A VERY INFORMAL INTRODUCTION TO WHITNEY TOWERS

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Joint work with Jim Conant and Rob Schneiderman

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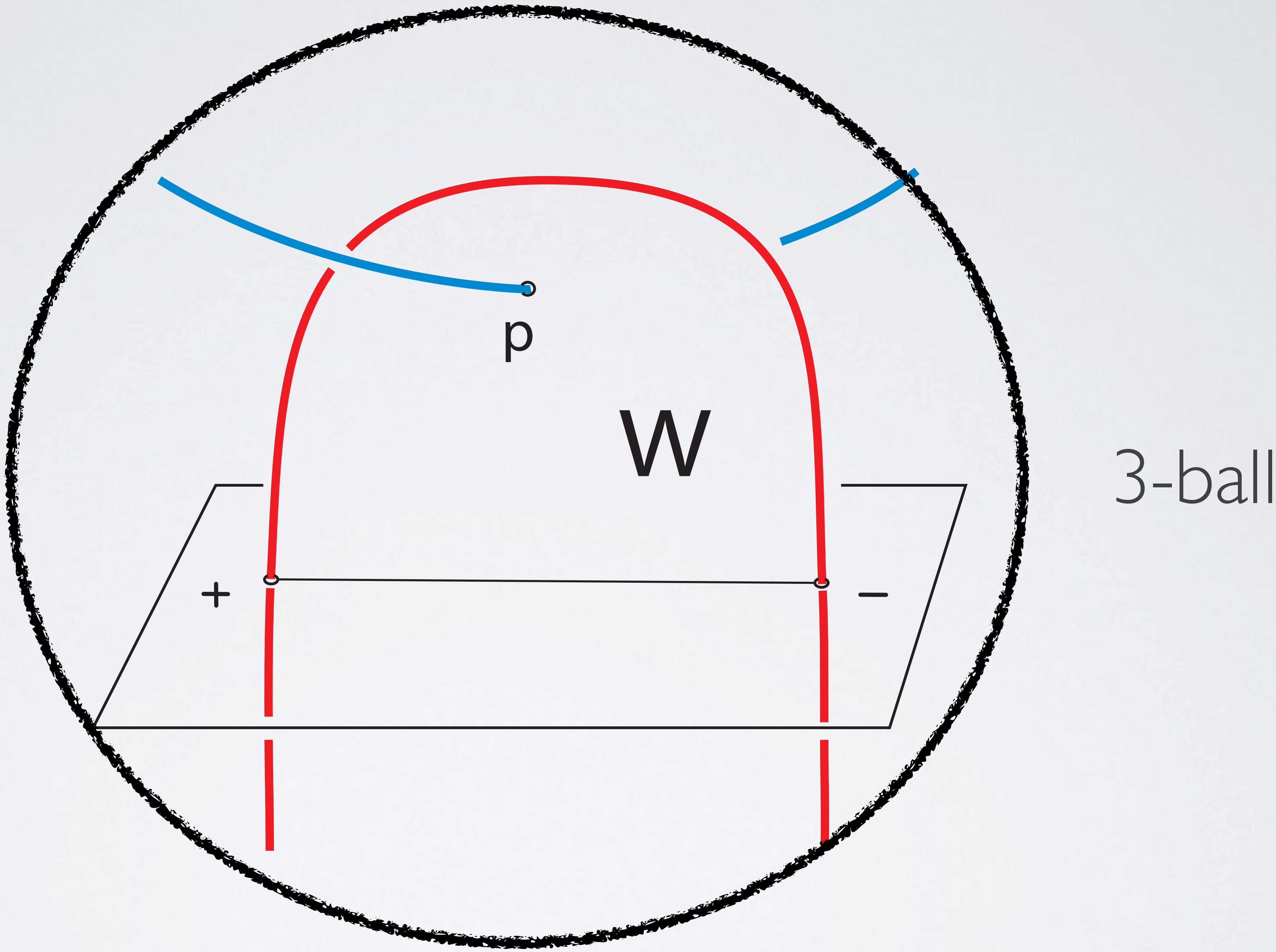
GOALS

- Motivate Whitney towers and their order.
- Understand those m-component links in the 3-sphere that bound Whitney towers of order n in the 4-ball.
- Compute $W_n(m)$ = the associated graded groups.

MAIN PROBLEM IN DIM. 4



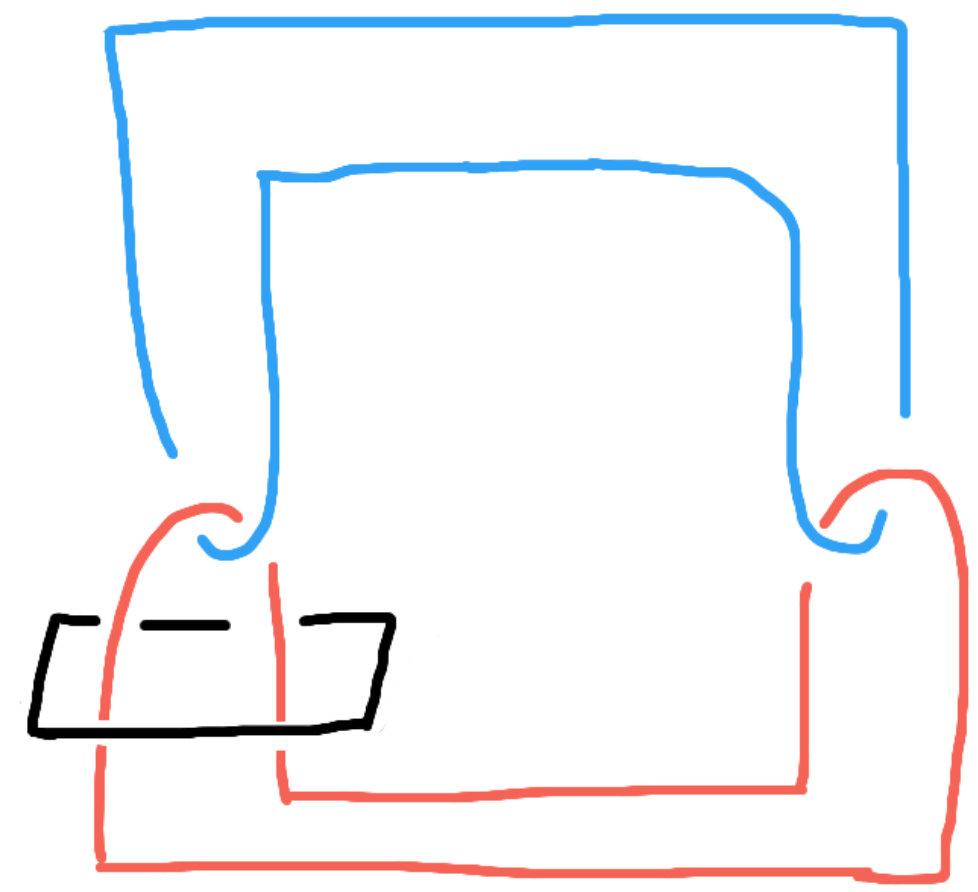
In 4-ball = neighborhood of Whitney disk W = 3-ball x time:
red and blue arcs become disks!

UGLY LINK ON THE BOUNDARY

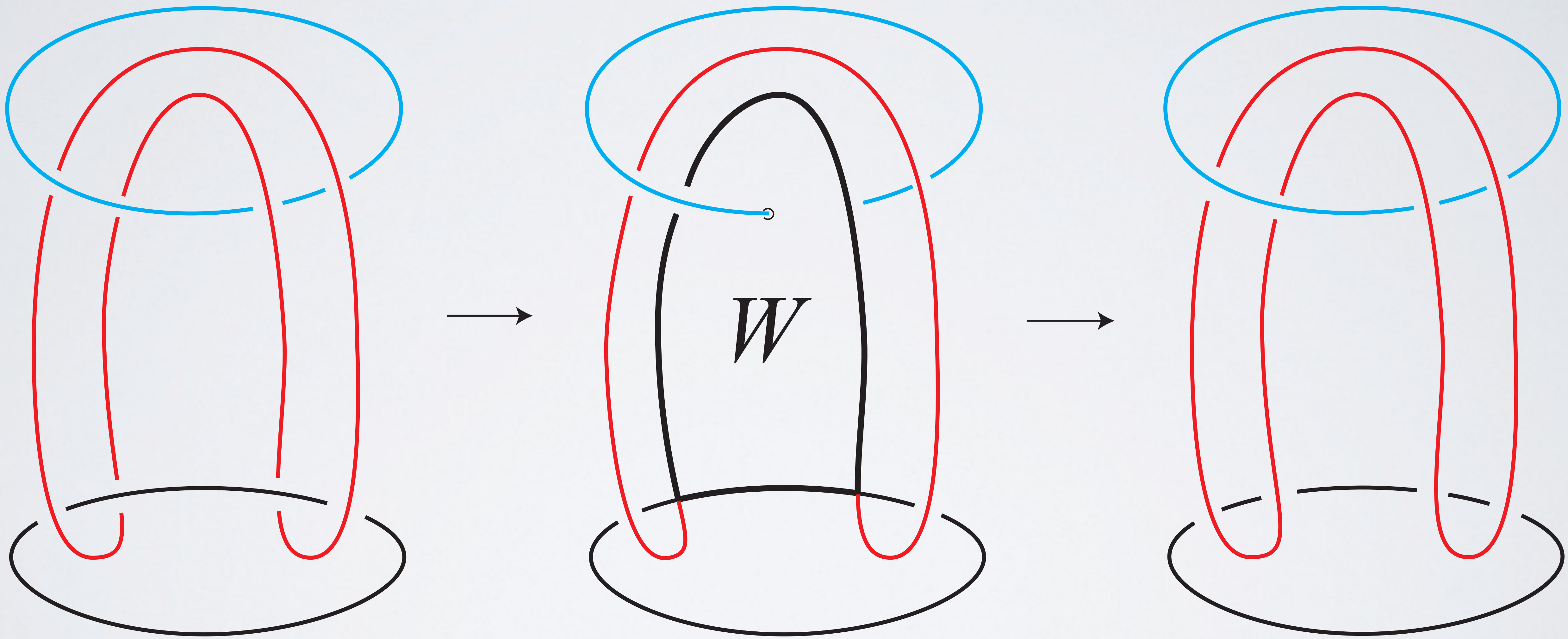
A neighborhood of the Whitney disk W is a 4-ball (3-ball x time) and its boundary is a 3-sphere.

The three disks in 4-ball have their boundary in this 3-sphere, the (ugly) Borromean rings.

What if they were slice ??



NICE LINK ON THE BOUNDARY



Our four disks in the 4-ball can be seen in this movie.

FIRST APPEARANCE OF THIS PROBLEM

Subtlety in Freedman's Theorem (1982):

Any odd unimodular form λ is realized as the intersection form of exactly two closed simply-connected topological 4–manifolds.

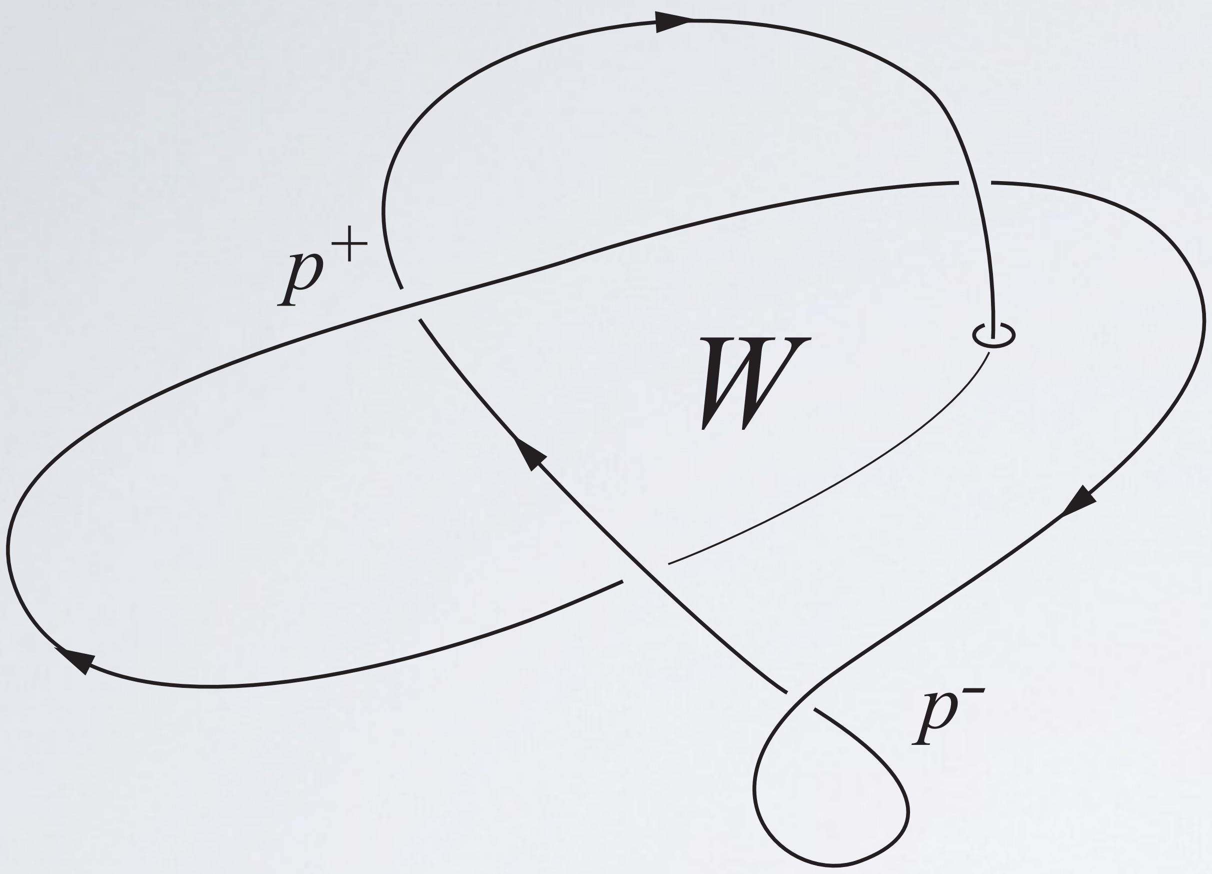
These manifolds are homotopy equivalent and are distinguished by the following equivalent criteria: Exactly one of the 4-manifolds....

... is smoothable after crossing with the real line,

... has vanishing Kirby-Siebenmann invariant,

... exhibits the following formula for its quadratic function τ:

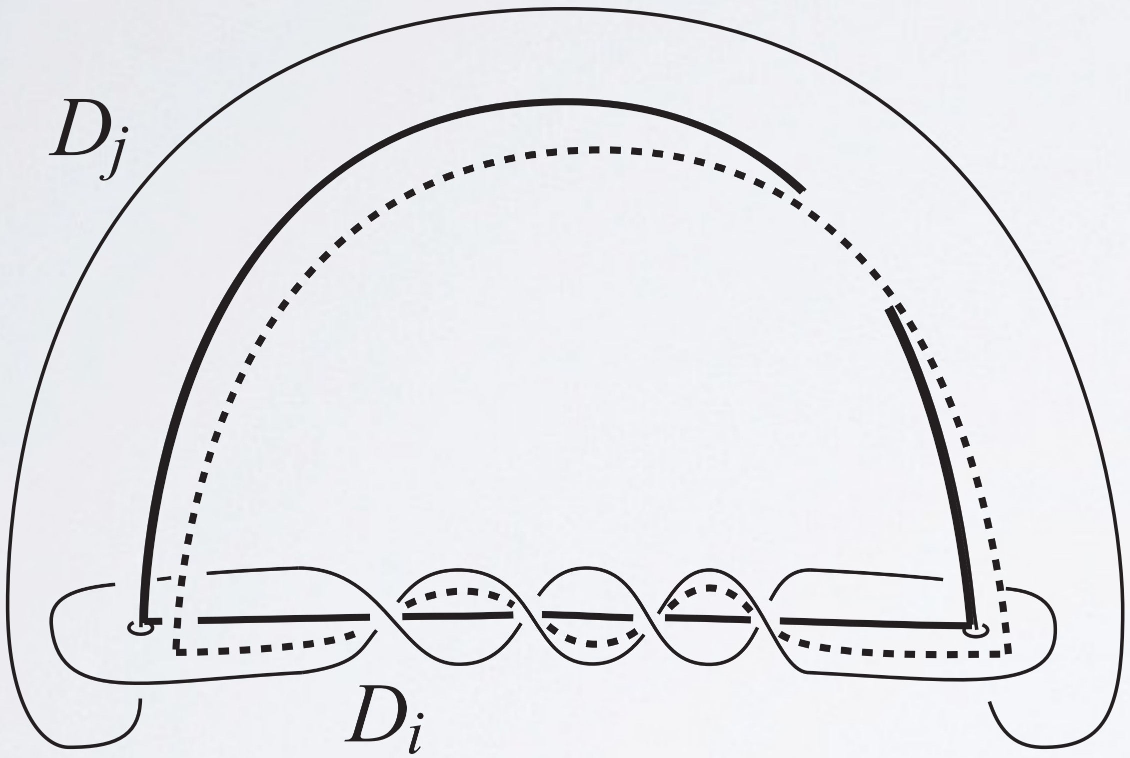
$$\tau(c) = (\lambda(c, c) - \text{signature } \lambda)/8 \mod 2 \quad \forall \text{ characteristic } c.$$

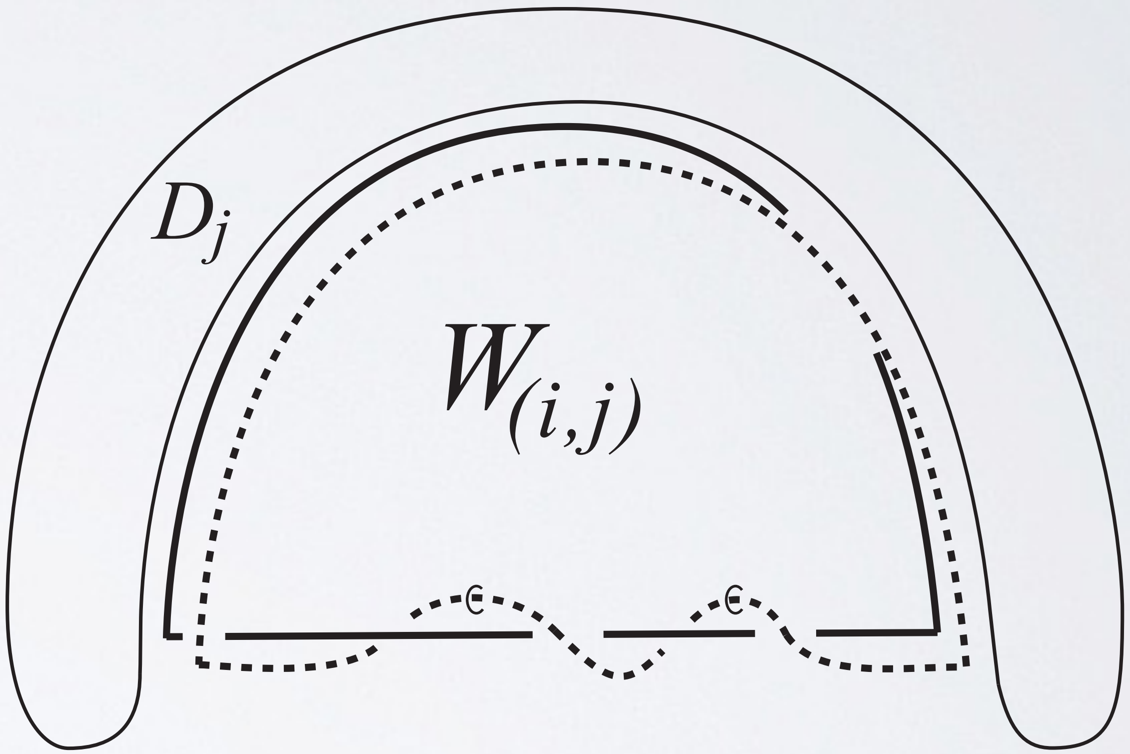


Sister projective plane: Attach a 1-framed 2-handle along trefoil and close off by the unique contractible 4-manifold.

The contractible manifold replaces a 4-handle and can be obtained by a topological plus construction from the homology sphere = 1-framed Dehn surgery on trefoil.

A TWISTED WHITNEY DISK





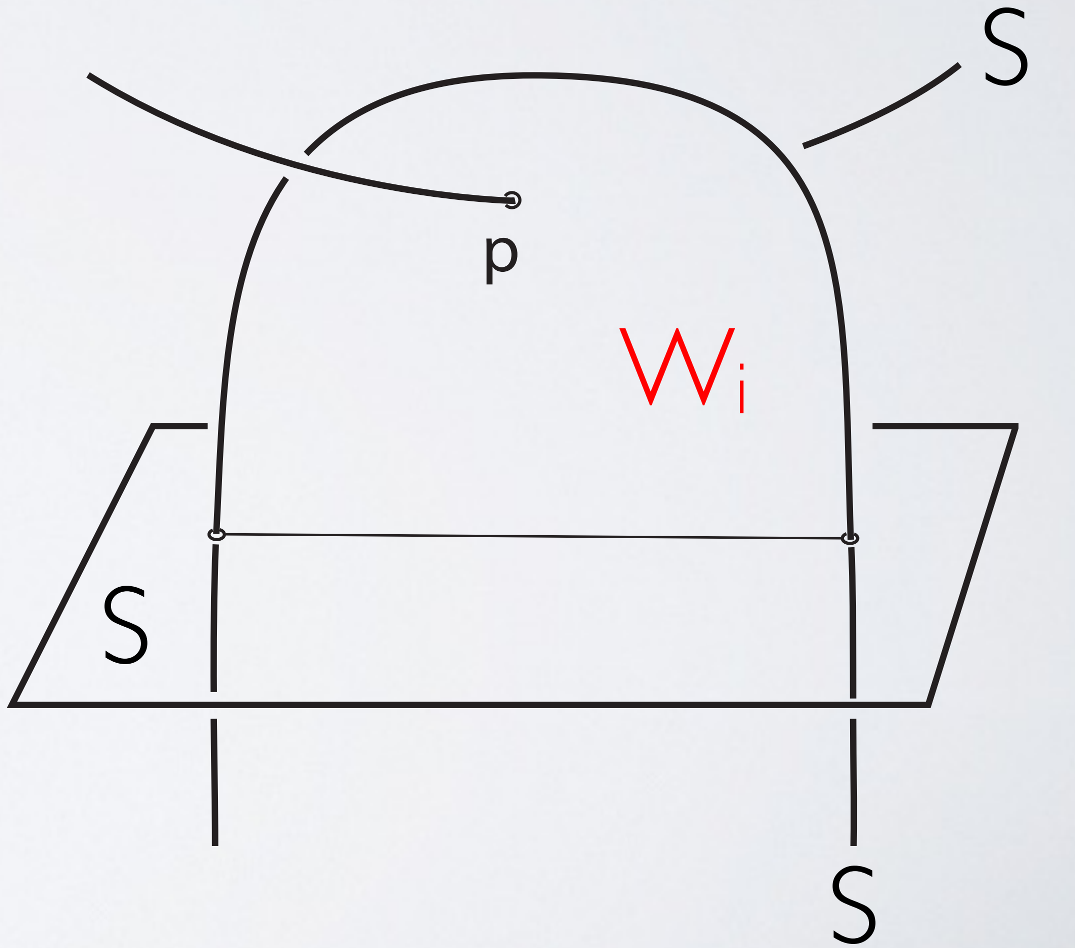
Twist link on the boundary is not slice unless twist = 0.
We work with framed Whitney disks for now.

HIGHER ORDER INTERSECTIONS

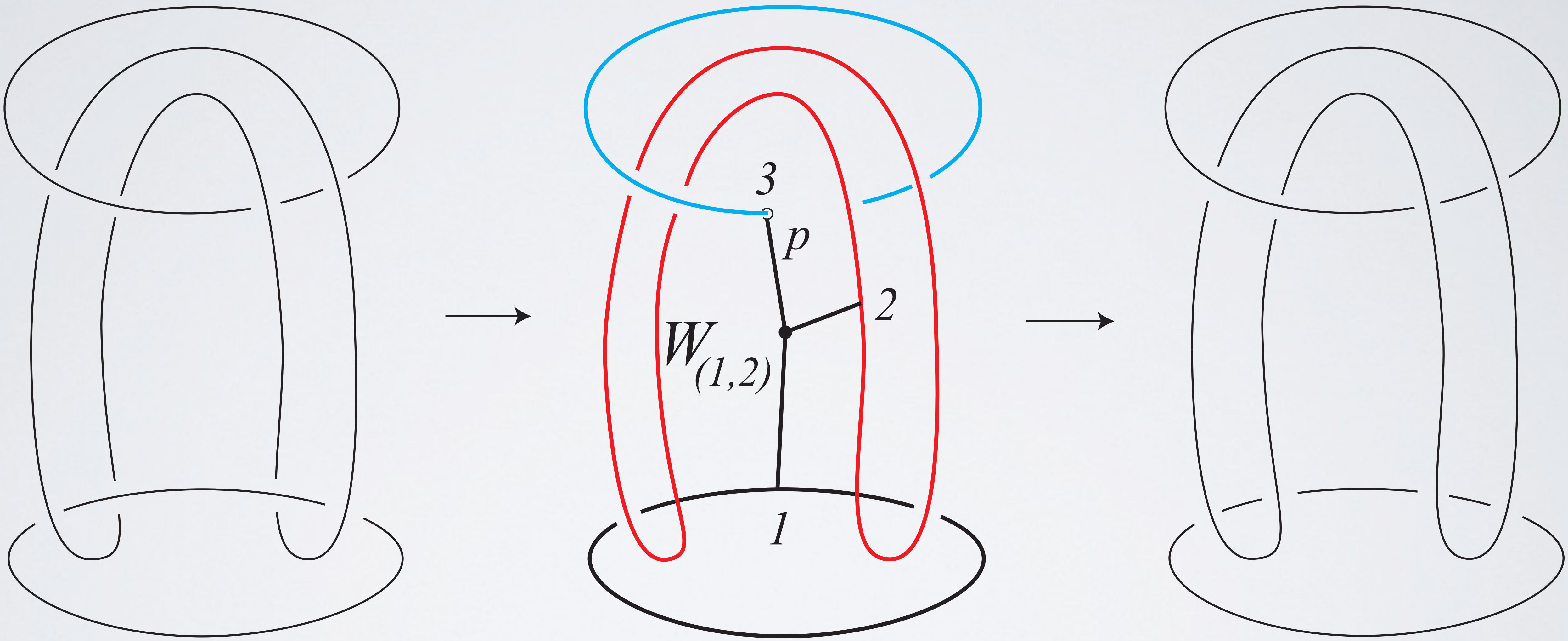
$$\tau(c) = \tau_1(S, W_i) := \sum_i \#\{S \pitchfork W_i\} \qquad \mod 2$$

S is a 2-sphere, immersed in the 4-manifold and representing the characteristic element c.

S has algebraically vanishing number of self-intersections, paired by Whitney disks W_i.

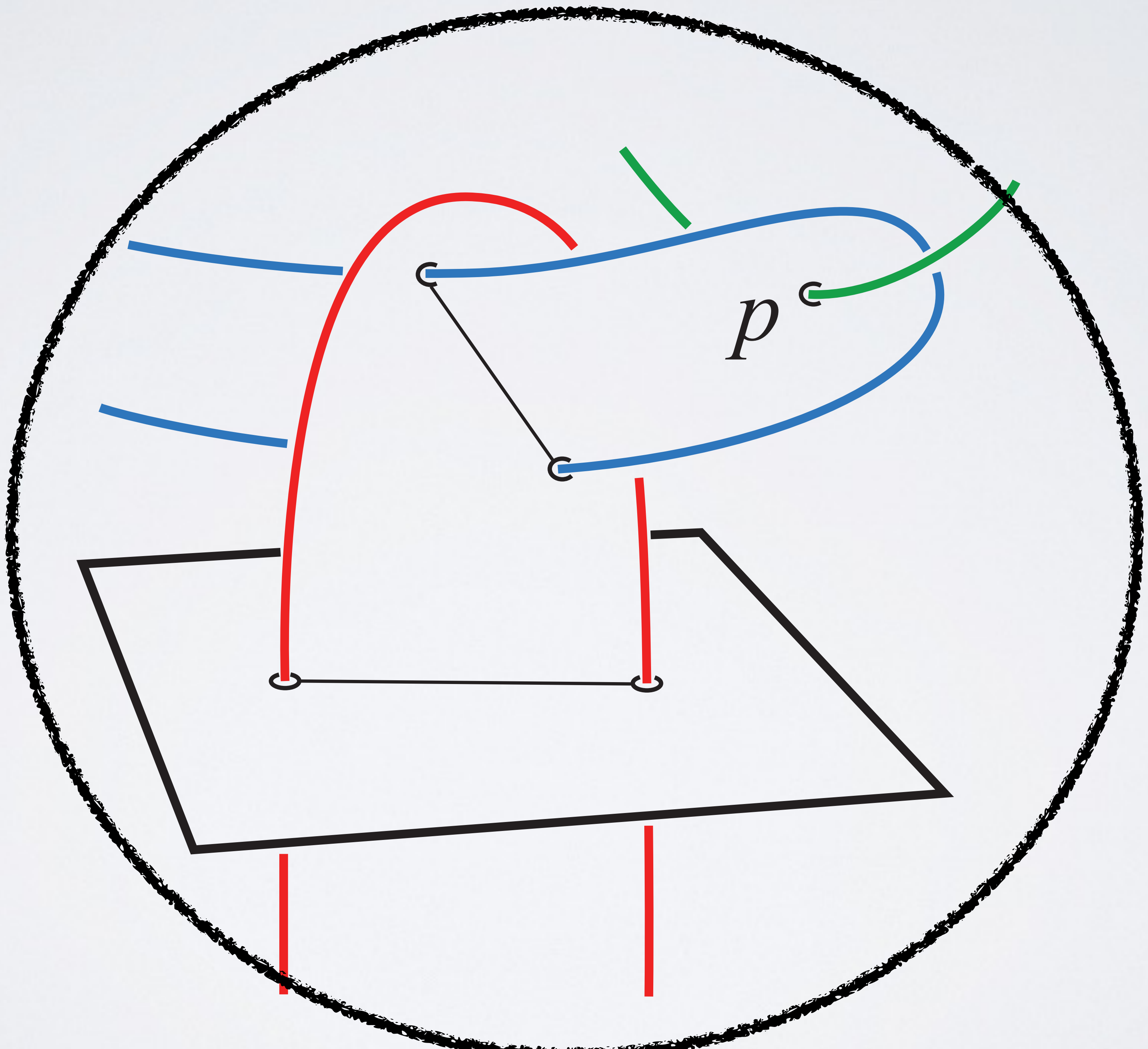


THE SIMPLEST TREE

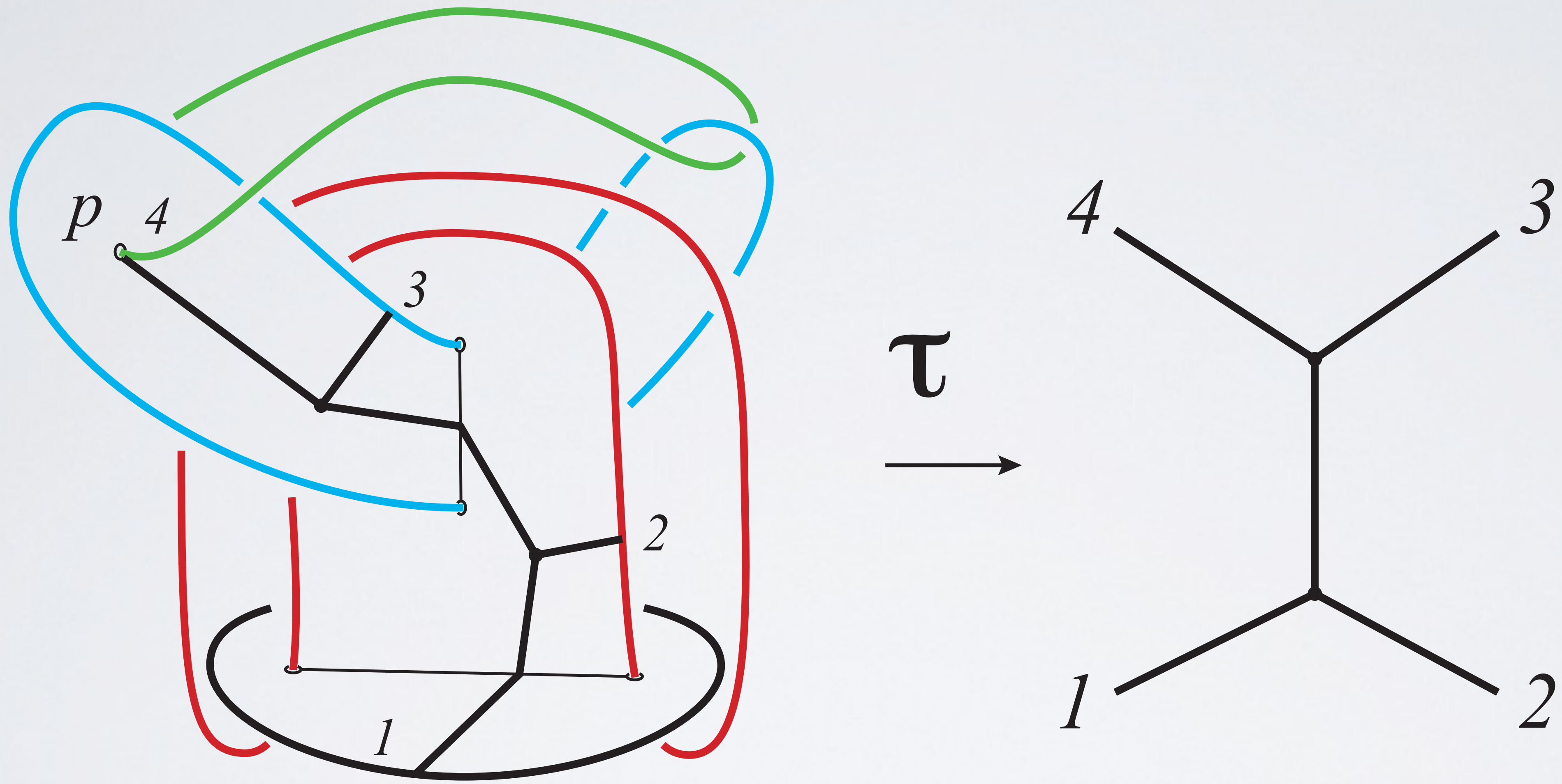


Combinatorics is described by labelled trivalent tree.
Freedman counts the number of such trees modulo 2.

SOLVE FIRST PROBLEM, GET A SECOND PROBLEM:

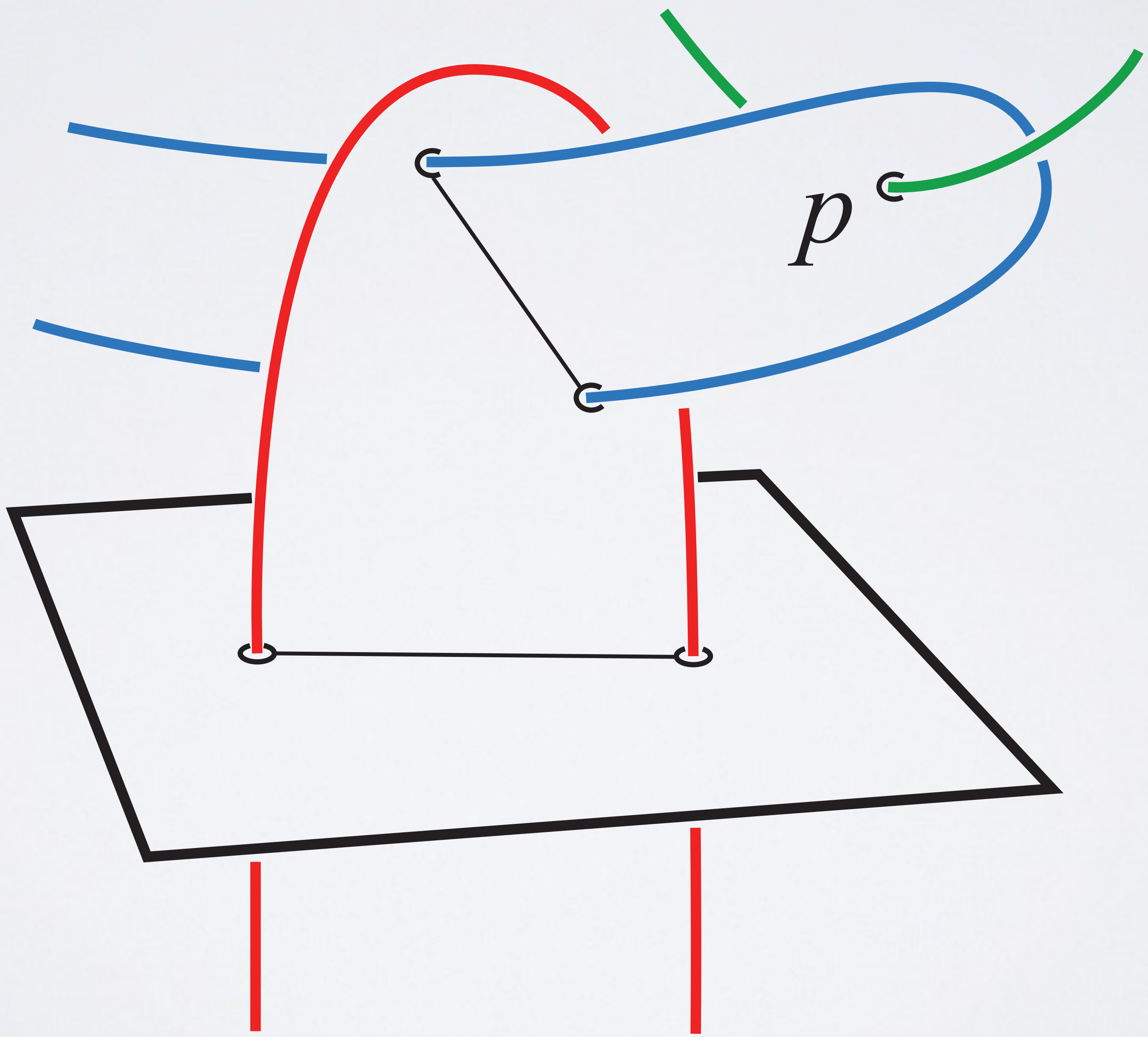


USE HIGHER ORDER TREES



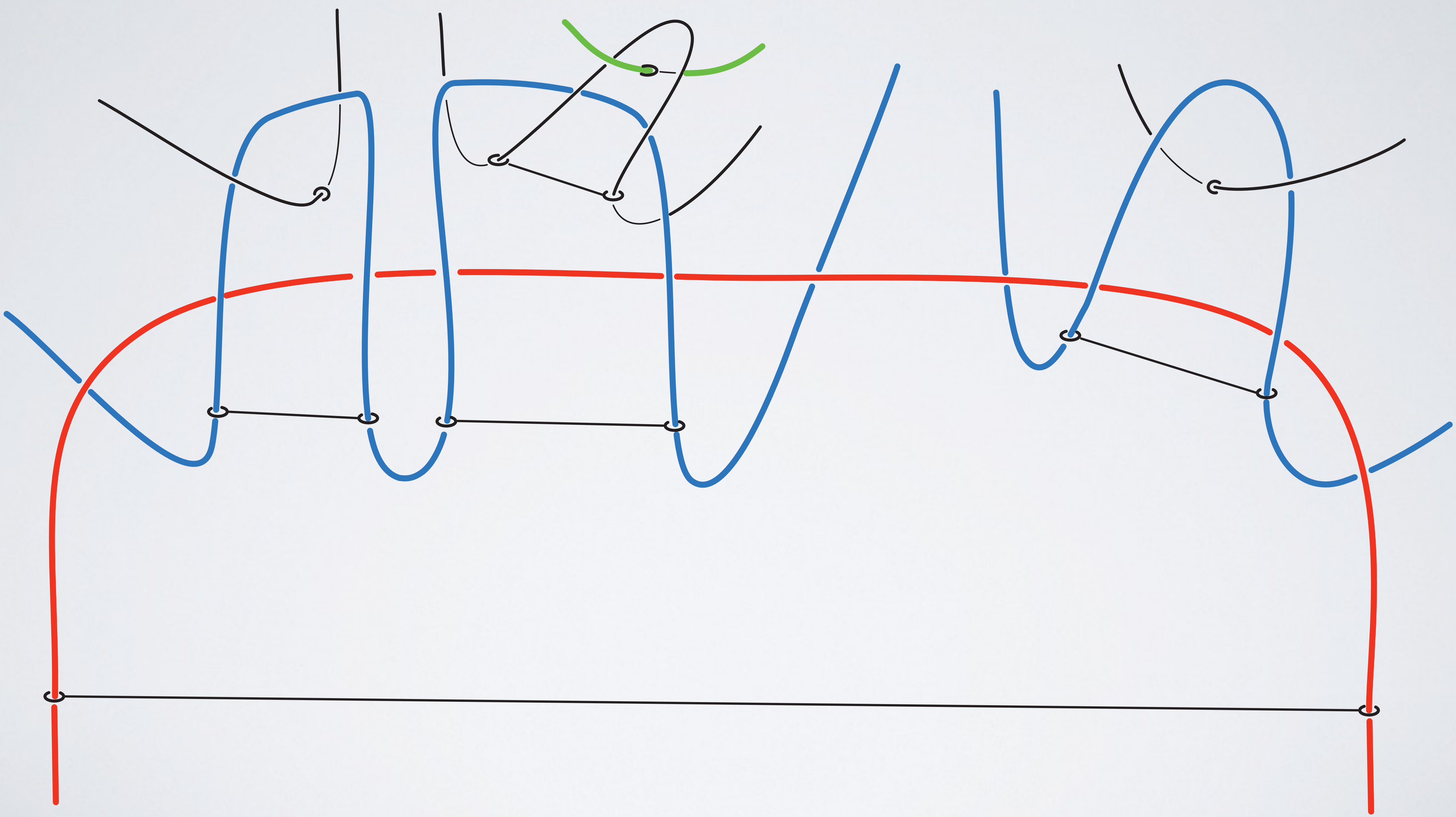
Theorem [C-S-T]: One can read off the lowest order Milnor invariants of a link from a Whitney tower in the 4-ball that bounds it, in fact, just from its intersection trees.

CORRESPONDING TO HIGHER ORDER WHITNEY DISKS ...

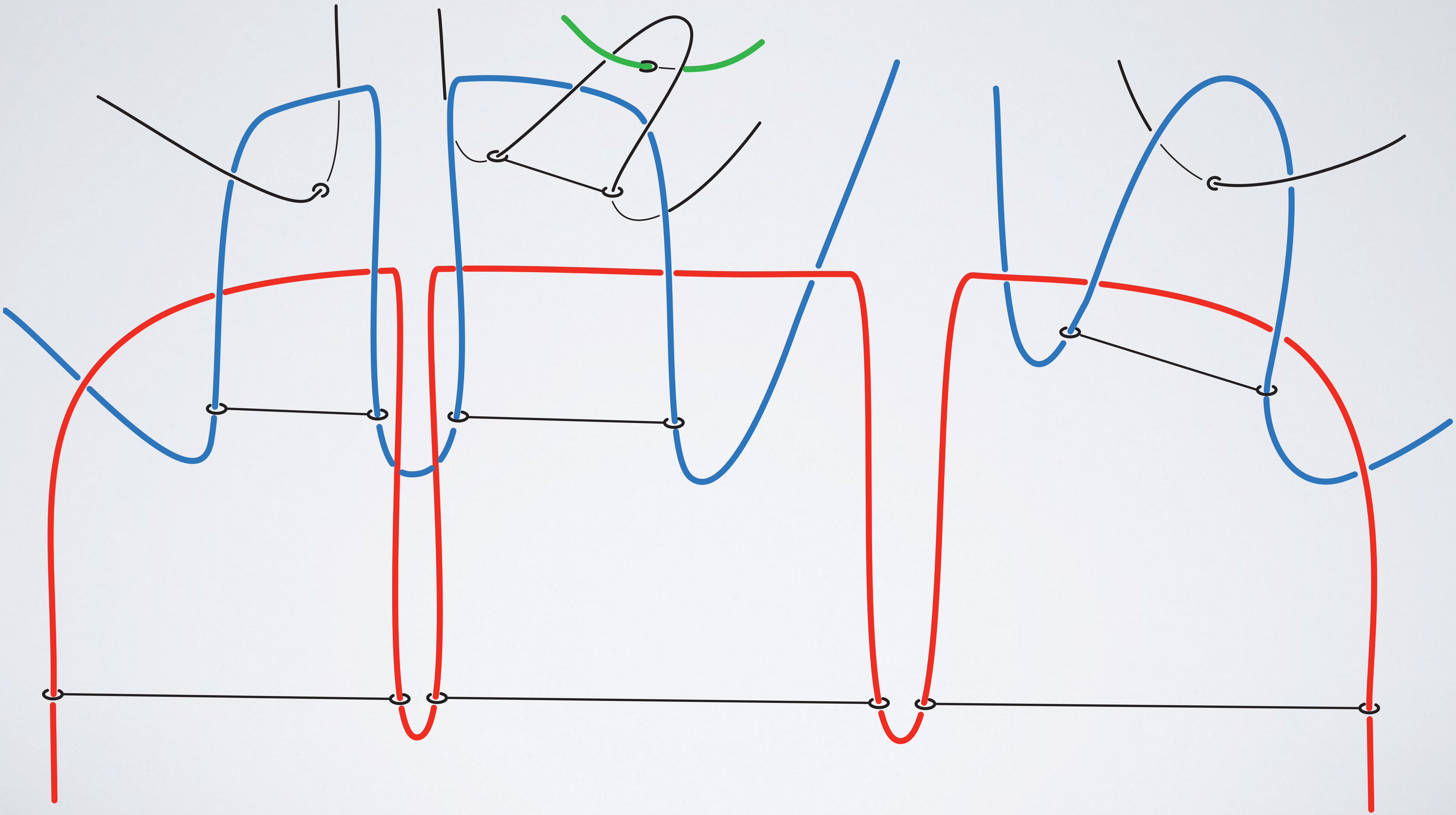


...WHICH MAKE A WHITNEY TOWER

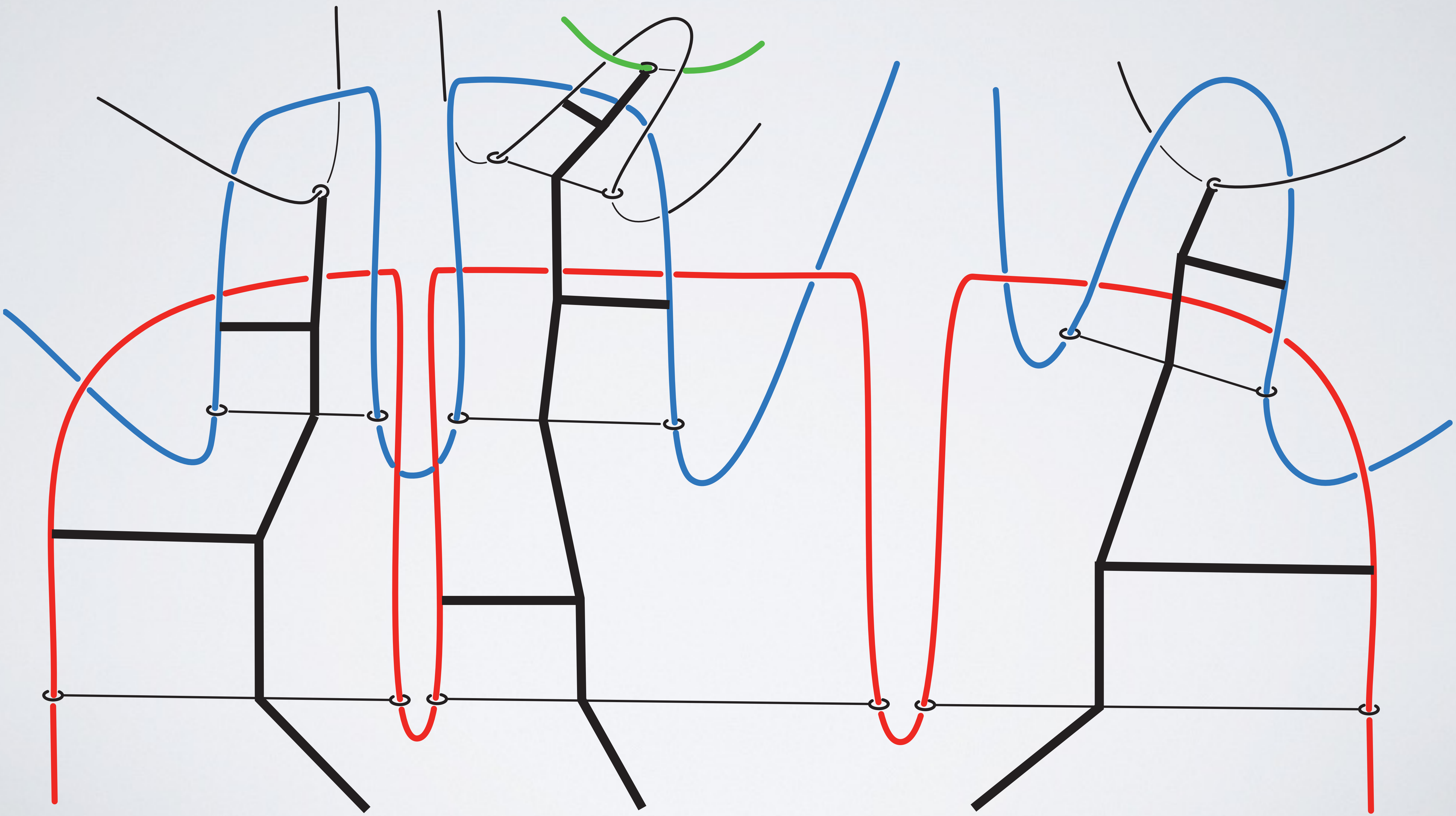
FINGER MOVES ....



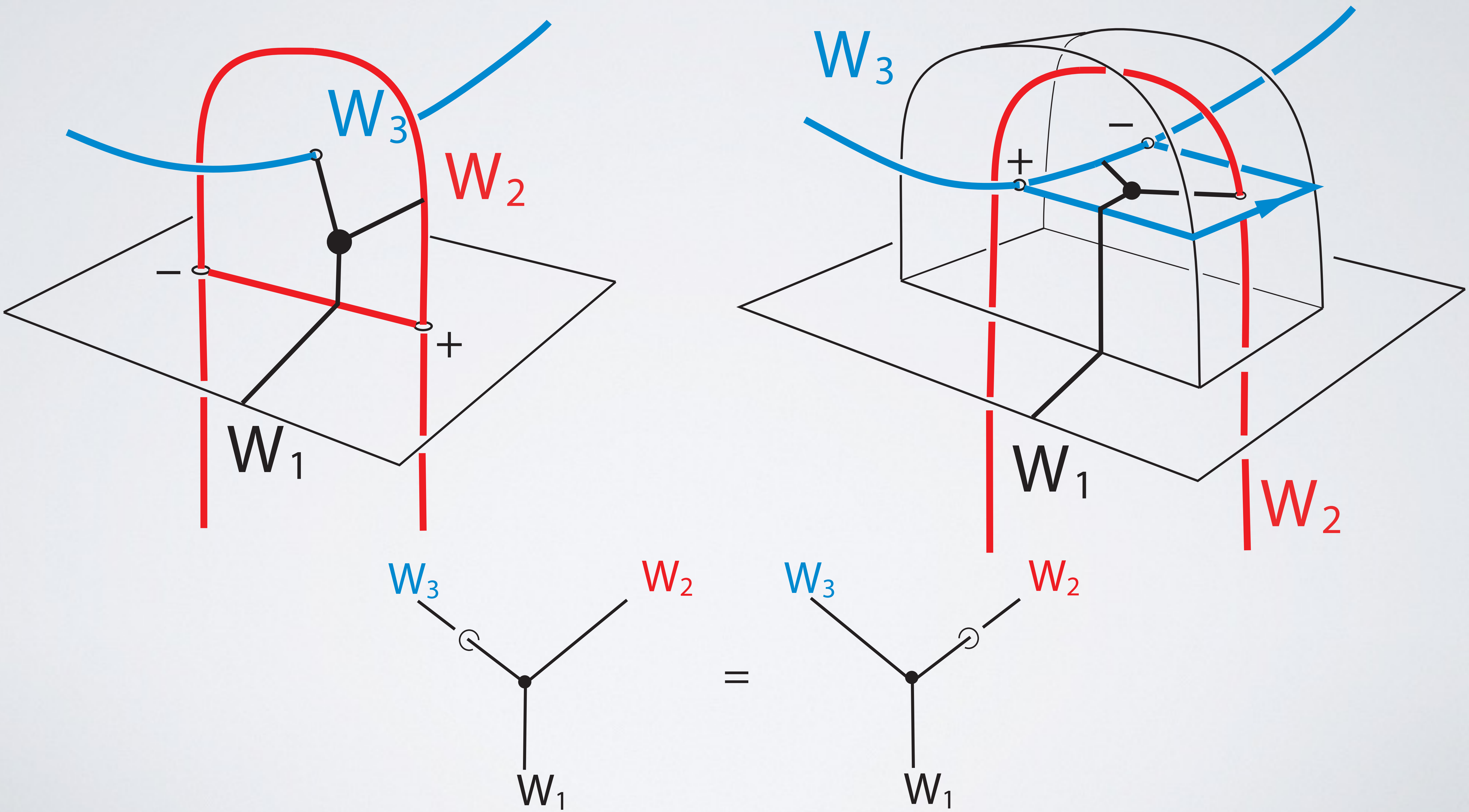
… SPLIT WHITNEY TOWERS



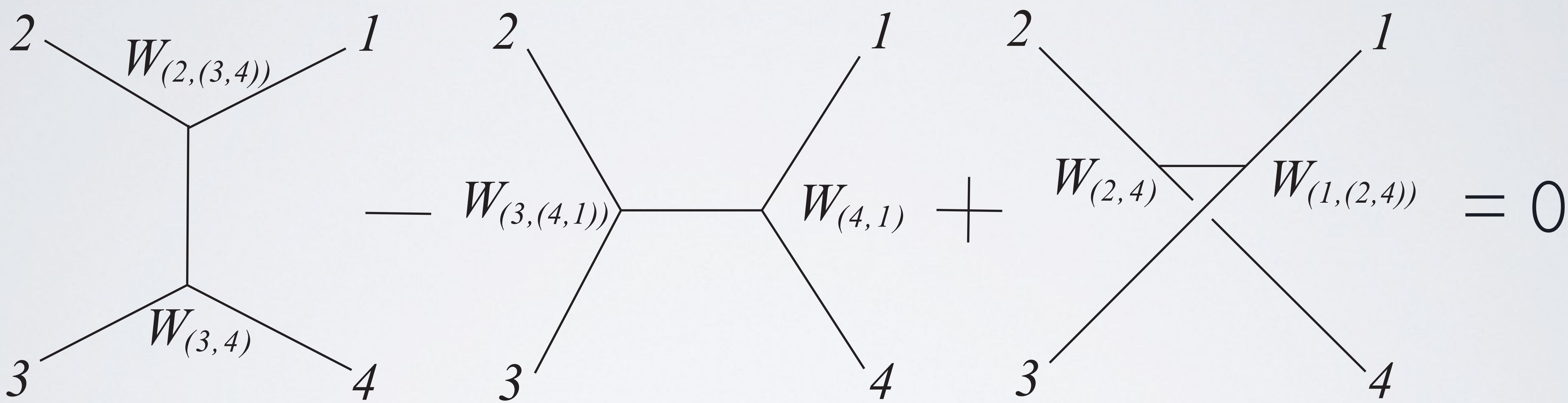
TREES ORGANIZE WHITNEY TOWERS



KEY FIGURE: TREE IS PRESERVED BY WHITNEY MOVE

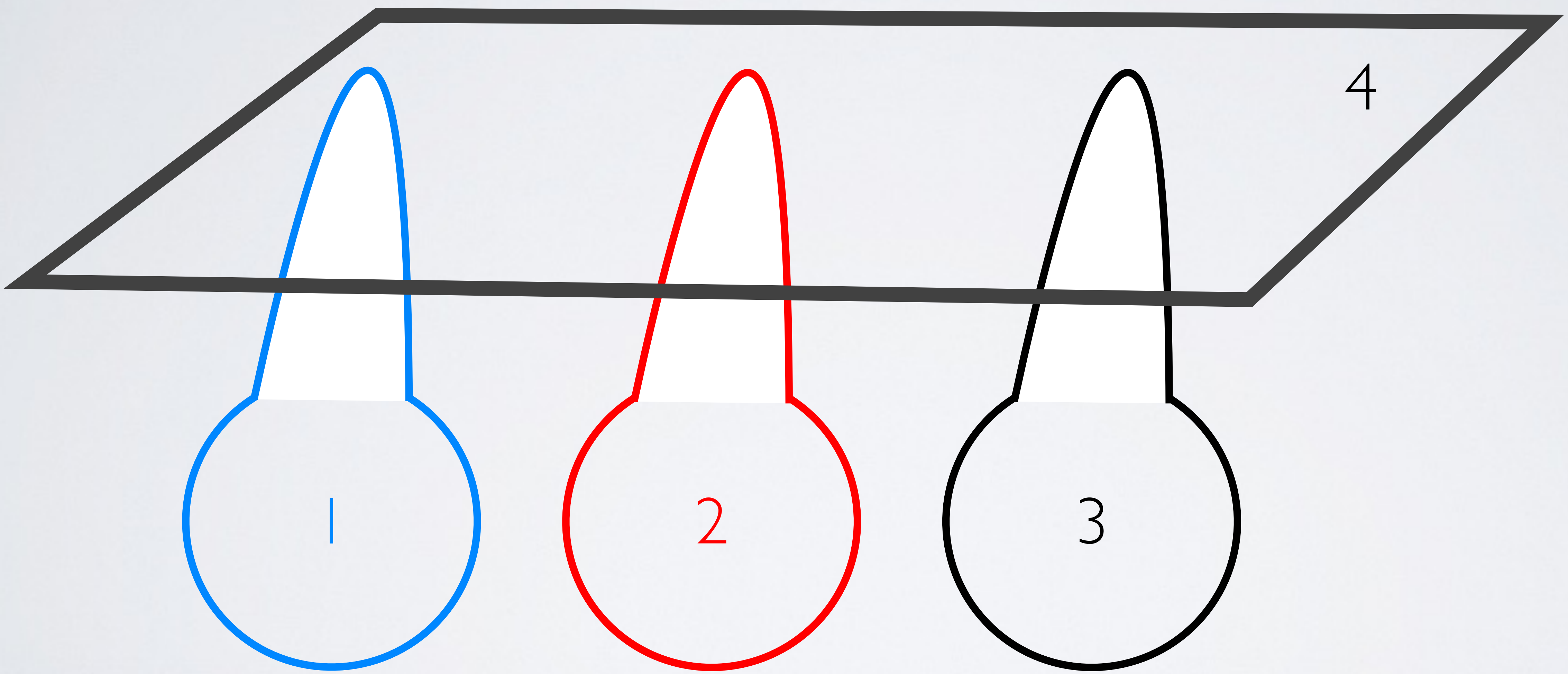


OUR 4-DIMENSIONAL JACOBI IDENTITY

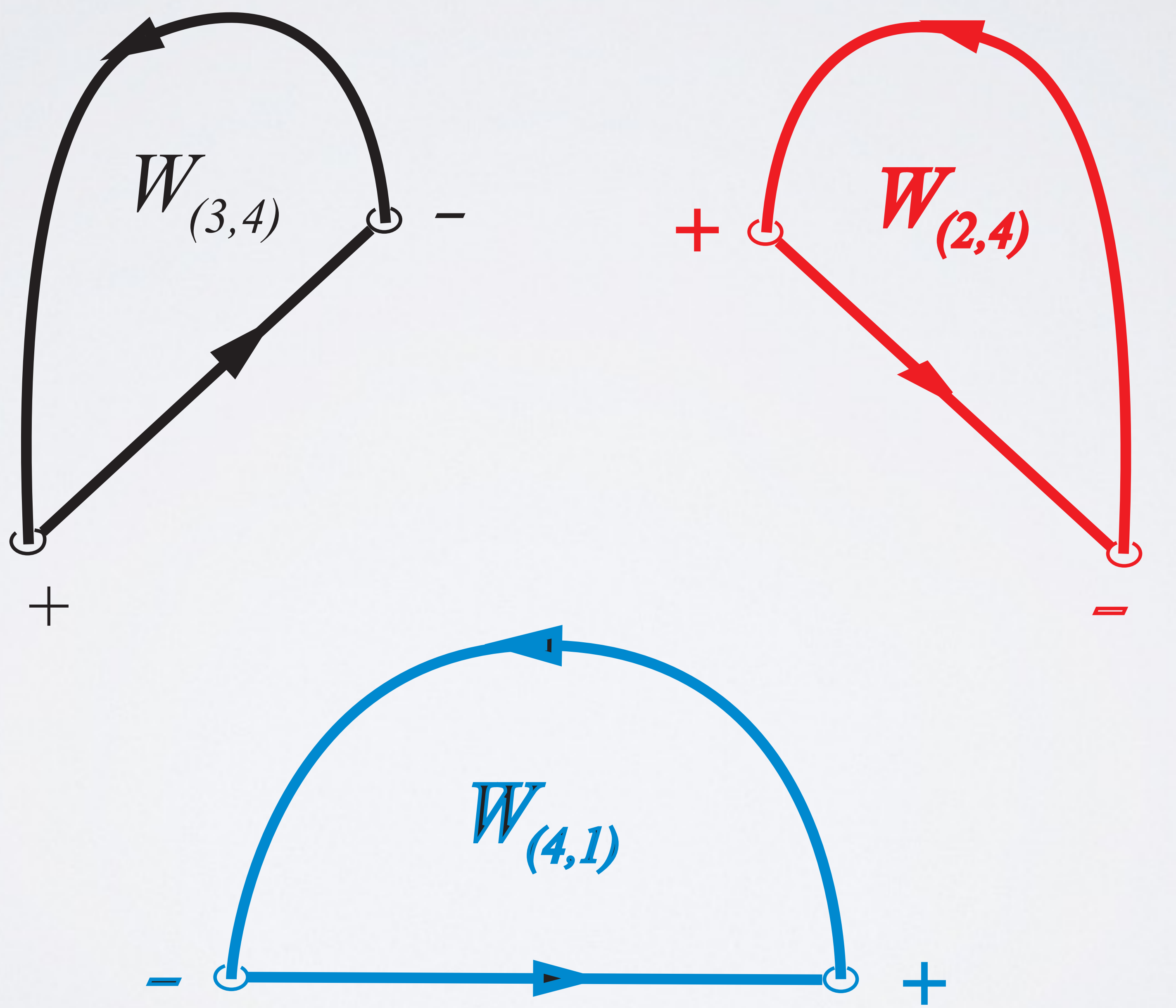


Proof is an exercise in visualization:

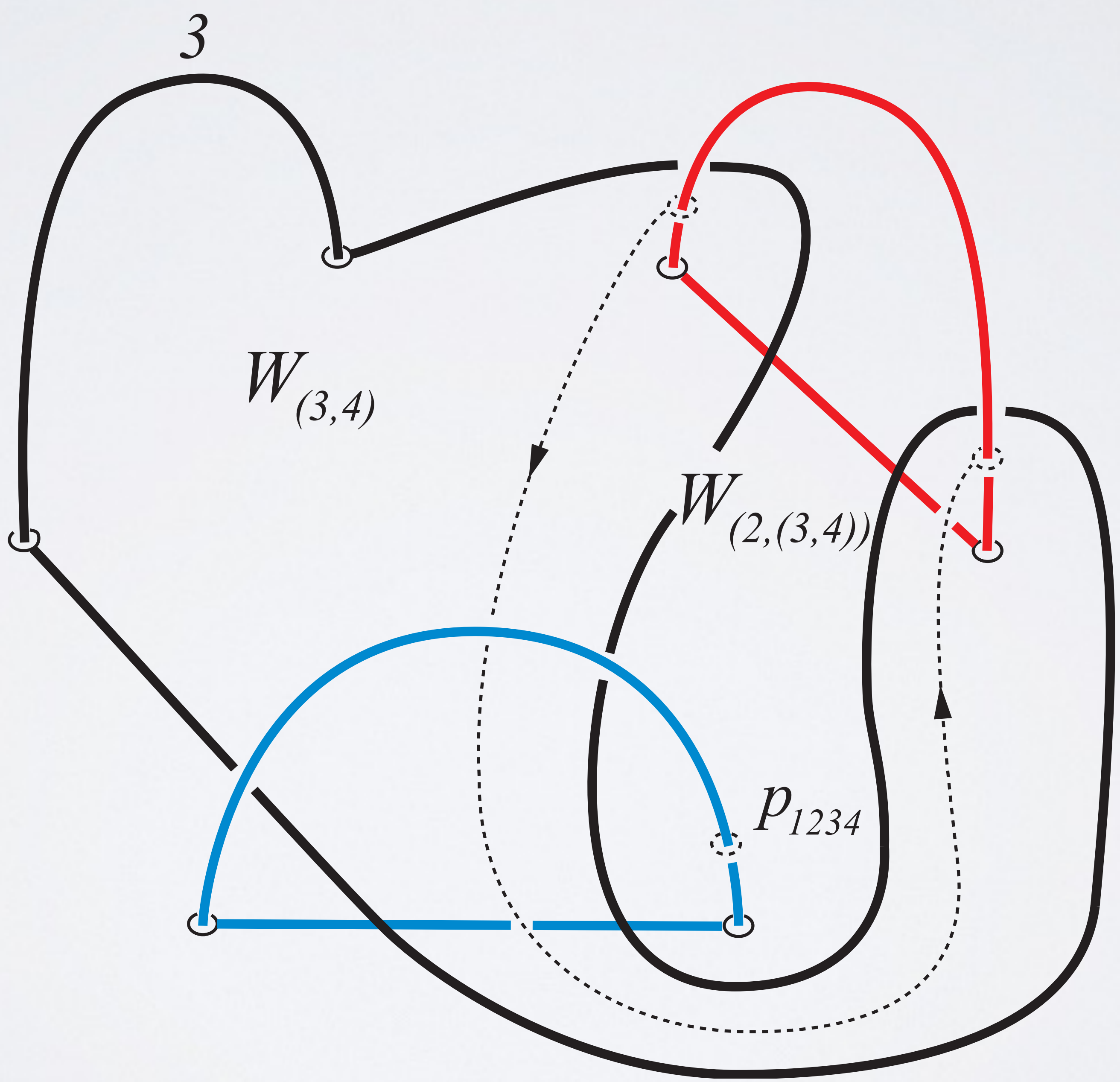
START WITH FOUR SMALL SPHERES



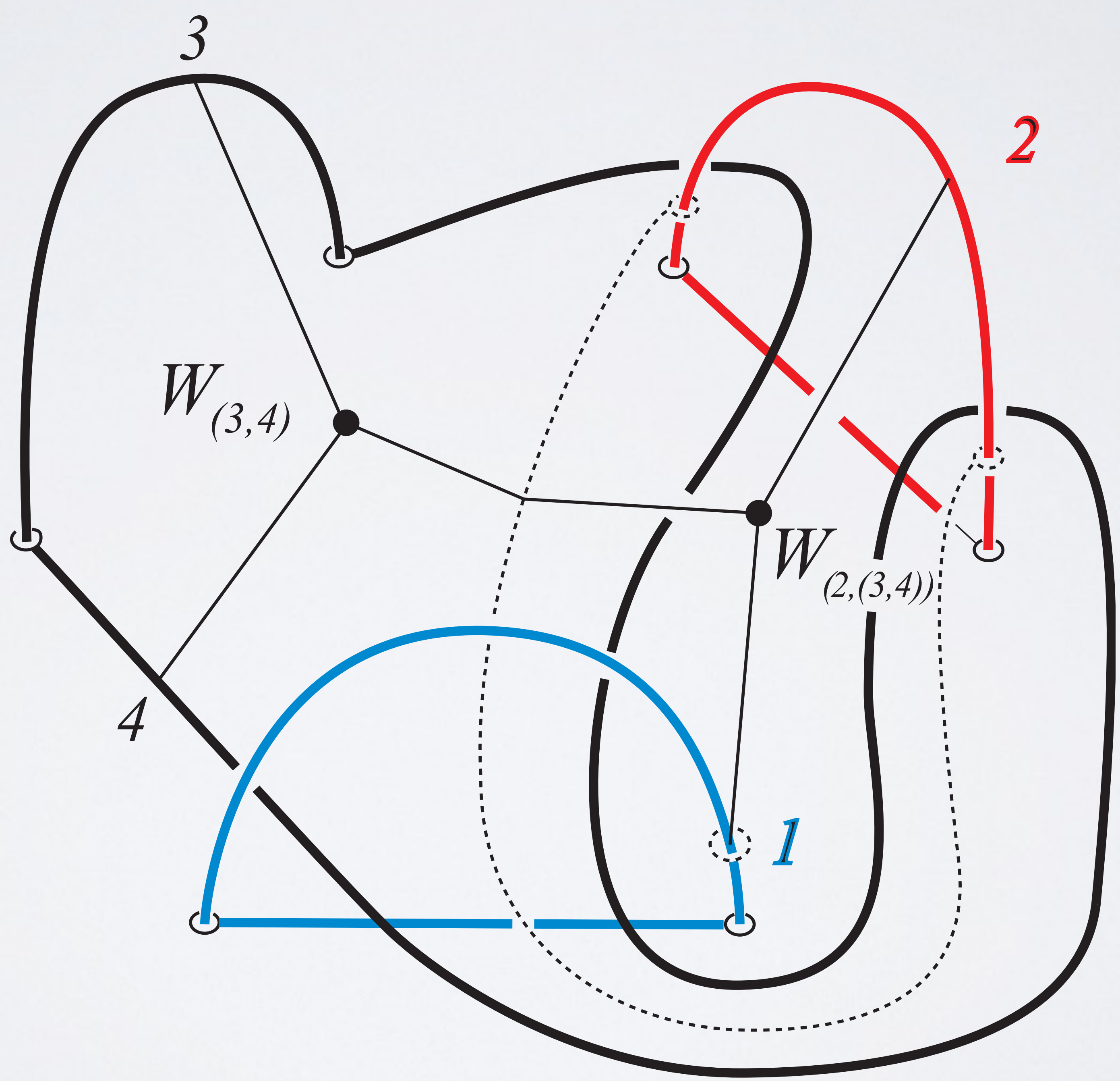
PICK THREE WHITNEY DISKS



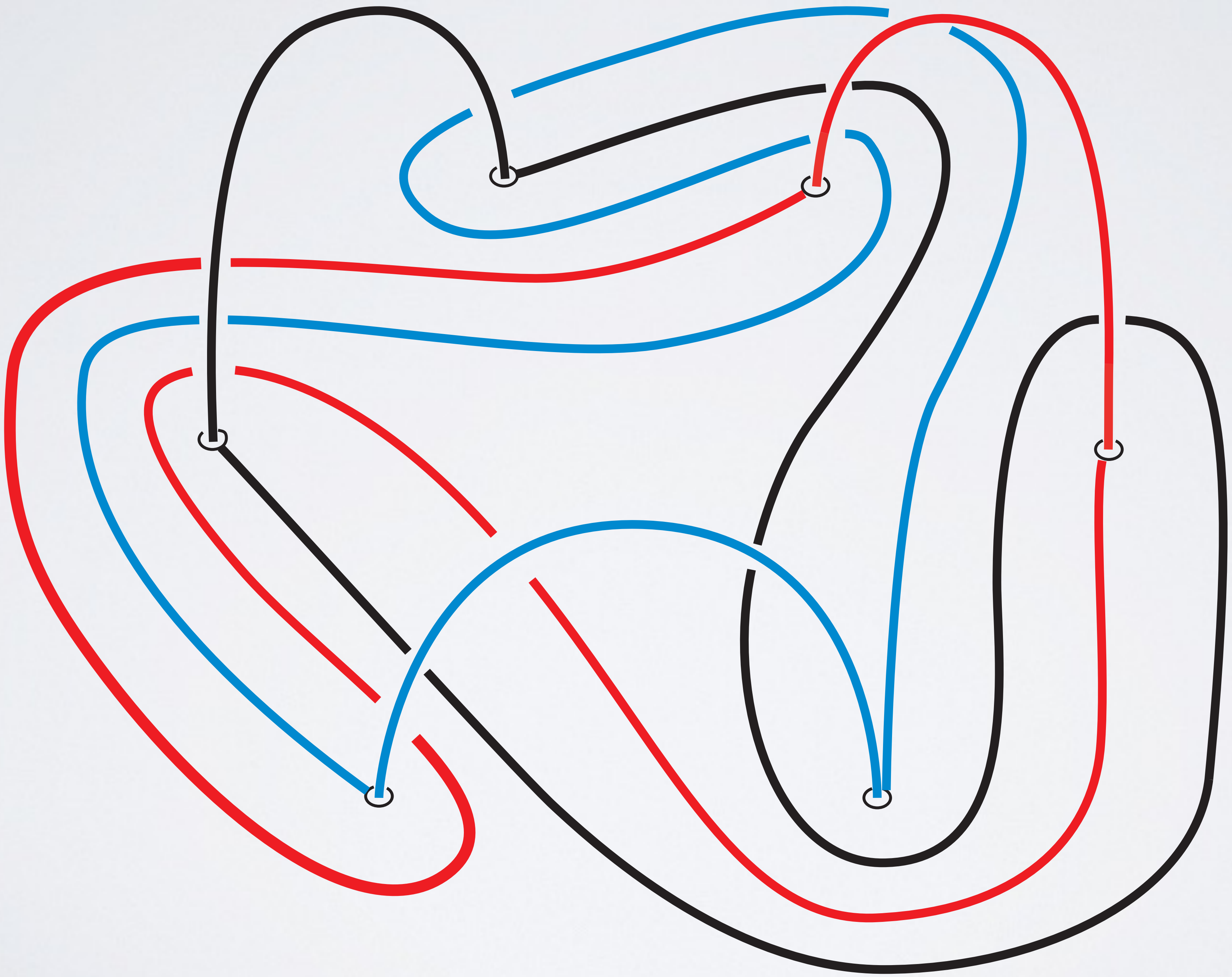
MOVE WHITNEY ARCS



GET A WHITNEY TOWER OF ORDER 2



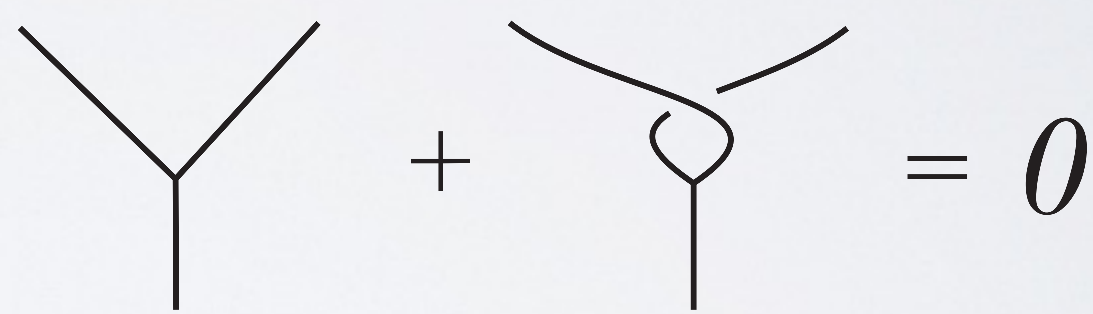
REMOVE INTERSECTIONS



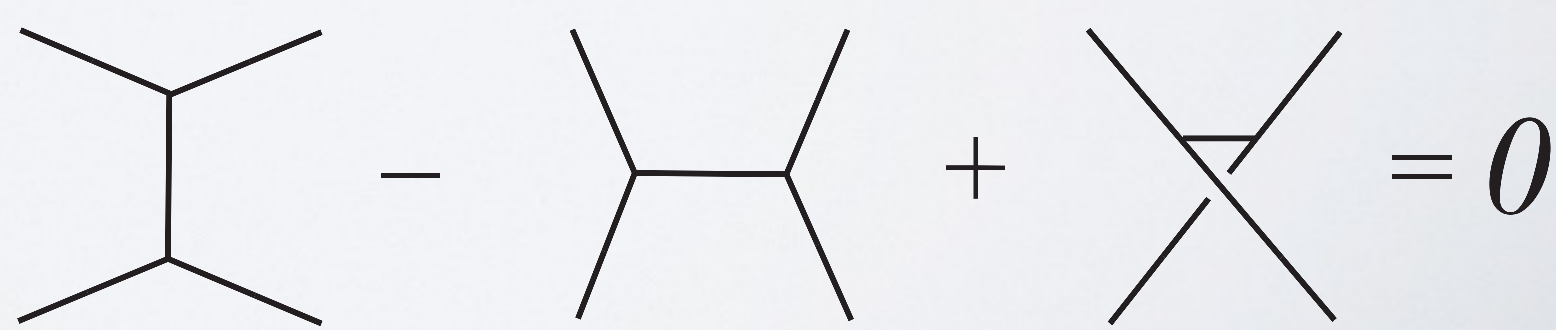
DEFINITION OF TREE GROUPS

$T_n = T_n(m)$ is the abelian group on oriented unitrivalent trees, with n trivalent vertices and univalent vertices labelled by $\{1, 2, \ldots, m\}$, modulo the two local relations:

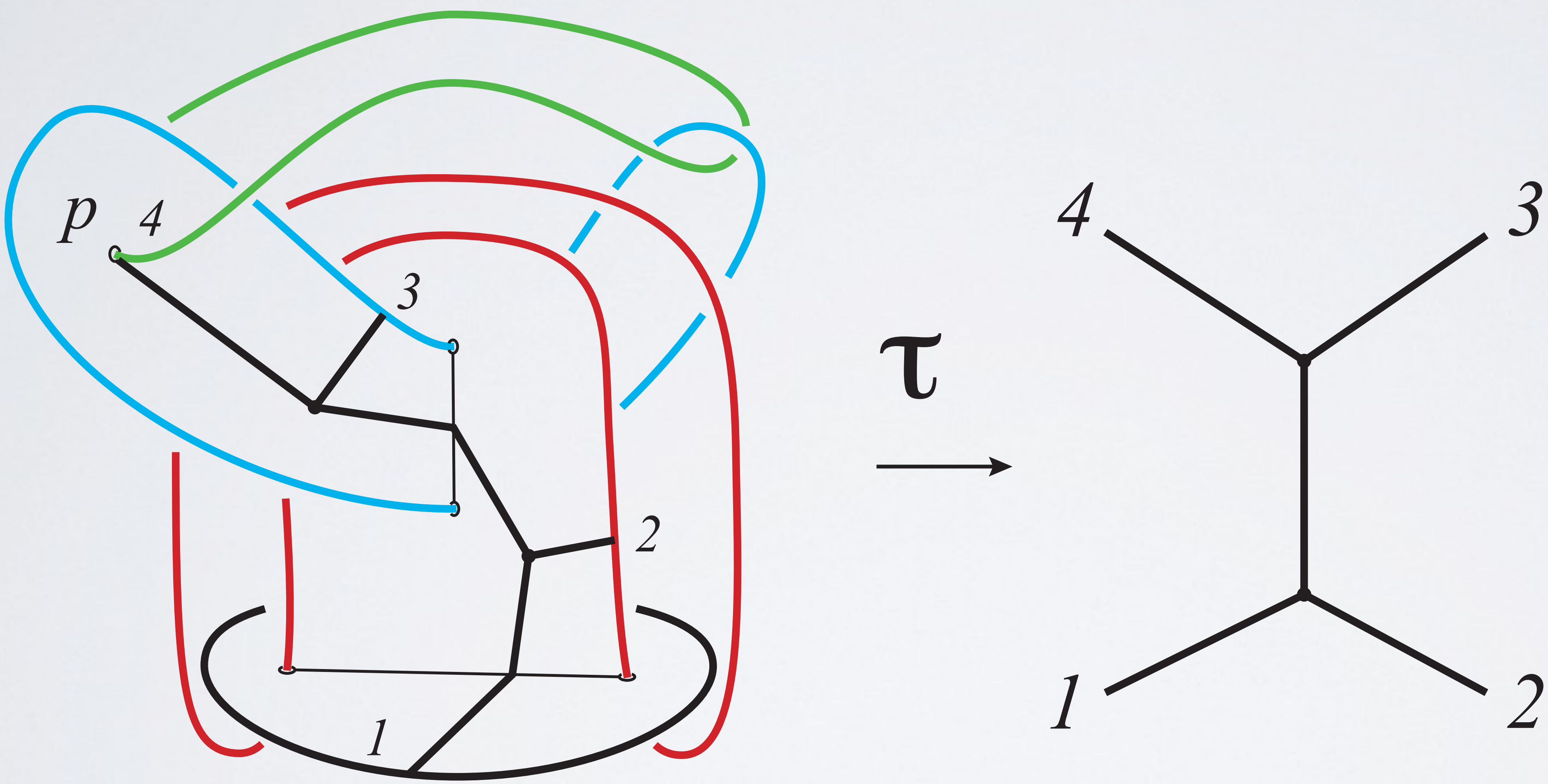
Anti-symmetry:



Jacobi Identity:



BING(HOPF)



$\in T_2(4)$

REALIZATION THEOREM

Consider the set of (framed, m-component) links in the 3-sphere bounding Whitney towers of order n in the 4-ball. Let $W_n = W_n(m)$ be the associated graded.

Theorem 1: There are surjective realization maps

$$R_n : T_n \twoheadrightarrow W_n$$

whose kernel consists of torsion.

Theorem 2: T_n (and hence W_n) are finitely generated abelian groups with at most 2-torsion.

MASTER DIAGRAM

order n Whitney towers
in the 4-ball, up to isotopy and Whitney moves

$\xrightarrow{\partial_n}$

links in the 3-sphere
that are boundaries of order n Whitney towers

intersection tree τ_n

associated graded

$\mathcal{T}_n$ = abelian group generated by trees of order n, up to the AS- and IHX-relations

$\xrightarrow{R_n}$

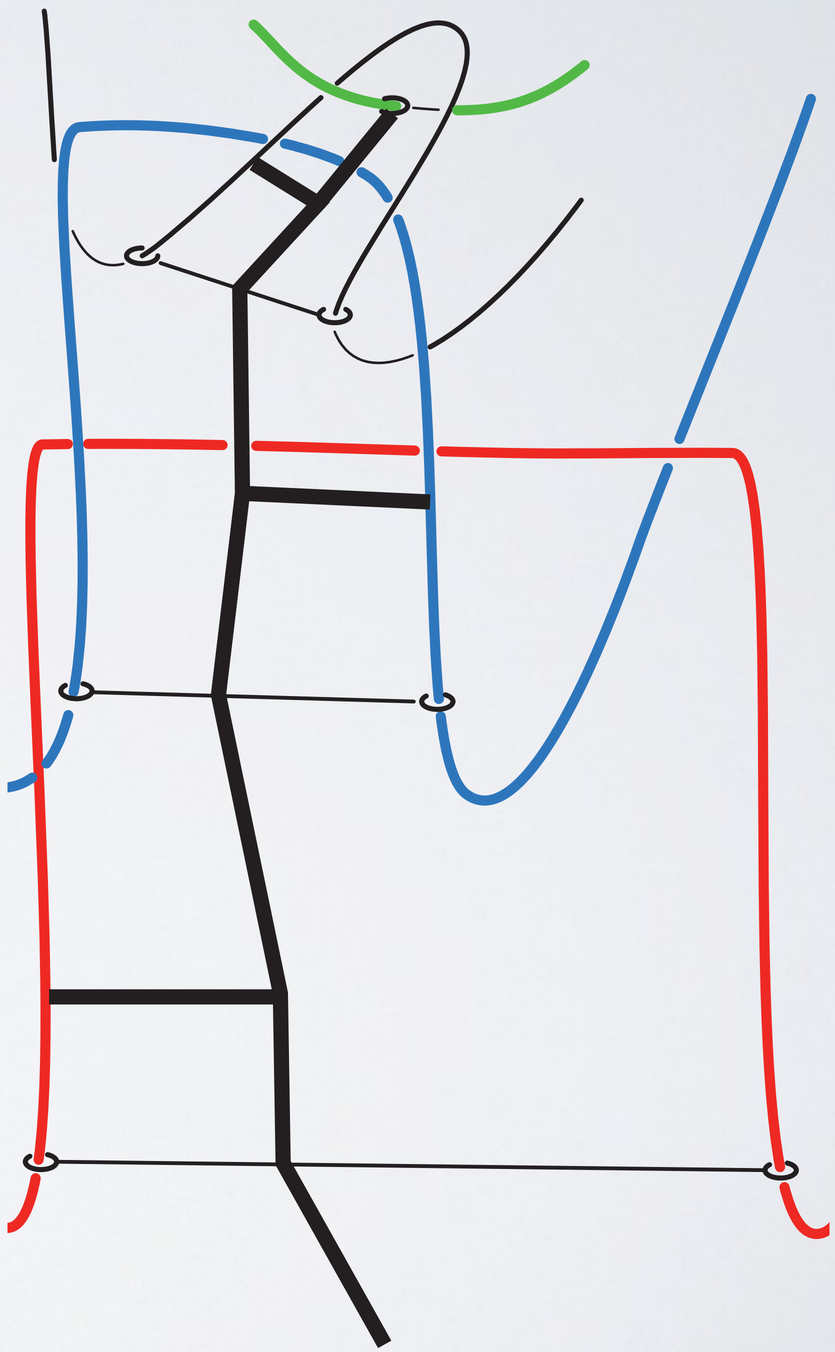
$\mathbb{W}_n$ = links that bound order n Whitney towers, up to those bounding order n+1.

geometric obstruction theory:

If a Whitney tower W of order n has vanishing intersection invariant, τ_n (W)=0, then it extends to order n+1 (up to Whitney moves).

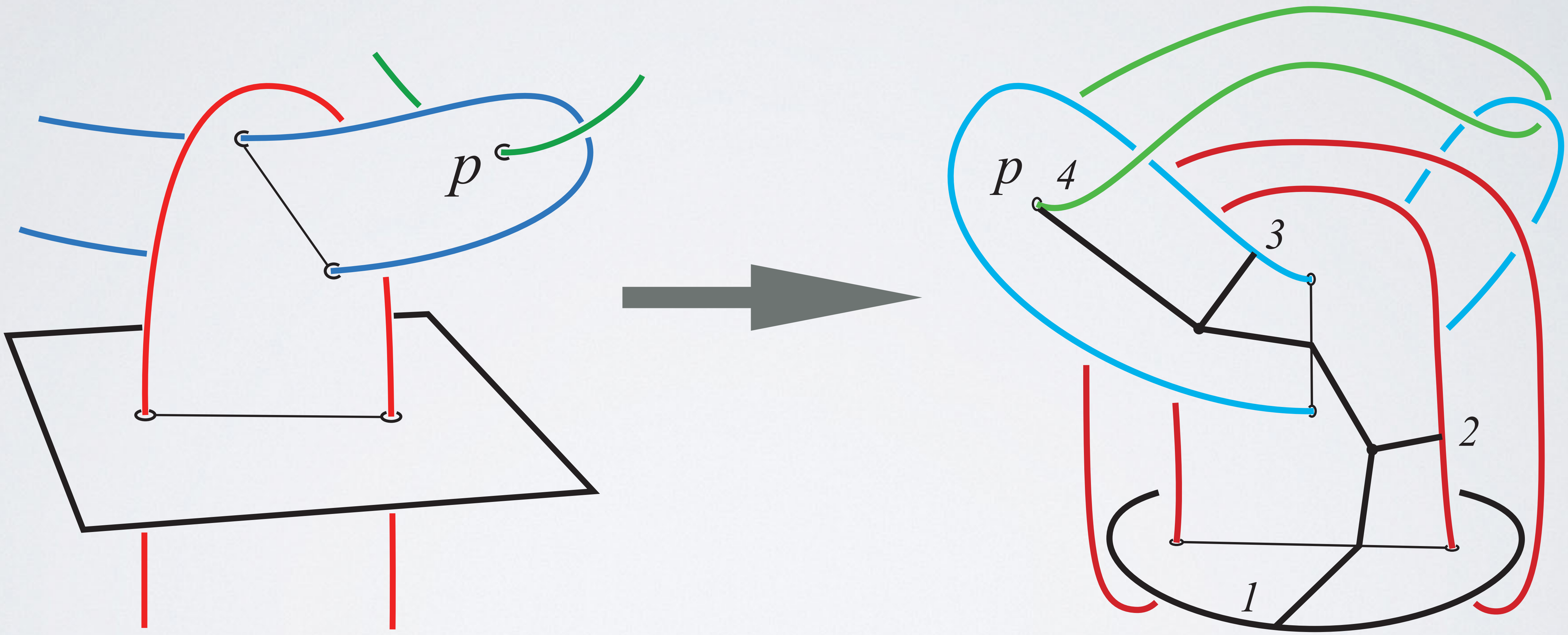
SURJECTIVITY OF INTERSECTION TREE

Take the Whitney towers W in our standard pictures:



Then $t = \boldsymbol{\tau}_n$ (W) runs through trees that generate T_n and the link on the boundary is $R_n(t)$.

LINK ON THE BOUNDARY



EXTENDED MASTER DIAGRAM

$\mathcal{T}_n$ = abelian group generated by trees of order n, up to the AS- and IHX-relations $\xrightarrow{R_n}$ $\mathcal{W}_n$ = links that bound order n Whitney towers, up to those bounding order n+1.

discrete Morse theory: $\cong$ $\quad \downarrow \eta_n$ = sum over roots $\qquad \downarrow \mu_n$

D'_n = subgroup of abelian group generated by rooted trees of order n, up to the AS- and IHX-relations $\xrightarrow{p_n}$ D_n = free abelian group of known rank, target of order n Milnor invariants μ_n

for odd n, kernel is 2-torsion: [x,x]

free quasi Lie algebra: [x,y]=-[y,x]

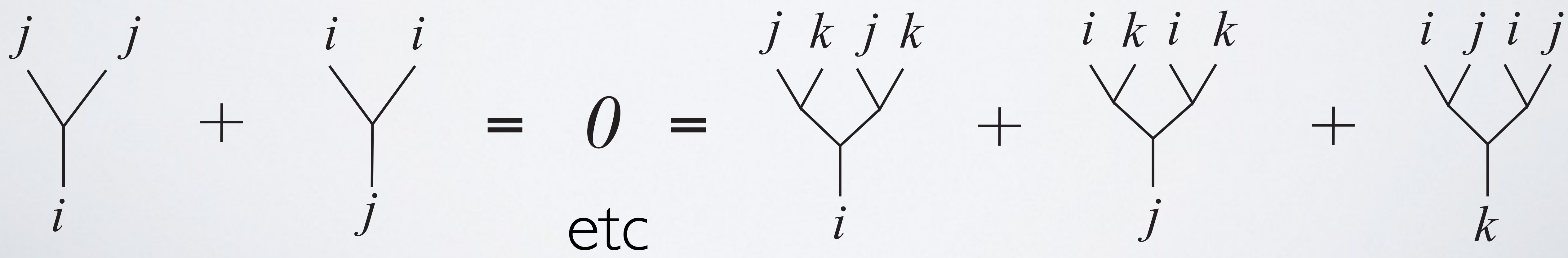
free Lie algebra: [x,x]=0

CLASSIFICATION RESULTS

Theorem 3: For even n, p_n and R_n are isomorphisms, $T_n \cong W_n \cong D_n$ are free abelian of known rank, detected by Milnor invariants μ_n.

Moreover, the μ_n detect the free part of W_n for all n.

Theorem 4: For odd n, R_n factor through the quotient T_n /fr by our framing relations:



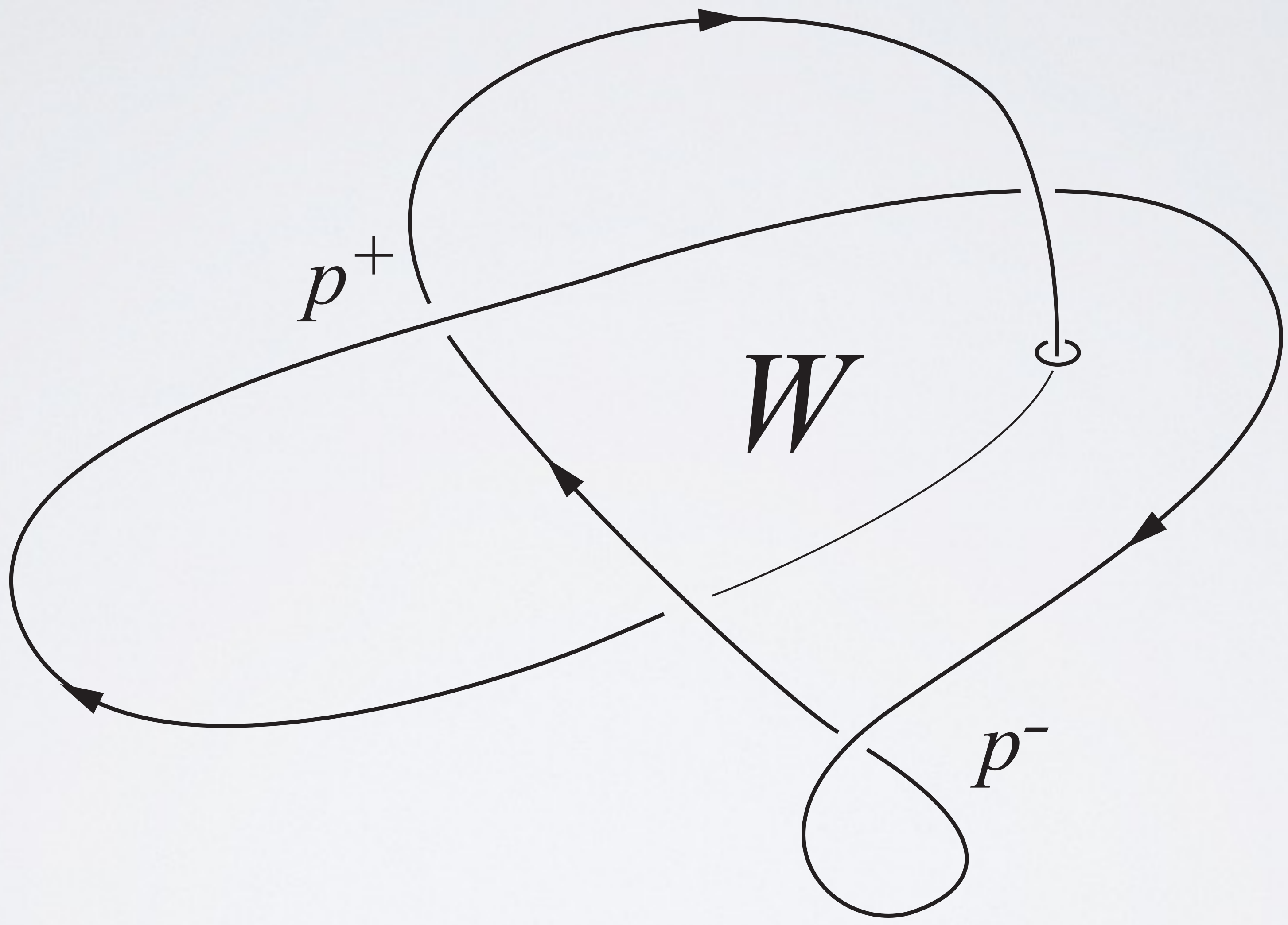
4-PERIODIC BEHAVIOUR

Theorem 5: R_{4k-1} induces $T_{4k-1}/fr \cong W_{4k-1}$

The 2-torsion of W_{4k-1} is known, it is detected by higher order Sato-Levine invariants.

Theorem 6: There is an upper bound on W_{4k-3} .
Its 2-torsion is detected by Sato-Levine and higher order Arf invariants Arf_k : $k=1,2,3,\ldots$..

TREFOIL IS DETECTED



Arf_1 gives classical Arf invariants of link components.

COMPUTATIONS

$W_n(m) =$ number m of components

order n

	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^{10}$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^{15}$
2	0	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{50}$
3	0	$\mathbb{Z}_2^2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^8$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{20}$	$\mathbb{Z}^{126} \oplus \mathbb{Z}_2^{40}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	$\mathbb{Z}_2^{e_2}$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{e_3}$	$\mathbb{Z}^{340} \oplus \mathbb{Z}_2^{e_4}$	$\mathbb{Z}^{1740} \oplus \mathbb{Z}_2^{e_5}$
6	0	$\mathbb{Z}^6$	$\mathbb{Z}^{126}$	$\mathbb{Z}^{1200}$	$\mathbb{Z}^{7050}$

$\mathrm{Arf}_2: \quad 3 \le e_2 \le 4, \quad 18 \le e_3 \le 21, \quad 60 \le e_4 \le 66, \quad 150 \le e_5 \le 160.$