcs that each puncture the plane twice are the three strands of a trivial string link. Add tubes around the arcs to turn the plane into a genus 3 surface  $\Sigma$ . This surface  $\Sigma$  is the bottom stage of our singular grope  $\bar{g}^c$ . We construct

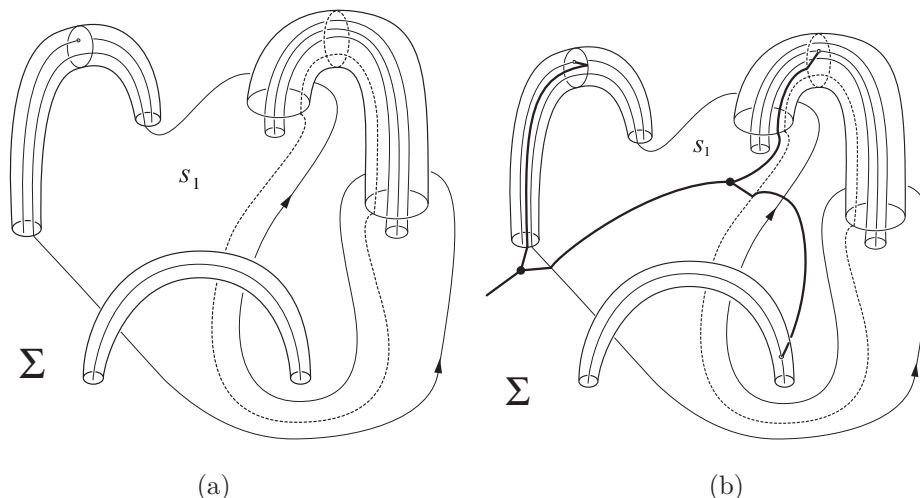


FIGURE 16. The construction of the capped surface  $s_1$  for the singular capped grope  $\bar{g}^c$  in Construction 31.

a symplectic basis for  $\Sigma$  as follows. Three of the curves are meridians to the tubes. To get the other three basis curves, connect the endpoints of each of the three pictured arcs in the plane (formerly Whitney arcs) by an untwisted arc that travels once over a tube. (Exercise: these three curves form a Borromean ring.) We fix surfaces bounding these latter three basis curves in the following way. Consider Figure 9, where a Whitney disk  $W_{(3,4)}$  is pictured. Thinking of the figure as being in a 3-ball (rather than a three-dimensional slice of 4-space), the Whitney disk has two intersections with an arc in strand 2 of the trivial string link, and adding a tube around this arc yields a surface  $s_1$  as illustrated in Figure 16(a). This surface has a pair of dual caps, whose boundaries are indicated by the dashed loops in Figure 16. One of these caps intersects the upper right strand 2, and the other intersects the bottom strand 1; these caps also have circles of intersection (not shown in the figure) with the tubes of  $\Sigma$  around these strands (but these circles of intersections will be eliminated during later applications of this construction). The curve dual to the attaching curve of  $s_1$  is a meridian to the strand 3 and so bounds a cap hitting strand 3 once. The tree structure for the stage  $s_1$  and its dual cap is  $[[1, 2], 3]$ , as shown in Figure 16(b).

Symmetrically, we can construct  $s_2$  and  $s_3$ , with trees  $[1, [2, 3]]$  and  $[[3, 1], 2]$ , by interpreting Figures 11 and 12 as both being in the 3-ball. Adding these three capped surfaces  $s_1, s_2, s_3$  to the surface  $\Sigma$  we get the desired singular capped grope  $\bar{g}^c$  bounded by strand 4. Including strand 4 as the root, the associated three trees give exactly the terms of the IHX relation. With a little extra effort in analyzing the orientations, one can verify that the signs of these three terms are correct.

*Proof of Theorem 6.* First consider the case where  $\ell = 4$  and  $(+t_I)\amalg(-t_H)\amalg(+t_X)$  is as in Figure 2. We will construct  $G^c$  as a grope cobordism of strand 4 together with trivial cobordisms (disks) of the other three strands. Take the 3-ball  $B$  from the above Construction 31 and remove regular neighborhoods of the three strands of the trivial string link in  $B$  to get a handlebody  $M$  which contains the uncapped body  $\bar{g}$  of the singular capped grope  $\bar{g}^c$ . Let  $m_i$  be a meridian to the  $i$ th strand on the surface of  $M$ . Now in the complement of a trivial four-component string link, embed  $M$  so that  $m_i$  is a meridian to strand  $i$ . Connect a parallel copy of the fourth strand by a band to the unknot  $\partial\bar{g}$  on the boundary surface of  $M$  calling the resulting strand  $4'$ . The embedding of  $M$  extends (by attaching disks to the  $m_i$ ) to an embedding of  $B$  into the 3-ball containing the four-component string link. Thus, 4 and  $4'$  cobound the singular capped grope  $\bar{g}^c$  from Construction 31 which sits inside  $B$ , where, by abuse of terminology, we let  $\bar{g}^c$  also denote the grope that has 4 and  $4'$  as its boundary.

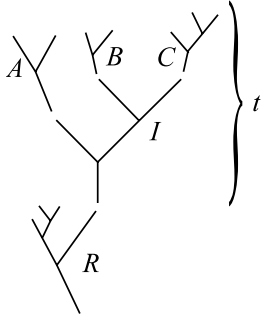


FIGURE 17.

Pick arcs  $\alpha$  and  $\beta$  contained in the bottom stage of  $\bar{g}^c$  and sharing endpoints with 4 and 4' such that  $\alpha \cup \beta$  splits  $\bar{g}^c$  into three capped grope cobordisms  $g_1^c$ ,  $g_2^c$  and  $g_3^c$ . If we number them appropriately,  $g_1^c$  modifies strand 4 to the strand  $\alpha$ ,  $g_2^c$  modifies  $\alpha$  to  $\beta$ , and  $g_3^c$  modifies  $\beta$  to 4'. Note that each of these three capped grope cobordisms is non-singular.

Examining the way in which the caps hit the strands, we see that

$$\prod_j \tilde{\tau}_3^c(G_j^c) = (+t_I) \prod (-t_H) \prod (+t_X),$$

where each  $G_j^c$  is just  $g_j^c$  together with trivial cobordisms on the first three strands.

In order to get the desired  $G^c$ , we wish to glue these cobordisms  $G_i^c$  back together so that the resulting grope is *embedded*. To do this, we use the transitivity argument from [CT04a], which is easily adapted to the current situation of arcs rel boundary (as opposed to knots). In that argument the individual gropes that are being glued together are homotoped inside the ambient 3-manifold until they match up. However, the homotopies are always isotopies when restricted to individual gropes (except in the framing correction move where some twists are introduced, which will not affect  $\tilde{\tau}^c(G^c)$ ). Thus  $\tilde{\tau}_3^c(G^c) = (+t_I) \prod (-t_H) \prod (+t_X)$  is not changed during this procedure.

Now consider the case where  $\ell = 4$  but the univalent vertices of the trees in the IHX relation are attached to strands  $j_1, j_2, j_3$  and  $j_4$  which are not necessarily distinct. Then the only modification needed in the above proof is to embed  $M$  so that the  $m_i$  are meridians to the  $j_i$ th strand arranged in the correct ordering ( $i = 1, 2, 3$ ), and make sure that the band from  $\partial\bar{g}$  attaches to the  $j_4$ th strand in the right place.

Finally, if there are more than four strands, add the rest of the strands to the picture away from the above construction. □

More generally, let us consider grope cobordisms of higher class.

**THEOREM 32.** *Let  $t_I, t_H$  and  $t_X$  be three trees which differ by the terms in an IHX relation in  $\hat{\mathcal{A}}_n^t(\ell)$ . Then there is a class  $n$  simple grope cobordism  $G^c$ , from the  $\ell$ -component trivial string link to itself, such that  $\tilde{\tau}_n^c(G^c) = (+t_I) \prod (-t_H) \prod (+t_X)$ .*

*Proof.* As in the proof of Theorem 6, we will construct a cobordism of one of the strands, extending the others by disks. As argued at the end of the proof, it is sufficient to assume that no two tips of any one tree are attached to the same component. Hence we may assume that  $\ell \geq n + 1$ . Further, as in that proof, we may assume that  $\ell = n + 1$  on the nose.

Decompose  $t_I$  into rooted trees  $I, A, B, C, R$ , where  $I$  represents the ‘I’ in the IHX relation, a chosen root of  $I$  is connected to  $R$ , and the tips of  $I$  connect to the roots of the trees  $A, B$  and  $C$ . Let the rooted tree given by the union of  $I, A, B$  and  $C$  be called  $t$  as illustrated in Figure 17. Think of the ball containing  $(n + 1)$  strands as a boundary-connected sum  $B_t \# B_R$ , where  $B_t$  is

a ball with strands which inherit the (distinct) labels of  $t$  and  $B_R$  is a ball with strands labeled distinctly from the rest of  $\{1, 2, \dots, n + 1\}$ .

Consider a capped grope  $g_t^c$  with one boundary component having geometric intersection tree equal to the tree  $t$  and contained in  $B_t$ . (To see that such a grope exists, note that a regular neighborhood of a grope is a handlebody, which can be thought of as a ball with a tubular neighborhood of some arcs removed. The tips are part of a spine for the handlebody, so that there is a bijection between tips and arcs, with each arc going through a single tip once. Thus, the tips bound disks that are punctured by distinct arcs. Now there is an embedding of this ball with arcs to  $B_t$  that takes the arcs to strands in  $B_t$  according to any bijection.)

Pruning the “I” part  $g_I^c$  of  $g_t^c$ , we get three capped gropes realizing the trees  $A, B, C$ , denoted  $g_A^c, g_B^c, g_C^c$  respectively. As in Theorem 6, consider the genus 3 handlebody  $M$  which is the complement of a trivial three-strand string link with  $m_i$  meridians to the strands on  $\partial M$ . Taking  $M$  to be a regular neighborhood of  $g_I^c$ , there is an embedding of  $M$  into  $B_t$  such that the  $m_i$  map to  $\partial g_A^c, \partial g_B^c$  and  $\partial g_C^c$ . Now, by Construction 31, there is a singular grope  $\bar{g}$  of class 3 inside  $M$  which bounds an unknot on the boundary of  $M$  such that the tips of  $\bar{g}$  bound parallel copies of  $g_A^c, g_B^c$  and  $g_C^c$ . (Note that these parallel copies intersect each other because the second stages of  $\bar{g}$  intersect each other, and because parallel gropes in dimension 3 intersect.)

Let  $g_R^c$  be a capped grope realizing the tree  $R$  inside  $B_R$ , such that the tip  $T_0$  of  $g_R^c$  corresponding to the tip of  $t$  that connects to the root of  $I$  bounds a cap that does not intersect any strands. Note that  $g_R^c$  can be surgered into a disk, so that its boundary is unknotted.

Tube the cap on the (unknotted) tip  $T_0$  of  $g_R^c$  to the (unknotted)  $\partial\bar{g}$  on the boundary of  $M$ . Connect sum the (unknotted)  $\partial g_R^c$  to (a push-off of) the strand in  $B_R$  corresponding to the root of  $R$ .

We get a singular capped grope cobordism  $\bar{G}^c$  taking the trivial  $(n + 1)$ -component string link to itself. The connected grope cobordism of the strand corresponding to the root of  $R$  is genus 3 at one stage and is embedded at that and all lower stages (the ‘ $R$  part’). Higher stages (the ‘ $A, B,$  and  $C$  parts’) that lie above different genus 1 subsurfaces of the genus 3 stage (in the ‘ $I$  part’) may intersect. Splitting the grope via Proposition 16 of [CT04a], we get three grope cobordisms, each separately embedded, which can then be reglued by transitivity, as in the proof of Theorem 6, to get a non-singular grope cobordism,  $G^c$ , with  $\widehat{\tau}_n^c(G^c) = (+t_I) \amalg (-t_H) \amalg (+t_X)$ . This completes the proof of Theorem 32.  $\square$

The previous theorem can be rephrased in the language of claspers and implies Theorem 7 of the Introduction.

A picture of three claspers of degree 3 as in Theorem 7 is given in Figure 9 of [CT04b]. This was derived from Theorem 6 using a mixture of claspers and gropes in the following way. First (using the notation in the proof of Theorem 6) the clasper representing  $g_1^c$  was drawn. Next, we modified strand 1 by  $g_1^c$  to the new position  $\alpha$ . We then drew in the clasper representing  $g_2^c$ . This clasper intersects the grope  $g_1^c$ , but using the usual pushing-down argument we pushed all the intersections down to the bottom stage. We then pushed them off the strand 0 boundary component of the grope, which is an isotopy in the complement of  $\alpha$ . This gave rise to two disjoint claspers, surgery on which moves strand 0 to the arc  $\beta$ . The process was repeated for the clasper representing  $g_3^c$ : it was pushed out of the trace of the first two grope cobordisms/clasper surgeries. We double-checked the result by performing surgery along these three claspers and verified that the result was isotopic to the original trivial four-component string link.

#### 4.2 General IHX relations and the map $\widehat{\tau}_n^g$

Next, we extend the realization of IHX relations from trees to arbitrary diagrams. Extending the map  $\widehat{\tau}_n^c$  to *uncapped* grope cobordisms involves some new wrinkles. First of all, in the absence of



caps bounding the grope tips, it will not be possible to attach the tips of the grope trees to  $\ell$  strands with a meaningful ordering; however, tips will still be associated to components of the string link according to the linking between the components and the corresponding tips. Secondly, non-trivial linking between certain tips will lead to the construction of graphs with non-zero Betti number which result from gluing together the corresponding tips.

The reader may wonder why we do not introduce a map  $\widehat{\tau}_n^g$  at the monoid level at this point. The reason is that  $\widehat{\tau}_n^g$  is well defined at the group level, by Proposition 36 below, but is not well defined at the monoid level, unless the choice of tips is included as part of the grope data.

**DEFINITION 33.** Consider the abelian group generated by connected *diagrams* (vertex-oriented univalent graphs) whose univalent vertices are labeled by the string link components  $1, \dots, \ell$  (possibly with repeats), modulo the AS antisymmetry relations. Also divide by the relation setting any diagram with a loop at a vertex to zero. Let  $\widehat{\mathcal{B}}_n^g(\ell)$  be the subgroup generated by such diagrams of grope degree  $n$ . (Recall that the grope degree is half the number of vertices plus the first Betti number.)

*Remark 34.* The fact that a loop at a vertex must be zero is a consequence of IHX relations, provided that  $n \geq 3$ . In the case  $n = 2$ , an AS relation implies that such a diagram is 2-torsion, and hence is zero over any ring where 2 is invertible.

Now we define  $\widehat{\tau}_n^g: \mathbb{G}_n(\ell) \rightarrow \widehat{\mathcal{B}}_n^g(\ell)$ . Let  $G$  be a grope cobordism of class  $n$ . First, choose a grope component  $g \subset G$ . As before, each genus 1 branch of  $g$  has an associated vertex-oriented trivalent rooted tree  $t$  whose tips  $L_i$  correspond to tips  $T_i$  of  $g$ . For each such  $T_i$ , choose either a component  $x_j$  of the string link, or another tip  $T_j$  of  $g$ , and label the corresponding tip  $L_i$  of  $t$  by  $(L_i, x_j)$ , or  $(L_i, T_j)$  respectively. The root of  $t$  is labeled by the string link component that the boundary of  $g$  meets. Now sum over all choices to get a formal sum of labeled trees denoted  $T(G)$ .

Now we proceed to glue together some of the tips on each of these labeled trees, based on the geometric information of how the tips link each other and the string link. Let  $t$  be a labeled tree. It has tips  $L_i$  labeled  $(L_i, T_j)$  or labeled  $(L_i, x_j)$ , where each tip  $L_k$  corresponds to the tip  $T_k$ . A *matching* of such a labeled tree  $t$  is a partition of the set of all the tips of  $t$  labeled by tips (and not string link components) into pairs, such that the labels on each pair are of the form  $(L_i, T_j), (L_j, T_i)$ . A matching determines a labeled connected graph  $\Gamma$ , obtained by gluing together matched tips of  $t$ , where each edge resulting from such a gluing assumes the coefficient  $\text{lk}(T_i, T_j) = \text{lk}(T_j, T_i)$ . Each of the remaining univalent vertices  $L_i$  is labeled by some component  $x_j$ , and assumes the coefficient  $\text{lk}(T_i, x_j)$ . Each such  $\Gamma$  determines a multiple of a generating diagram of  $\widehat{\mathcal{B}}_n^g(\ell)$ , where the coefficient of the diagram is the product of the coefficients on the tips and edges of  $\Gamma$ . Define  $\langle t \rangle$  as the sum of these elements in  $\widehat{\mathcal{B}}_n^g(\ell)$  over all matchings of  $t$ . If there are no matchings, then  $\langle t \rangle = 0$  by definition. Extend  $\langle \cdot \rangle$  to linear combinations of trees linearly. Now define  $\widehat{\tau}_n^g(G)$  to be  $\langle T(G) \rangle$ .

*Remark 35.*

- (i) If  $G$  extends to a simple grope cobordism  $G \subset G^c$ , then  $\widehat{\tau}_n^g(G)$  is just the image of  $\widehat{\tau}_n^g(G^c)$  under the map *pull-off*:  $\widehat{\mathcal{A}}_n^t(\ell) \rightarrow \widehat{\mathcal{B}}_n^g(\ell)$  that pulls the trees off the components of  $\ell$  and labels their univalent vertices accordingly.
- (ii) If one translates a grope cobordism into a union of rooted clasper surgeries, the map  $\widehat{\tau}_n^g$  can be calculated as follows. Instead of  $T(G)$ , consider the associated tree of each clasper with root labeled by the strand linked by the clasper's root, and then apply  $\langle \cdot \rangle$  as before. Then sum over all of the clasps. If the rooted clasper,  $C$ , can be turned into a simple clasper,  $C'$ , by turning Hopf pairs of tips into edges, then  $\widehat{\tau}_n^g(C)$  is the diagram which is the associated graph of  $C'$ , with univalent vertices labeled according to where the capped tips of  $C'$  meet the string link.

PROPOSITION 36. *The map  $\widehat{\tau}_n^g$  is well defined.*

We prove this in Appendix A, where we consider the more general situation of gropes which may not be of genus 1.

Finally, we show that the IHX relation can be realized in the world of graphs by uncapped gropes.

THEOREM 37. *Let  $D_I, D_H, D_X \in \widehat{\mathcal{B}}_n^g(\ell)$  be diagrams differing by the terms in an IHX relation. Then there is a grope cobordism  $G$ , from the trivial  $\ell$ -string link to itself, such that  $\widehat{\tau}_n^g(G) = D_I - D_H + D_X$ .*

*Proof.* First, cut some edges (not contained in the ‘I’ part) of  $D_I$  to make a tree  $D_I^t$ . Pick a univalent vertex that did not come from a cut as the root. Let  $\ell$  be the number of tips. As before, think of the complement of a trivial  $\ell$ -string link as a handlebody,  $M$ , with special curves  $\{m_i\}_{i=1}^\ell$  on its boundary. Let the tips of  $D_I^t$  be placed in correspondence with the curves  $m_i$ . Embed  $M$  in the complement of a trivial string link, such that, if a tip  $L_i$  of  $D_I^t$  is labeled by a component  $x$  of the string link, then the corresponding  $m_i$  links  $x$  exactly once. Also, tips resulting from cuts of  $D_I$  should have the corresponding  $m_i$  linking exactly once. Take a trivial subarc of the component of the string link corresponding to the root of  $D_I^t$  and perform a finger move so that it goes through  $M$  as a trivial subarc  $\eta$ . Now the proof of Theorem 32 yields a ‘weak’ capped grope cobordism  $g^{\bar{c}}$  (with  $g \subset M$ ) which modifies  $\eta$ , where the weakness comes from the fact that here the linking pairs of tips have intersecting caps. Ignoring this defect,  $g^{\bar{c}}$  extends (as in the proof of Theorem 32) to a (weak) capped grope cobordism  $G^{\bar{c}}$  of the trivial string link such that  $\widehat{\tau}_n^g(G^{\bar{c}}) = D_I - D_H + D_X$ . This can be seen as follows. Note that in this case  $\widehat{\tau}^g$  behaves just like  $\widehat{\tau}^c$ , except that it identifies tips corresponding to Hopf-linked tips (where the caps intersect), and hence glues the cut edges back together. The three different genus 1 pieces of  $G^{\bar{c}}$  link with each other in rather a complicated way but this is not seen by the map  $\widehat{\tau}_n^g$ . Also note that the tips of  $G$  are parallel to the curves  $m_i$ , so that the map  $\widehat{\tau}_n^g$  labels the univalent vertices appropriately.  $\square$

### 5. Grope cobordism of string links

Let  $\mathcal{L}(\ell)$  be the set of isotopy classes of string links in  $D^3$  with  $\ell$  components (which is a monoid with respect to the usual ‘stacking’ operation). Its quotient by the relation of grope cobordism (respectively capped grope cobordism) of class  $n$  is denoted  $\mathcal{L}(\ell)/G_n$  (respectively  $\mathcal{L}(\ell)/G_n^c$ ); compare Definition 18. The submonoid of  $\mathcal{L}(\ell)$  consisting of those string links which cobound a class  $n$  grope (respectively capped grope) with the trivial string link is denoted by  $G_n(\ell)$  (respectively  $G_n^c(\ell)$ ).

*Proof of Theorem 8.* Let us begin with the statements for the capped case. Then  $\mathcal{L}(\ell)/G_n^c$  can be identified with the quotient of  $\mathcal{L}(\ell)$  modulo the relation of simple clasper surgery of class  $n$ . This translation works just like for knots where it was explained in [CT04a]. All the results then follow from [Hab00, Theorem 5.4]. For example, the fact that the iterated quotients are central is proven by showing that  $ab = ba$ , modulo simple clasper surgery of class  $(n + 1)$ , if  $a$  is a string link that is simple clasper  $n$ -equivalent to the trivial string link. This follows by sliding the claspers (that turn the trivial string link into  $a$ ) past another string link  $b$ .

In the absence of caps one has to translate into rooted clasper surgery of grope degree  $n$  instead, as explained in [CT04a]. Just as above, all results follow from the techniques of Habiro [Hab00].  $\square$

This result makes it possible to try to compute the *abelian* iterated quotients in terms of diagrams, which we proceed to do. We shall first define the map from diagrams to string links modulo grope cobordism:

$$\Phi_n: \mathcal{B}_n^g(\ell) \rightarrow G_n(\ell)/G_{n+1}.$$

Indeed, we defined this for  $\ell = 1$  in [CT04b] in the following way. Given a diagram  $D \in \widehat{\mathcal{B}}_n^g(\ell)$ , find a grope cobordism  $G$  of class  $n$ , corresponding to a simple clasper, such that  $\widehat{\tau}_n^g(G) = D$ . Then define

$$\widehat{\Phi}_n(D) = \partial_1 G (\partial_0 G)^{-1},$$

where  $\partial G = \partial_0 G \cup \partial_1 G$ . One must show that the map is well defined, i.e. that the choice of embedding of the simple clasper does not matter. The argument given in [CT04b] works with little modification for all  $\ell \geq 1$ .

The next proposition implies that we can take any grope cobordism  $G$  satisfying  $\widehat{\tau}_n^g(G) = D$  in the above definition, not having to restrict to those corresponding to simple claspers.

PROPOSITION 38. *Given any grope cobordism  $G$  of class  $n$ ,  $\partial_1 G (\partial_0 G)^{-1} = \widehat{\Phi}_n \circ \widehat{\tau}_n^g(G) \in G_n(\ell) / G_{n+1}$ .*

*Proof.* Any grope cobordism can be refined to a sequence of genus 1 grope cobordisms by Proposition 16 of [CT04a] and this refinement evidently commutes with  $\widehat{\tau}_n^g$ . Then, using Theorem 35 of [CT04a], each of these cobordisms can be refined into a sequence of simple clasper surgeries and clasper surgeries of higher degree, and this refinement commutes with  $\widehat{\tau}_n^g$ . (To see this it suffices to notice that the ‘zip construction’ commutes with  $\widehat{\tau}_n^g$ .) Thus

$$\partial_1 G (\partial_0 G)^{-1} = (\partial_1 G)(L_k)^{-1}(L_k)(L_{k-1})^{-1} \cdots (L_1)(\partial_0 G)^{-1},$$

where the  $L_i$  are string links modified by successive simple clasper surgeries. Note that we can omit any pairs  $(L_i)(L_{i-1})^{-1}$  corresponding to clasper surgeries of higher degree, since this product is trivial in  $\mathcal{L}(\ell) / G_{n+1}$ . On the other hand, we know that for pairs differing by simple claspers  $C_i$  of degree  $n$ ,  $(L_i)(L_{i-1})^{-1} = \widehat{\Phi}_n(\widehat{\tau}_n^g(C_i))$ , by definition of  $\widehat{\Phi}_n$ . Thus

$$\begin{aligned} \partial_1 G (\partial_0 G)^{-1} &= \#_i \widehat{\Phi}_n(\widehat{\tau}_n^g(C_i)) \\ &= \widehat{\Phi}_n\left(\widehat{\tau}_n^g\left(\sum C_i\right)\right) \\ &= \widehat{\Phi}_n(\widehat{\tau}_n^g(G)), \end{aligned}$$

which completes the proof. □

We next show that  $\widehat{\Phi}_n$  vanishes on all IHX relations and hence descends to a well-defined map  $\Phi_n$ .

THEOREM 39. *The map  $\Phi_n: \mathcal{B}_n^g(\ell) \rightarrow G_n(\ell) / G_{n+1}$  is a well-defined surjective homomorphism.*

*Proof of Theorem 39.* By Theorem 37, any IHX relation,  $R_{\text{IHX}}$ , is the image under  $\widehat{\tau}_n^g$  of a grope cobordism,  $G$ , from a trivial string link to another trivial string link, denoted  $1_\ell$ . So by Proposition 38,

$$\begin{aligned} \widehat{\Phi}_n(R_{\text{IHX}}) &= \widehat{\Phi}_n(\widehat{\tau}_n^g(G)) \\ &= (\partial_1 G)(\partial_0 G)^{-1} \\ &= 1_\ell \# 1_\ell^{-1} \\ &= 1_\ell. \end{aligned}$$

Next we consider surjectivity of  $\Phi_n$ . The elements of  $G_n(\ell)$  are by definition of the form  $\partial_1 G$  where  $G$  is a class  $n$  grope cobordism with  $\partial_0 G = 1_\ell$ . By Proposition 38,  $\partial_1 G = \Phi_n \circ \widehat{\tau}_n^g(G)$ . □

Using the Kontsevich integral as a rational inverse, we are now able to prove Theorem 9, which says that  $\Phi_n$  turns into an isomorphism after tensoring with  $\mathbb{Q}$ .

*Sketch of proof of Theorem 9.* This was proven in full detail in [CT04b] for the case when  $\ell = 1$ . One sets up the (logarithm of the) Kontsevich integral as an inverse. Using the Aarhus integral [BGRT02], it is easy to show that the bottom degree term of the Kontsevich integral coincides with our map  $\widehat{\tau}_n^g$ . More precisely, if  $G$  is a grope cobordism, then Aarhus surgery formulae show that

$$(\log Z)_n(\partial_1 G(\partial_0 G)^{-1}) = \widehat{\tau}_n^g(G),$$

where  $(\log Z)_n$  is of grope degree  $n$ . Thus we get  $\Phi_n((\log Z)_n(\partial_1 G(\partial_0 G)^{-1})) = \partial_1 G(\partial_0 G)^{-1}$ , or  $\Phi_n \circ (\log Z)_n = \text{id}$ . On the other hand  $(\log Z)_n(\Phi_n(D)) = (\log Z)_n(\partial_1 G(\partial_0 G)^{-1})$  for a grope  $G$  satisfying  $\widehat{\tau}_n(G) = D$ . But then, by the above highlighted formula we can conclude that  $(\log Z)_n \circ \Phi_n = \text{id}$ .

Also, the Kontsevich integral of grope cobordisms of class  $(n + 1)$  will lie in degree  $(n + 1)$ , so that the Kontsevich integral indeed factors through  $G_n(\ell)/G_{n+1} \otimes \mathbb{Q}$ . (Here we use the fact that the Kontsevich integral of string links preserves the loop (and hence grope) degree.) The fact that  $\log Z_n$  is a homomorphism is straightforward using the Aarhus formula. (In [CT04b] we used the Wheeling isomorphism to show this for knots, but that was unnecessary. The lowest degree part of the Wheeling isomorphism is just the identity.)  $\square$

It is unknown whether the analogous statements for the relation of *capped* grope cobordism of string links are true. There are two difficulties. One is the question of whether one can realize the STU relations in  $\mathcal{A}_n(\ell)$  by capped grope cobordisms. The other is the question of whether Habiro’s main theorem [Hab00] generalizes from knots to string links. Does the Vassiliev filtration of string links agree with the relation generated by simple clasper surgery? It follows from the techniques of [CT04a] that the latter agrees with capped grope cobordism.

We conclude this section by carefully proving Lemma 3.11(c) from [CT04b], which we restate here for convenience.

LEMMA 40. *Let  $U$  be the unknot. Suppose three claspers  $C_i$  of grope degree  $n$  on  $U$  differ according to the IHX relation. Then  $U_{C_1} \# U_{C_2} \# U_{C_3} \in G_{n+1}(1)$ .*

*Proof.* Let  $K = U_{C_1} \# U_{C_2} \# U_{C_3}$ . The union of the three claspers corresponds to a grope cobordism,  $g$ , of class  $n$  between the unknot and  $K$ , where the bottom stage is of genus 3. By Proposition 38, we have that  $K = \Phi_n \circ \widehat{\tau}_n^g(g)$ . However  $\widehat{\tau}_n^g(g)$  is an IHX relator, and so, by Theorem 39,  $\Phi_n$  vanishes on it. Thus  $K$  is trivial in  $G_n(1)/G_{n+1}$ , implying that  $K \in G_{n+1}(1)$ .  $\square$

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### Appendix A. Associating a linear combination of graphs to an arbitrary grope

In this appendix we consider the set of class  $n$  grope cobordisms of  $\ell$ -string links, which may not be of genus 1. Let this set be denoted  $\widehat{\mathcal{G}}_n(\ell)$ .

Now we define  $\widehat{\tau}_n^g: \widehat{\mathcal{G}}_n(\ell) \longrightarrow \widehat{\mathcal{B}}_n^g(\ell)$ . Let  $G$  be a grope cobordism of class  $n$ . First, choose a grope component  $g \subset G$ . Choose tips for the grope component. Associate a linear combination of trees to  $g$  as follows. Each stage of  $g$  has a set of hyperbolic pairs of basis elements which bound further stages of the grope, or are tips. A branch of the grope is defined to be a choice of such a pair at the bottom stage, followed by a choice of hyperbolic pair at each stage which is bounded by the first pair, and so on. Each branch of the grope has an evident tree associated with it, whose tips  $L_i$  correspond to the tips  $T_i$  of the branch of the grope. As in the construction after Definition 33 in §4.2, for each such  $T_i$ , choose either a component  $x_j$  of the string link, or another tip  $T_j$  of  $g$ , and

label the corresponding tip  $L_i$  of  $t$  by  $(L_i, x_j)$  or  $(L_i, T_j)$  respectively. The root of  $t$  is labeled by the component of the string link that the boundary of  $g$  meets. Now sum over all choices, including all choices of branch of  $g$ , to get a formal sum of labeled trees denoted  $T(G)$ .

Now define  $\widehat{\tau}_n^g(G)$  to be  $\langle T(G) \rangle$ , as before.

PROPOSITION A1. *The map  $\widehat{\tau}_n^g$  is well defined.*

*Proof.* The first ambiguity is the orientation. As in Lemma 25, changing a positive quadrant results in a change of orientation all of the higher stages, including pushing annuli. Changing the orientation of a pushing annulus changes the sign of every term in  $\widehat{\tau}_n^g$ , either by reversing the sign of the linking number with another tip, or by changing the sign of the linking number with a string link component. Thus, as in the proof of Lemma 25, there are an even number of sign changes.

The second ambiguity is the choice of pushing annuli. Every tamely embedded grope can be extended to include pushing annuli, but this extension may not be unique. At a top stage of the grope, there will be two choices for every hyperbolic pair of tips, according to whether a given annulus extends ‘up’ or ‘down’ from the surface stage. Changing the choice of pushing annuli at a hyperbolic pair of tips has the effect of switching which quadrants are positive. However, the cyclic order of the vertex does not change. The induced orientations of the pushing annuli are either the same, or they are both reversed, resulting in no net change in sign.

The third ambiguity arises from choosing different tips for a grope component  $g \subset G$ . Notice that  $\widehat{\tau}_n^g$  never sees the linking of tips on the same stage of  $g$ . Either they belong to different branches and hence will be part of different tree summands, or they are dual to each other, in which case a graph with a loop at a vertex would result. Thus on a single surface stage, the linking number with objects  $c_i$  is all that matters, where  $c_i$  is either a component of the string link or another tip of  $g$  on a different stage.

Suppose we are not at a top stage. Then at least one curve in every hyperbolic pair bounds a higher surface stage. Removing a regular neighborhood of the higher surface stages, we get a planar surface. The tips become arcs joining some pairs of boundary components. Different choices of tips are related by Dehn twists on curves in the planar surface. Note that the boundary components of the planar surface are all null-homologous in the complement of  $\bigcup c_i$ . (They bound surfaces, and if the surfaces are slightly perturbed, they avoid  $c_i$ .) Hence choices of tips differ by multiples of curves which link the  $c_i$  trivially and hence do not change the contribution of  $g$  to  $\widehat{\tau}_n^g(G)$ .

Now suppose we are at a top stage of genus  $m$ . Any two choices of tips = symplectic bases  $(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m)$  are related by an element of  $Sp(2m, \mathbb{Z})$ , which is generated by the following automorphisms:

- (a) for some  $i$ ,  $\alpha_i \mapsto \alpha_i + \beta_i$  and everything else is fixed;
- (b) for some  $i$ ,  $\beta_i \mapsto \alpha_i + \beta_i$  and everything else is fixed;
- (c) for some  $i \neq j$ ,  $\begin{cases} \alpha_i \mapsto \alpha_i + \alpha_j \\ \beta_j \mapsto -\beta_i + \beta_j \end{cases}$  and everything else is fixed;
- (d) for some  $i \neq j$ ,  $\begin{cases} \alpha_i \mapsto \alpha_i + \beta_j \\ \alpha_j \mapsto \beta_i + \alpha_j \end{cases}$  and everything else is fixed;
- (e) for some  $i \neq j$ ,  $\begin{cases} \beta_i \mapsto \beta_i + \alpha_j \\ \beta_j \mapsto \alpha_i + \beta_j \end{cases}$  and everything else is fixed;
- (f) for some  $i \neq j$ ,  $\begin{cases} \beta_i \mapsto \beta_i + \beta_j \\ \beta_j \mapsto -\alpha_i + \alpha_j \end{cases}$  and everything else is fixed.

Let us adopt the following notation for expressing the contribution  $T(g)$  of  $g$  to  $T(G)$ . Compute the disjoint union of trees where the tips correspond to the tips of  $g$ , and label each tip  $L_i$  by a linear combination  $\sum_r n_r \mathbf{c}_r$  where the labels  $\mathbf{c}_r$  correspond to components of the string link and tips  $T_j$  of  $g$  with  $j \neq i$  (and  $n_r$  is the corresponding linking number with  $T_i$ ). This represents  $T(g)$  by expanding the trees linearly in the labels. Note that if any labeled trees in  $T(g)$  represent zero modulo AS or IHX relations, then these relations will still be present upon gluing, so that the corresponding contribution to  $\widehat{\tau}_n^g(G) = \langle T(G) \rangle$  will also be zero.

The trees in  $T(g)$  before and after applying the first automorphism (a) above only differ in a subtree isomorphic to a ‘Y’, which we can represent by a bracket  $[\cdot]$ . The difference is then represented by

$$\left[ \sum_r \text{lk}(\alpha_i, c_r) \mathbf{c}_r, \sum_r \text{lk}(\beta_i, c_r) \mathbf{c}_r \right] - \left[ \sum_r \text{lk}(\alpha_i + \beta_i, c_r) \mathbf{c}_r, \sum_r \text{lk}(\beta_i, c_r) \mathbf{c}_r \right].$$

Breaking the second summand into two terms, we see that

$$\left[ \sum_r \text{lk}(\beta_i, c_r) \mathbf{c}_r, \sum_r \text{lk}(\beta_i, c_r) \mathbf{c}_r \right] = 0$$

is sufficient to show that  $T(g)$ , and hence  $\widehat{\tau}_n^g(G)$ , remains unchanged. The fact that  $[x, x] = 0$  corresponds to the statement that a loop at a vertex is zero. The case of the second automorphism (b) is handled in the same way.

Let us consider the third automorphism (c). Abbreviate the notations  $\sum_r \text{lk}(\alpha, c_r) \mathbf{c}_r$  by  $\text{lk}(\alpha, c)$ . Then notice that the difference in  $T(g)$  only occurs in the  $i$  and  $j$  trees, and this difference is

$$[\text{lk}(\alpha_i, c), \text{lk}(\beta_i, c)] + [\text{lk}(\alpha_j, c), \text{lk}(\beta_j, c)] - [\text{lk}(\alpha_i + \alpha_j, c), \text{lk}(\beta_i, c)] - [\text{lk}(\alpha_j, c), \text{lk}(-\beta_i + \beta_j, c)],$$

which is easily seen to be zero. The cases of the last three automorphisms (d)–(f) are handled identically.  $\square$

We remark that Proposition 38 is still true for this extended definition of  $\widehat{\tau}_n^g$ .

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