HIGHER CATEGORIES IN A NUT-SHELL

These are notes for a talk at the Max Planck Institute for Mathematics in Spring 2013. The goal was to convince the audience that the necessary definitions for a good model of (symmetric monoidal) (∞, n)-categories can be presented in one lecture. These gadgets naturally form a *relative category* and examples include n-dimensional bordism categories and deloopings of the symmetric monoidal category of vector spaces. Maps between these objects describe local (functorial, homotopical) field theories, more precisely, the *space* of such field theories is the simplicial mapping space in this relative category.

We will ignore set theoretic issues in the following, refering to Mike Shulman's article [Sh].

1. Strict n-categories via iterated enrichments

Let A be a category with products, the letter A standing for *ambient*. Then there is a new category A-Cat with products, namely A-enriched categories, or shorter A-categories:

Definition 1. An A-category C consists of a set (of objects) C_0 and objects $\mathsf{C}(X, Y)$ in A (of morphisms) for each pair $X, Y \in \mathsf{C}_0$. Moreover, there are composition maps

$$\circ : \mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \to \mathsf{C}(X, Z)$$

which are (strictly) associative and have (strict) identities.

Morphisms in the category A-Cat are (strict) functors and the product is given objectwise. Iterating this construction, one obtains strict versions of higher categories.

Definition 2. A strict n-category is a category enriched in strict $(n-1)$-categories. These are the objects in the category (with products) n-Cat^{st}, defined inductively, starting at 0-$\mathsf{Cat}^{st} = \mathsf{Set}$. Similarly, starting with $\mathsf{A} = \mathsf{Top}$, the category of compactly generated Hausdorff spaces, leads to strict *topological n-categories* which are categories enriched in strict topological $(n-1)$-categories.

A product preserving functor $\mathsf{A} \to \mathsf{B}$ induces a product preserving functor A-$\mathsf{Cat} \to \mathsf{B}$-$\mathsf{Cat}$. Starting with the known adjunctions between Cat and Set thus leads to adjunctions between n-Cat^{st} and $(n-1)$-Cat^{st}. These can be combined to the functor of *isomorphism classes of objects* $\pi_0 : n$-$\mathsf{Cat}^{st} \to \mathsf{Set}$ and to the *discrete n-category* functor $\mathsf{Set} \to n$-Cat^{st}.

Grothendieck's *homotopy hypothesis* requires that a good notion of n-groupoids includes a classifying space functor which induces an "equivalence" between n-groupoids and n-types (spaces whose homotopy groups vanish above dimension n). If one defines the notion of a strict n-groupoid inductively (similarly to the weak case below) from strict n-categories, it was already known to Grothendieck that no essentially surjective classifying space functor exists, see [Si]. In fact, Dmitri Ara showed [A] that simply connected strict n-groupoids lead exactly to products of Eilenberg MacLane spaces.

This is one reason to generalize definitions from strict to weak n-categories. There are many ways to do this, see e.g. Tom Leinster's beautiful survey [L], and we present an iterated construction, where enriched categories are simply replaced by enriched simplicial sets.

2. Enriched simplicial sets

Definition 3. An A-simplicial set S consists of a set (of vertices) S_0 and objects $\mathsf{S}(X_0, \dots, X_k)$ in A (of k-simplices) for each sequence $X_0, \dots, X_k \in \mathsf{S}_0$. Moreover, there are structure maps

$$\sigma^* : \mathsf{S}(X_0, \dots, X_k) \to \mathsf{S}(X_{\sigma(0)}, \dots, X_{\sigma(m)})$$

for every order preserving map $\sigma : [m] \to [k] := \{0, \dots, k\}$. These maps are required to compose: $(\sigma_1 \circ \sigma_2)^* = \sigma_2^* \circ \sigma_1^*$ ("higher associativity") and there is the "identity axiom" saying that $\mathsf{S}(X_0) \in \mathsf{A}$ are terminal objects. The last condition implies that the unique map $[m] \to [0]$ induces "unit" maps from a terminal object to $S(X_0, \dots, X_0) \in \mathsf{A}$.

Remark 4. The above notion of an A-simplicial set (or A-enriched simplicial set) compares to a simplicial object in A exactly as an A-enriched category compares to a category internal to A. It is important for the considerations below that S_0 is a set and not an object in A.

We get a category of A-simplicial sets with products (taken simplex-wise). A morphism $f : \mathsf{S} \to \mathsf{S}'$ is a map between vertex sets $f_0 : \mathsf{S}_0 \to \mathsf{S}'_0$, together with morphisms in A, $f_k : \mathsf{S}(X_0, \dots, X_k) \to \mathsf{S}'(f(X_0), \dots, f(X_k))$ that are compatible with the structure maps.

Example 5. If $\mathsf{C} = (\mathsf{C}_0, \mathsf{C}(X, Y), \circ)$ is an A-category, we define its *nerve* N_C to be the A-simplicial set with vertices C_0 and k-simplices

$$\mathsf{N}_\mathsf{C}(X_0, \dots, X_k) := \mathsf{C}(X_0, X_1) \times \mathsf{C}(X_1, X_2) \times \dots \times \mathsf{C}(X_{k-1}, X_k)$$

The structure maps are induced by compositions $\circ$ and projections, just like for the ordinary nerve of a category. For example, the k projections from the right hand side of the above equation to $\mathsf{C}(X_i, X_{i+1})$ correspond to the inclusions of the edge $\langle i, i+1 \rangle$ into a k-simplex. Another important example is the first face map $d_1^* : \mathsf{N}_\mathsf{C}(X_0, X_1, X_2) \to \mathsf{N}_\mathsf{C}(X_0, X_2)$ which is simply given by the composition $\circ$ in C.

One should think of the nerve as having k-simplices determined by the longest path from the vertex X_0 to the vertex X_k. It is an important excercise to check that the associativity of composition $\circ$ is equivalent to a simplicial identity for maps $[1] \to [3]$ (which in turn implies all required simplicial identities).

The nerve construction will later lead to an inclusion of strict n-categories into weak n-categories and similarly of topological n-categories into (∞, n)-categories.

The goal of the next notion is to weaken the condition that nerves of A-categories satisfy by definition. For this purpose, we need to assume that the ambient category A allows a notion of *equivalences* $X \xrightarrow{\sim} Y$. For example, we use bijections for $\mathsf{A} = \mathsf{Set}$ and π_*-isomorphisms for $\mathsf{A} = \mathsf{Top}$.

Definition 6. An A-simplicial set S satisfies the *Segal conditions* if for each $k \geq 2$, the product of inclusions of the edges $\langle i, i+1 \rangle$ into a k-simplex induces an equivalence

$$\mathsf{S}(X_0, \dots, X_k) \xrightarrow{\sim} \mathsf{S}(X_0, X_1) \times \mathsf{S}(X_1, X_2) \times \dots \times \mathsf{S}(X_{k-1}, X_k) \tag{Seg}$$

For $\mathsf{A} = \mathsf{Set}$ with bijections $\sim$, we see that simplicial sets satisfy the Segal conditions if and only if they are nerves of categories. Below a *weak* 1-category will be a simplicial set satisfying the Segal conditions and hence $\mathsf{Cat} \simeq 1\text{-}\mathsf{Cat}$ as desired. A *Segal category* is a simplicial space satisfying the Segal conditions, i.e. one uses $(\mathsf{A}, \sim) = (\mathsf{Top}, \sim_{\pi_*})$ in (Seg).

3. A model for weak n-categories and (∞, n)-categories

Definition 7. Let $(\infty,0)$-$\mathsf{Cat} := \mathsf{Top}$ with π_*-isomorphisms and "homotopy category" equal to the fundamental groupoid $hX := \pi_{\leq 1}(X)$. Assume inductively that we defined a category (∞,n)-Cat with products together with

- a notion of equivalences $\sim$,
- a functor $h : (\infty,n)\text{-}\mathsf{Cat} \to \mathsf{Cat}$ that preserves products and equivalences.

Let $(\infty,n+1)$-Cat be the category of (∞,n)-Cat-simplicial sets that satisfy the Segal conditions (Seg). The (nerve of the) homotopy category $h\mathsf{S}$ of an $(\infty,n+1)$-category S is obtained by applying $\pi_0 \circ h$ the each k-simplex. An equivalence $f : \mathsf{S} \xrightarrow{\sim} \mathsf{S}'$ is an enriched simplicial map inducing equivalences

$$f_k : \mathsf{S}(X_0,\ldots,X_k) \xrightarrow{\sim} \mathsf{S}'(f(X_0),\ldots,f(X_k)) \quad \text{and} \quad hf : h\mathsf{S} \xrightarrow{\sim} h\mathsf{S}'$$

This is probably the simplest model for (∞,n)-categories, also called *Segal* n-categories. If we start with 0-$\mathsf{Cat} := \mathsf{Set}$ with bijections as equivalences then the same inductive procedure leads us to n-Cat, the category of (weak) n-categories, with products, a notion of equivalences and a functor $h : n\text{-}\mathsf{Cat} \to \mathsf{Cat}$. This beautiful inductive approach goes back to Simpson [Si] and Tamsamani.

The functors $\pi_0 : \mathsf{Top} \rightleftharpoons \mathsf{Set}$ lead to functors $(\infty,n)\text{-}\mathsf{Cat} \rightleftharpoons n\text{-}\mathsf{Cat}$ and in fact, there is a whole filtration between these two notions given by $(n+k,n)$-categories: These are $(n+k)$-categories with all r-morphisms (weakly) invertible for $r > n$. We thus require a careful analysis of invertibility which we shall only formulate in the ∞-setting.

Definition 8. An (∞,n)-*groupoid* is an (∞,n)-category S such that all k-simplices $\mathsf{S}(X_0,\ldots,X_k) \in (\infty,n-1)$-$\mathsf{Cat}$ are $(\infty,n-1)$-groupoids and moreover, the homotopy category $h\mathsf{S}$ is a groupoid. We define $\pi_0(\mathsf{S}) := \pi_0(h\mathsf{S})$, as isomorphism classes of objects, and inductively $\pi_k(\mathsf{S},X) := \pi_{k-1}(\mathsf{S}(X,X),1_X)$ for $k > 0$ and a basepoint $X \in \mathsf{S}_0$. As usual, these are groups for $k \geq 1$, abelian for $k > 1$.

This gives full subcategories (∞,n)-Grp and n-Grp whose objects are (∞,n)-groupoids respectively (weak) n-groupoids (defined inductively exactly as in the ∞-case). They inherit products, notions of equivalence and their homotopy category functors take values in groupoids $\mathsf{Grp} \simeq 1\text{-}\mathsf{Grp}$.

We shall next explain the relation between n-groupoids and n-types which uses the classifying space functor. It can be defined for an $(\infty,n+1)$-category S inductively as follows: Assume $B : (\infty,n)\text{-}\mathsf{Cat} \to \mathsf{Top}$ is constructed and preserves products and equivalences. Define $B(\mathsf{S})$ as the usual classifying space of the simplicial space with k-simplices $B(\mathsf{S}(X_0,\ldots,X_k))$. Segal's work [Se] implies that B preserves products and equivalences and that on $(\infty,n+1)$-groupoids, it also preserves homotopy groups:

$$\pi_k(B(\mathsf{S}),X) \cong \pi_k(\mathsf{S},X)$$

Theorem 9 (Grothendieck's homotopy hypothesis)**.** *The classifying space functor induces*

$$B : n\text{-}\mathsf{Grp} \xrightarrow{\sim} n\text{-}\mathsf{Typ}$$

The right hand side is the full subcategory of Top *consisting of spaces with vanishing homotopy groups above dimension n. Both sides are naturally $(\infty,1)$-categories and our statement is that B induces an equivalence between them.*

The most subtle point is the correct notion of "mapping space" between two objects of the above categories. On the right hand side this is routine: One replaces the domain n-type by a CW-complex (cofibrant replacement) and then takes the (compactly generated) compact-open topology on the mapping space. It is easy to check that the homotopy groups of these mapping spaces vanish above dimension n and hence the right hand side is in fact an $(n+1,1)$-category.

On the left hand side, there is no obvious model structure and in fact, we don't want to define the structure of a topological category on n-groupoids but rather the corresponding weak notion, an $(\infty,1)$-category.

There is a beautiful approach going back to Clark Barwick and Dan Kan who showed that any *relative category* has a well defined "homotopy theory" which can be expressed by saying that it defines an $(\infty,1)$-category, unique up to equivalence in the Joyal model structure. In our setting, the equivalences we defined inductively give n-Grp the structure of a relative category and hence lead to an $(\infty,1)$-category (which is $(n+1,1)$ by the theorem).

4. Symmetric monoidal higher categories

These are the key to a precise definition of functorial field theories. Our approach again goes back to Segal [Se] who first introduced the category Γ^{op} of finite pointed sets into the discussion. The object $\{0,1,\dots,k\}$ is denoted by k^+, where $0 \mathrel{\widehat{=}} +$ is the basepoint.

Definition 10. Let A be a category with products and equivalences. A *weakly symmetric* A-*monoid* is a functor $F:\Gamma^{op}\to\mathsf{A}$ satisfying the *Segal conditions*: The product of the k maps $p_i : k^+ \to 1^+$ sending $i\in\{1,\dots,k\}$ to 1 and everything else to $+$ induces an equivalence

$$F_k \xrightarrow{\sim} F_1^{\times k} \tag{Segal}$$

Here we wrote $F_k := F(k^+)$. For $k=0$ this condition says that $F_0\in\mathsf{A}$ is terminal.

For $\mathsf{A}=(\infty,n)$-Cat we obtain the notion of a *symmetric monoidal* (∞,n)-category and for $\mathsf{A}=n$-Cat we get *symmetric monoidal* n-categories (using our notion of equivalence $\sim$).

Note that there is a third map $p:2^+\to 1^+$ with $p(1)=p(2)=1$ and it induces the weakly symmetric multiplication $F_1\times F_1 \xrightarrow{\iota} F_2 \xrightarrow{p_*} F_1$, after the choice of "inverse" ι for the Segal map $F_2 \xrightarrow{\sim} F_1\times F_1$.

This is just like the third edge $\langle 0,2\rangle$ in the 2-simplex for the Segal conditions (Seg). In fact, there is a functor $\Delta^{op}\to\Gamma^{op}$ which associates to each weakly symmetric A-monoid a weakly associative A-monoid. The latter is simply an A-simplicial set with one object satisfying the Segal conditions.

The functor $\Delta^{op}\to\Gamma^{op}$ is easiest to describe by noting that Δ^{op} is equivalent to the following category: Objects are finite ordered sets with two basepoints, one being the smallest element and the other the largest. Morphisms are order preserving maps which preserve both basepoints. Then the functor to Γ^{op} just ignores the order and sends both basepoints to the single basepoint.

Example 11. Symmetric monoidal 1-categories can be shown to form a category equivalent to (the usual notion of) symmetric monoidal categories.

Symmetric monoidal $(\infty,0)$-categories or symmetric monoidal ∞-groupoids are usually referred to as Γ-spaces. They turn out to be Quillen equivalent to E_∞-spaces [M].

A Γ-spaces is *grouplike* if the monoid structure on $\pi_0(F_1)$ induced by p is a group. These are also called *Picard* ∞-groupoids and they give a very simple model for *connective spectra*.

In particular, the inner Hom in connective spectra is homotopy equivalent to the inner Hom in Picard ∞-groupoids and this is the way one can calculate the space of invertible field theories. Sometimes it may be advantagous to compute the enriched Hom in spectra whose connective cover is the inner Hom.

References

[A] Dimitri Ara, *On homotopy types modelized by strict ∞-groupoids.* ArXiv 2012

[BK] Clark Barwick and Dan Kan, *Relative categories: Another model for the homotopy theory of homotopy theories.*

[L] Tom Leinster, *A survey of definitions of n-category.*

[M] Mike Mandell, *An Inverse K-Theory Functor.*

[Se] Graeme Segal, *Categories and Cohomology Theories.*

[Sh] Mike Shulman, *Set theory and Category theory.* ArXiv 2012

[Si] Carlos Simpson, *The Homotopy of Higher Categories.* Book 2012