$\mathbb{CP}^2$-STABLE CLASSIFICATION OF 4-MANIFOLDS WITH FINITE FUNDAMENTAL GROUP

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ABSTRACT. We show that two closed, connected 4-manifolds with finite fundamental groups are $\mathbb{CP}^2$-stably homeomorphic if and only if their quadratic 2-types are stably isomorphic and their Kirby-Siebenmann invariant agrees.

1. INTRODUCTION

Two 4-manifolds are $\mathbb{CP}^2$-stably homeomorphic if they become homeomorphic after taking connected sum with finitely many copies of $\mathbb{CP}^2$. If the manifolds are orientable, we allow connected sums with both orientations. We will show that the $\mathbb{CP}^2$-stable classification for closed connected 4-manifolds with finite fundamental group is determined by their (stable) quadratic 2-type.

The quadratic 2-type Q_M consists of the Postnikov 2-type of the 4-manifold M, determined by $\pi_1 M, \pi_2 M$ (as a $\pi_1 M$-module) and the k-invariant in $H^3(\pi_1 M; \pi_2 M)$, together with the orientation character $w_1 M : \pi_1 M \to \{\pm 1\}$ and the equivariant intersection form $\lambda_M : \pi_2 M \times \pi_2 M \longrightarrow \mathbb{Z}[\pi_1 M]$:

$$Q_M := (\pi_1 M, w_1 M, \pi_2 M, k_M, \lambda_M).$$

We say that the quadratic 2-types of two manifolds are stably isomorphic if they become isomorphic after finitely many stabilizations as follows:

$$Q_M \mapsto Q_{M\#\pm\mathbb{CP}^2} \cong (\pi_1 M, w_1 M, \pi_2 M \oplus \mathbb{Z}[\pi_1 M], (k_M, 0), \lambda_M \perp \langle \pm 1 \rangle)$$

Theorem 1.1. *Let M_1, M_2 be closed, connected 4-manifolds with finite fundamental group π, orientation character $w : \pi \to \{\pm 1\}$ and equal Kirby-Siebenmann invariants. Then the following are equivalent:*

(1) $c_*[M_1] = c_*[M_2] \in H_4(\pi; \mathbb{Z}^w)/\pm \operatorname{Out}(\pi)$;
(2) M_1 and M_2 are $\mathbb{CP}^2$-stably homeomorphic;
(3) the quadratic 2-types Q_{M_1} and Q_{M_2} are stably isomorphic.

If follows from Kreck's modified surgery [Kre99] that (1) and (2) are equivalent for arbitrary fundamental groups, see also [KPT, Thm 1.1]. Moreover, (2) implies (3) by definition and hence the interesting implication for us is (3)⇒(1).

In [KPT, Thm A] we showed that already the quadruple (π_1, π_2, k, w_1) determines the $\mathbb{CP}^2$-stable classification, for groups that have one end or are torsion-free.

We also described two 4-manifolds of the form lens space times circle which both have fundamental group $\mathbb{Z}/n \times \mathbb{Z}$ and $\pi_2 = 0$ but are not $\mathbb{CP}^2$-stably homeomorphic [KPT, Ex. 1.4]. Hence for infinite groups (with two ends and torsion), the implication (3)⇒(1) is false in general.

Date: July 27, 2019.
2010 *Mathematics Subject Classification.* 57N13.
Key words and phrases. Whitehead's Gamma group; stable classification of 4-manifolds.

The following question remains open, compare [KPT, Question 1.5], even though we show there that the answer is "no" if $H_4(\pi;\mathbb{Z}^w)$ is annihilated by 4 or 6.

Question 1.2. Are there finite groups such that the equivariant intersection form λ is needed in Theorem 1.1 .

Note that Theorem 1.1 is true in the smooth category (where Kirby-Siebenmann vanishes), compare [KPT, p.3]. In the topological case, one could also allow connected sums with Freedman's manifold $*\mathbb{CP}^2$. Then the Kirby-Siebenmann invariant can be changed and Theorem 1.1 has a version where one does not have to control it.

Our proof is based on the following results. Let X be a finite connected Poincaré 4-complex with finite fundamental group π, orientation character w and chosen fundamental class $[X] \in H_4(X;\mathbb{Z}^w) \cong \mathbb{Z}$. If B is a 3-coconnected CW-complex then X is *B-polarized* if it is equipped with a 3-equivalence $f\colon X \to B$. The set of 4-dimensional B-polarized Poincaré complexes with orientation character $w : \pi_1 B \to \{\pm 1\}$, up to homotopy equivalence over B, is denoted by $\mathcal{S}_4^{\mathrm{PD}}(B,w)$.

Theorem 1.3. *Assume that $\mathcal{S}_4^{\mathrm{PD}}(B,w)$ is non-empty and that $\pi := \pi_1 B$ is finite. Then there is an exact sequence*

$$0 \to \mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(B))) \to \mathcal{S}_4^{\mathrm{PD}}(B,w) \to \mathbb{Z}/|\pi| \times \mathrm{Herm}(\pi_2(B)).$$

Hambleton-Kreck [HK88, Thm 1.1] prove this result in the oriented case $w = 0$ but we need the above generalization to the non-orientable case. The general case as well as Theorem 1.4 below were obtained by the second author in his PhD-thesis [Tei92] but have not yet been published, so we'll review them in this paper.

For a hermitian form λ on $\pi_2 B$, let $\mathcal{S}_4^{\mathrm{PD}}(B,w,\lambda)$ be the subset of $\mathcal{S}_4^{\mathrm{PD}}(B,w)$ of B-polarized Poincaré complexes (X,f) such that λ is mapped to the intersection form λ_X via f^*.

Theorem 1.4. *Assume that $\mathcal{S}_4^{\mathrm{PD}}(B,w,\lambda)$ is non-empty. Then there is a bijection*

$$\mathcal{S}_4^{\mathrm{PD}}(B,w,\lambda) \longleftrightarrow \mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B)).$$

Corollary 1.5. *If $\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X)) = 0$, the quadratic 2-type Q_X of X determines the homotopy type of X.*

In the orientable case, the above torsion group vanishes if π has cyclic or quaternion 2-Sylow-subgroups [HK88, Bau88] whereas in the non-orientable case, the following is a simple non-vanishing result:

Proposition 1.6. *Let (X,w) be a Poincaré 4-complex with fundamental group π. If $\pi_2(X)$ splits off a free module and there exists an element $g \in \pi$ with $g^2 = 1$ and $w(g) = -1$, then $\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X)) \neq 0$.*

Remark 1.7. This applies to simply 4-manifolds such as $X = \mathbb{RP}^4 \# k\mathbb{CP}^2$ and $X = (\mathbb{RP}^2 \times S^2)\# k\mathbb{CP}^2$ for $k \geq 1$. In these cases the quadratic 2-type still determines the homotopy type by [HKT94, Theorem 3] when we restrict to manifolds instead of Poincaré complexes.

Considering $X = \mathbb{RP}^4 \# \mathbb{CP}^2$ we have $\pi_2(X) \cong \mathbb{Z}[\mathbb{Z}/2]$ and by the proof of Proposition 1.6 we have $\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2])) \cong \mathbb{Z}/2$. We will show in Proposition 3.13 that there is a Poincaré complex with the same quadratic 2-type as X which is not homotopy equivalent to X. By the above this implies that it cannot be homotopy equivalent to a manifold.

Remark 1.8. For $X = \mathbb{RP}^2 \times S^2$ we have that $\pi_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}^w$ and thus $\Gamma(\pi_2(X)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^w$ and $\mathbb{Z}^w \otimes \Gamma(\pi_2(X)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$. Hence up to homotopy equivalence there are at most four Poincaré complexes with the same quadratic 2-type as X. For manifolds this case was studied by Kim, Kojima and Raymond [KKR92] and Hambleton and Kreck together with the second author [HKT94]. Among the four homotopy types of Poincaré complexes there are three that are realized by manifolds. These are distinguished by their intersection form on $\mathbb{Z}/2$-homology and either a $\mathbb{Z}/4$-valued quadratic refinement of the equivariant intersection form [KKR92] or a $\mathbb{Z}/8$-valued Arf invariant [HKT94]. Hambleton and Milgram [HM78, §3] showed that there is a Poincaré complex with this quadratic 2-type which is not homotopy equivalent to a manifold. Hence up to homotopy equivalence there are precisely four Poincaré with the quadratic 2-type of $\mathbb{RP}^2 \times S^2$.

We will recall the necessary background on the equivariant intersection form and its connection to Whitehead's quadratic functor in Section 2. In Section 3, we will then prove the statements from the introduction.

Acknowledgements: The authors thank Matthias Kreck for insightful conversations and Mark Powell for helpful comments on a previous version - this all happened at the Max Planck Institute for Mathematics in Bonn. The first author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

2. Background

Let X be a finite connected Poincaré 4-complex with finite fundamental group π, orientation character $w : \pi_1 X \to \{\pm 1\}$ and chosen fundamental class $[X] \in H_4(X; \mathbb{Z}^w) \cong \mathbb{Z}$. Here the coefficient group $\mathbb{Z}^w$ is $\mathbb{Z}$ viewed as a $\mathbb{Z}\pi$-module with action $gz = w(g)z$

We have an involution on $\mathbb{Z}\pi$ given by $g \mapsto \bar{g} := w(g)g^{-1}$ and we use this involution to consider a right $\mathbb{Z}\pi$-module as a left $\mathbb{Z}\pi$-module if necessary. We fix a point $x_0 \in X$ and a lift $\widetilde{x}_0 \in \widetilde{X}$. We abbreviate $\pi_2(X, x_0)$ by $\pi_2(X)$ and write $H_*(-)$ for $H_*(-;\mathbb{Z})$. Using the isomorphism $H^2_{cs}(\widetilde{X}) \xrightarrow{-\cap[X]} H_2(\widetilde{X}) \xleftarrow{\cong} \pi_2(\widetilde{X}) \xrightarrow{\cong} \pi_2(X)$, we can view the equivariant intersection form as a pairing on $\pi_2 X$, or, as we prefer in this paper, as a pairing $H^2_{cs}(\widetilde{X}) \times H^2_{cs}(\widetilde{X}) \to \mathbb{Z}\pi$. It is given by

$$\lambda_X(\alpha, \beta) = \lambda_X(\beta)(\alpha) = \sum_{g \in \pi} (g\alpha \cup \beta) \cap [X])g^{-1} = \sum_{g \in \pi} g\alpha \cap ((\beta \cap [X]))g^{-1}.$$

The form λ_X is hermitian with respect to our involution: For $h_i \in \pi$ we have

$$\begin{aligned}
\lambda_X(h_1\alpha, h_2\beta) &= \sum_{g \in \pi} ((gh_1\alpha \cup h_2\beta) \cap [X])g^{-1} \\
&= \sum_{g \in \pi} ((h_2^{-1}gh_1\alpha \cup \beta) \cap w(h_2)[X])g^{-1} \\
&= \sum_{g \in \pi} ((g\alpha \cup \beta) \cap w(h_2)[X])h_1 g^{-1} h_2^{-1} \\
&= h_1 \lambda(\alpha, \beta) w(h_2) h_2^{-1} = h_1 \lambda(\alpha, \beta) \bar{h}_2
\end{aligned}$$

and $\lambda_X(\alpha, \beta) = \overline{\lambda_X(\beta, \alpha)}$.

The 2-type B of X is the 2nd stage of a Postnikov tower for X. This means that there is a 3-equivalence $f\colon X \to B$ and B fibers over $K(\pi,1)$ with fiber $K(\pi_2(X),2)$. Such fibrations are classified by the unique obstruction k_X for finding a section.

$$k_X \in H^3(K(\pi,1);\pi_2(K(\pi_2(X),2))) = H^3(\pi;\pi_2(X))$$

is by definition the *k-invariant* of X. The *quadratic 2-type* of X is

$$Q_X := (\pi, w, \pi_2(X), k_X, \lambda_X).$$

Definition 2.1. Let A be an abelian group. Then $\Gamma(A)$ is the abelian group with generators $v(a)$ for all $a \in A$ and the following relations:

$$\{v(-a) - v(a) \mid a \in A\} \quad \text{and}$$
$$\{v(a+b+c) - v(a+b) - v(b+c) - v(c+a) + v(a) + v(b) + v(c) \mid a,b,c \in A\}.$$

Lemma 2.2 ([Whi50, page 62])**.** *If A is free abelian with basis B, then $\Gamma(A)$ is free abelian with basis $\{v(b), v(b+b') - v(b) - v(b') \mid b, b' \in B\}$. Another important case is $\Gamma(\mathbb{Z}/2) \cong \mathbb{Z}/4$.*

Let Y be a simply-connected CW-complex and let $\eta\colon S^3 \to S^2$ be the Hopf map. It induces a map $\widehat{\eta}\colon \Gamma(\pi_2(Y)) \to \pi_3(Y)$ given by $v(\alpha) \mapsto \alpha \circ \eta$.

Theorem 2.3 ([Whi50, Sections 10 and 13])**.** *For every simply-connected CW-complex Y we have* Whitehead's exact sequence

$$\dots \pi_4(Y) \to H_4(Y) \xrightarrow{b_4} \Gamma(\pi_2(Y)) \xrightarrow{\widehat{\eta}} \pi_3(Y) \to H_3(Y;Z) \to 0.$$

We will now explain how the equivariant intersection form λ_X for a finite Poincaré 4-complex with finite fundamental group can be viewed as an element of $\Gamma(\pi_2(X))$. This is well-known, but for convenience and to fix notation we sketch the argument here.

Let A free abelian. An element $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z}),\mathbb{Z}), A)$ is symmetric if for all $f, g \in \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ we have $f(\varphi(g)) = g(\varphi(f))$. And we denote the subgroup of symmetric homomorphisms by $\operatorname{Sym}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z}), A)$. Let Y be a finite simply-connected CW-complex with $H_2(Y)$ free abelian. We define $\operatorname{Sym}_{\mathbb{Z}}(H^2(Y), H_2(Y))$ as $\operatorname{Sym}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(H_2(Y),\mathbb{Z}), H_2(Y))$ using the canonical isomorphism $H^2(Y) \to \operatorname{Hom}_{\mathbb{Z}}(H_2(Y),\mathbb{Z})$.

Theorem 2.4 ([Whi50, p. 96])**.** *Let Y be a finite simply-connected CW-complex with $H_2(Y)$ free abelian. There is an isomorphism*

$$\zeta\colon \operatorname{Sym}_{\mathbb{Z}}(H^2(Y), H_2(Y)) \to \Gamma(\pi_2(Y))$$

such that for every class $z \in H_4(Y)$ we have $\zeta(- \cap z) = b_4(z)$.

Let X be a finite Poincaré 4-complex with finite fundamental group π and orientation character w. Note that by Poincaré duality $H_2(\widetilde{X}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(\widetilde{X}),\mathbb{Z})$ and thus $H_2(\widetilde{X})$ is free abelian. Let $\operatorname{Herm}(H^2(\widetilde{X}))$ denote the hermitian forms on $H^2(\widetilde{X})$. We have a map

$$\operatorname{Herm}(H^2(\widetilde{X})) \to \operatorname{Sym}_{\mathbb{Z}}(H^2(\widetilde{X}), \operatorname{Hom}_{\mathbb{Z}}(H^2(\widetilde{X}),\mathbb{Z})$$

given by

$$\lambda \mapsto (\beta \mapsto (\alpha \mapsto \operatorname{ev}_0 \lambda(\alpha,\beta))),$$

where $\operatorname{ev}_0\colon \mathbb{Z}\pi \to \mathbb{Z}$ takes the coefficient at the neutral element. This map is injective with image the π-linear homomorphisms, where we view $\operatorname{Hom}_{\mathbb{Z}}(H^2(\widetilde{X}),\mathbb{Z})$

as a left $\mathbb{Z}\pi$-module with the involution on $\mathbb{Z}\pi$ coming from w. Under identifying $\operatorname{Hom}_{\mathbb{Z}}(H^2(\widetilde{X}),\mathbb{Z})$ with $H_2(\widetilde{X})$, the intersection form $\lambda_X \in \operatorname{Herm}(H^2(\widetilde{X}))$ maps to $-\cap[\widetilde{X}]$. Therefore, we consider the intersection form as an element of $\Gamma(\pi_2(X))$ as follows.

Definition 2.5. Let X be a finite Poincaré 4-complex with finite fundamental group π and orientation character w. Then we can consider the element

$$\lambda_X := \zeta(-\cap[\widetilde{X}]) = b_4([\widetilde{X}]) \in \Gamma(\pi_2(X)).$$

Corollary 2.6. *Let X be a finite Poincaré 4-complex with finite fundamental group and 2-type B. Let $f\colon X \to B$ be a 3-equivalence. Then there is an isomorphism $\Gamma(\pi_2(X)) \cong H_4(\widetilde{B},\mathbb{Z})$ mapping λ_X to $\widetilde{f}_*[\widetilde{X}]$.*

Proof. Since B is the 2-type of X, we have $\pi_3(B) = \pi_4(B) = 0$. Consider the diagram:

$$\begin{CD}
\pi_4(X) @>>> H_4(\widetilde{X}) @>>> \Gamma(\pi_2(X,x_0)) @>>> \pi_3(X) @>>> 0\\
@VVV @VV{\widetilde{f}_*}V @VV{\cong}V @VVV\\
0 @>>> H_4(\widetilde{B}) @>{\cong}>> \Gamma(\pi_2(B,f(x_0))) @>>> 0
\end{CD}$$

From the commutativity of the diagram and Definition 2.5 it follows that λ_X is mapped to $\widetilde{f}_*[\widetilde{X}]$ under the composition of the two isomorphisms. □

Corollary 2.7. *Let $X = K \cup_\alpha D^4$ be a Poincaré complex with finite fundamental group, with orientation character w and with 3-skeleton K. Under the map $\Gamma(\pi_2 X) = \Gamma(\pi_2 K) \to \pi_3(K)$ the element λ_X is mapped to $N^w\alpha$ (up to a unit), where $N^w = \sum_{g\in\pi} w(g)g \in \mathbb{Z}\pi$.*

Proof. We can assume that $\pi_2 X$ is nontrivial since otherwise X is homotopy equivalent to either a sphere S^4 or a projective plane $\mathbb{RP}^4$. The Whitehead sequences for X and K fit into the following commutative diagram:

$$\begin{CD}
H_4(\widetilde{X}) @>{b_4}>> \Gamma(\pi_2 X) @>{\widehat{\eta}}>> \pi_3(X) @>>> H_3(\widetilde{X}) @>>> 0\\
@AAA @| @AAA @AAA\\
0 = H_4(\widetilde{K}) @>>> \Gamma(\pi_2 K) @>{\widehat{\eta}}>> \pi_3(K) @>>> H_3(\widetilde{K}) @>>> 0\\
@. @. @AAA @AAA\\
@. @. \pi_4(X,K) @>{\cong}>> H_4(\widetilde{X},\widetilde{K}) \cong \mathbb{Z}\pi\\
@. @. @AAA @A{N^w}AA\\
@. @. \pi_4(X) @>>> H_4(\widetilde{X}) \cong \mathbb{Z}
\end{CD}$$

There is a choice of isomorphism $\mathbb{Z}\pi \cong \pi_4(X,K)$ such that 1 is mapped to α under $\pi_4(X,K) \to \pi_3(K)$. Since λ_X comes from $H_4(\widetilde{X})$, the image $\widehat{\eta}(\lambda_X) \in \pi_3(K)$ lies in the kernel of the maps to $\pi_3(X)$ and $H_3(\widetilde{K})$. Therefore, by a quick diagram chase, $\widehat{\eta}(\lambda_X)$ is a multiple of $N^w\alpha$ and since $(-\cap[\widetilde{X}]) \in \operatorname{Sym}_{\mathbb{Z}}(H^2(\widetilde{X}), H_2(\widetilde{X},\mathbb{Z}))$ is an isomorphism and hence primitive, it is mapped to $N^w\alpha$ up to a unit. □

In Section 3 we will need the following results about the transfer map.

Let $U \leq \pi$ have finite index. Let X be a CW-complex with fundamental group π and let $\widehat{X}$ denote the finite covering of X with fundamental group U. Then X and $\widehat{X}$ have the same universal covering and let C_* denote its cellular $\mathbb{Z}\pi$-chain complex. We can consider it as a right $\mathbb{Z}\pi$-module using the involution on $\mathbb{Z}\pi$. If M is a $\mathbb{Z}\pi$-module we can restrict the action to U and consider the projection

$$C_* \otimes_{\mathbb{Z}U} M \xrightarrow{p} C_* \otimes_{\mathbb{Z}\pi} M$$

inducing

$$p_*\colon H_*(\widehat{X};M) \to H_*(X;M).$$

On the chain level we obtain a map in the other direction by

$$\mathrm{tr}\colon C_* \otimes_{\mathbb{Z}\pi} M \to C_* \otimes_{\mathbb{Z}U} M, \quad c \otimes m \mapsto \sum_{Ug \in U\backslash\pi} cg^{-1} \otimes gm.$$

This map induces a map

$$\mathrm{tr}_*\colon H_*(X;M) \to H_*(\widehat{X};M)$$

and similarly one constructs a map

$$\mathrm{tr}^*\colon H^*(\widehat{X};M) \to H^*(X;M).$$

Lemma 2.8. *These transfer maps have the following properties:*

(1) $p_* \circ \mathrm{tr}_*$ *and* $\mathrm{tr}^* \circ p^*$ *are multiplication by the index* $[U:\pi]$.
(2) If $U \leq \pi$ *is normal, then* π/U *acts on* $\widehat{X}$ *and also on* $H_*(\widehat{X};M)$ *and* $H^*(\widehat{X};M)$. *In this case,* $tr_* \circ p_*$ *and* $p^* \circ tr^*$ *are multiplication by* $\sum_{g\in\pi/U} g$.
(3) If π *acts trivially on a commutative ring* R, *then* tr^* *is an* $H^*(X;R)$*-module homomorphism, i.e.*

$$tr^*(p^*(x) \cup y) = x \cup tr^*(y)$$

for all $x \in H^*(X;R)$ *and* $y \in H^*(\widehat{X};R)$.
(4) The transfer commutes with Kronecker-products and Steenrod-squares:

$$\langle tr^*(y), b\rangle = \langle y, tr_* b\rangle \in R$$

and

$$Sq^i(tr^*(y)) = tr^*(Sq^i(y)) \in H^*(X;\mathbb{Z}/2).$$

Proof. (1) and (2) follow directly from the definitions. For (3) and (4) one uses that for trivial modules tr_* and tr^* are induced by a stable map $tr\colon \Sigma^\infty X \to \Sigma^\infty \widehat{X}$ of suspension spectra, see [Ada78, Chapter 4]. □

3. The $\mathbb{CP}^2$-stable classification

In this section we will recall the proof of [HK88, Theorem 1.1] and then prove Theorems 1.1, 1.3 and 1.4. At the end of the section we will prove Proposition 1.6 and Proposition 3.13 which was mentioned in the introduction.

Let X, Y be Poincaré 4-complexes with finite fundamental group π. Assume that X and Y have isomorphic quadratic 2-types:

$$Q_X = (\pi, w, \pi_2(X), k, \lambda_X) \cong Q_Y.$$

In the following we will always assume that $\pi_2(X) \neq 0$ since otherwise X is homotopy equivalent to either a sphere S^4 or a projective plane $\mathbb{RP}^4$. Let B be the

2-type of X and let $f\colon X \to B$ be a 3-equivalence. From the results of Section 2, in particular Corollary 2.6, it follows that there is a 3-equivalence $g\colon Y \to B$ such that $\widetilde{g}_*[\widetilde{Y}] = \widetilde{f}_*[\widetilde{X}]$.

If in this situation there is a map $h\colon X \to Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & B & \end{array}$$

homotopy commutative, then h is an isomorphism on π_1, π_2. By Corollary 2.6 there is an isomorphism $\Gamma(\pi_2(Y)) \cong H_4(\widetilde{B})$ mapping λ_Y to $\widetilde{g}_*[\widetilde{Y}]$ and by assumption, $\widetilde{g}_*[\widetilde{Y}] = \widetilde{f}_*[\widetilde{X}] = \widetilde{g}_*\widetilde{h}_*[\widetilde{X}]$. Since $H_4(\widetilde{Y}) \cong \mathbb{Z}$, $H_4(\widetilde{B})$ is torsion-free and $\widetilde{g}_*[\widetilde{Y}]$ is non-trivial, this implies $\widetilde{h}_*[\widetilde{X}] = [\widetilde{Y}]$. Thus by Poincaré duality $\widetilde{h}$ is an isomorphism on homology of universal coverings in all degrees and hence h is a homotopy equivalence.

Lemma 3.1. *A map h as above exists if and only if $f_*[X] = g_*[Y] \in H_4(B; \mathbb{Z}^w)$.*

Proof. This result is proved by Hambleton and Kreck [HK88, Lemma 1.3] in the oriented case. Since the proof in the non-oriented case works the same we will not give any details here. However, we want to point out the following two things.

Firstly, there is a minor mistake at the beginning of the proof of [HK88, Lemma 1.3], where the authors construct B by attaching cells of dimension 4 and higher to Y, i.e. $g\colon Y \to B$ is the inclusion. Then they say that they want to extend

$$X^{(3)} \xrightarrow{f^{(3)}} B^{(3)} = Y^{(3)} \subseteq Y$$

over the 4-cell of X.

In fact, using the vanishing of obstructions in cohomology they can only extend the map $X^{(2)} \to Y$, because they might have to change the given map on the 3-cells in order to extend over the 4-cell. But also such an extension h suffices for the proof because the given (trivial) homotopy between f and $g \circ h$ on $X^{(2)} \times I$ extends to $X \times I$ using $\pi_3(B) = \pi_4(B) = 0$.

Secondly, to get the non-oriented case, note that Hambleton and Kreck show for $w \equiv 1$ that the obstruction in

$$H^4(X; \pi_3 Y) \cong H_0(X; \pi_3 Y \underset{\mathbb{Z}\pi}{\otimes} \mathbb{Z}^w) \cong \pi_3 Y \underset{\mathbb{Z}\pi}{\otimes} \mathbb{Z}^w \cong \pi_4(B, Y) \underset{\mathbb{Z}\pi}{\otimes} \mathbb{Z}^w \cong H_4(B, Y; \mathbb{Z}^w)$$

is given by the image of $f_*[X]$ under $H_4(B; \mathbb{Z}^w) \to H_4(B, Y; \mathbb{Z}^w)$ and therefore vanishes if $f_*[X] = g_*[Y]$. All these isomorphisms continue to hold for non-trivial w. □

Let X be a Poincaré complex with orientation character w, finite fundamental group π, intersection form λ_X and with a fixed 3-equivalence $f\colon X \to B$, where B is 3-coconnected.

Let λ be a hermitian form on $\pi_2(B)$. Recall that $\mathcal{S}_4^{\mathrm{PD}}(B, w, \lambda)$ denotes the set of 4-dimensional B-polarized Poincaré complexes (Y, g) with orientation character w, such that λ maps to λ_Y, up to homotopy equivalence over B. Assume $(X, f) \in \mathcal{S}_4^{\mathrm{PD}}(B, w, \lambda)$. By Lemma 3.1, sending (Y, g) to $g_*[Y] - f_*[X]$ defines an injection

$$\mathcal{S}_4^{\mathrm{PD}}(B, w, \lambda) \to H_4^w(B) := H_4(B; \mathbb{Z}^w).$$

We want to compute the image of this map. Under the isomorphism $H_4(\widetilde{B}) \cong \Gamma\pi_2(B)$, the element $\widetilde{f}_*[\widetilde{X}]$ is determined by $\lambda = f^*\lambda_X$ by Corollary 2.6. Furthermore, recall that $\mathrm{tr}_*[X] = [\widetilde{X}]$. Therefore $|\pi|(g_*[Y] - f_*[X]) = p_* \,\mathrm{tr}_*(g_*[Y] - f_*[X]) = 0$, i.e. the above image lies in the torsion subgroup of $H_4^w(B)$.

Write X as $K \cup_\alpha D^4$, where K is a 3-complex. Since K is a 3-dimensional complex, $H_3^w(K)$ is a subgroup of a free abelian group and thus torsion-free. Furthermore, $H^1(X) \cong H^1(\pi) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi, \mathbb{Z}) = 0$. Using this, from the exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & H_4^w(X) & \longrightarrow & H_4^w(X,K) & \longrightarrow & H_3^w(K) \longrightarrow & H_3^w(X) \\
& \| & & \| & & & \downarrow{\scriptstyle \mathrm{PD}^{-1}}\,\cong \\
& \mathbb{Z} & & \mathbb{Z} & & & H^1(X)
\end{array}$$

we obtain $H_3^w(K) = 0$. Therefore, $H_4^w(B) \to H_4^w(B,K)$ is an isomorphism. Since B is 3-coconnected, $\pi_4(B,K) \to \pi_3(K)$ is an isomorphism. Combining these, we obtain the very useful isomorphism

$$H_4^w(B) \xrightarrow{\cong} H_4^w(B,K) \xleftarrow{\cong} \pi_4(B,K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \xrightarrow{\cong} \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w. \tag{3.2}$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
H_4^w(X) & \longrightarrow & H_4^w(X,K) & \longleftarrow & \pi_4(X,K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \overset{\partial}{\searrow} & \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
H_4^w(B) & \xrightarrow{\cong} & H^w(B,K) & \xleftarrow{\cong} & \pi_4(B,K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \underset{\partial}{\xrightarrow{\cong}} & \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w
\end{array}$$

and view the top cell of X as an element in $\pi_4(X,K)$. Then it has boundary α and its image in $H_4^w(X,K)$ is also the image of $[X] \in H_4^w(X)$. Hence the composition of the horizontal isomorphisms (3.2) maps $f_*[X]$ to $[\alpha] \otimes 1$.

Finally, we use Whitehead sequences for $\widetilde{X}$ and $\widetilde{K}$:

$$\begin{array}{ccccccccccc}
\pi_4(\widetilde{X}) & \xrightarrow{0} & H_4(\widetilde{X}) & \xrightarrow{b_4} & \Gamma(\pi_2(X)) & \xrightarrow{\widehat{\eta}} & \pi_3(X) & \longrightarrow & H_3(\widetilde{X}) & = & 0 \\
& & \uparrow & & \| & & \uparrow & & \uparrow & & \\
& & 0 & \longrightarrow & \Gamma(\pi_2(K)) & \xrightarrow{\widehat{\eta}} & \pi_3(K) & \longrightarrow & H_3(\widetilde{K}) & \longrightarrow & 0 \\
& & & & & & \uparrow & & \uparrow & & \\
& & & & & & \pi_4(\widetilde{X},\widetilde{K}) & \xrightarrow{\cong} & H_4(\widetilde{X},\widetilde{K}) & \xrightarrow{\cong} & \mathbb{Z}\pi \\
& & & & & & \uparrow{\scriptstyle 0} & & \uparrow & & \\
& & & & & & \pi_4(\widetilde{X}) & \xrightarrow{0} & H_4(\widetilde{X}) & &
\end{array}$$

The image of the compositon $H_4(\widetilde{X}) \xrightarrow{b_4} \Gamma(\pi_2(X)) = \Gamma(\pi_2(K)) \xrightarrow{\widehat{\eta}} \pi_3(K)$ lies in the kernel of the map to $\pi_3(X)$ and thus inside $\pi_4(\widetilde{X},\widetilde{K}) \cong H_4(\widetilde{X},\widetilde{K}) \cong \mathbb{Z}\pi$. We see that it fits into a short exact sequence

$$0 \to H_4(\widetilde{X}) \to H_4(\widetilde{X},\widetilde{K}) \to H_3(\widetilde{K}) \to 0$$

and we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_4(\widetilde{X}) & \longrightarrow & \mathbb{Z}\pi & \longrightarrow & H_3(\widetilde{K}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b_4} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Gamma(\pi_2(K)) & \longrightarrow & \pi_3(K) & \longrightarrow & H_3(\widetilde{K}) & \longrightarrow & 0 \end{array}$$

The fundamental class $[\widetilde{X}]$ maps downward to $\lambda_X \in \Gamma(\pi_2 X) = \Gamma(\pi_2 K)$ and, by Corollary 2.7, to the right to $\pm N^w := \sum_{g\in\pi} w(g)g$. Note that $H_4(\widetilde{X}) \cong \mathbb{Z}^w$. By the upper short exact sequence above, $H_3(\widetilde{K}) \cong \mathbb{Z}\pi/N^w$. Tensoring the diagram with $\mathbb{Z}^w$ over $\mathbb{Z}\pi$ we obtain the following diagram of the long exact Tor-sequences:

$$\begin{array}{ccccccccccc} & & & & \mathbb{Z} & \xrightarrow{\cdot|\pi|} & \mathbb{Z} & & & & \\ & & & & \| & & \| & & & & \\ 0 & \to & \operatorname{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w,\mathbb{Z}^w) & \longrightarrow & \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \xrightarrow{N^w\otimes \mathrm{Id}} & \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \longrightarrow & \mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \operatorname{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w,\mathbb{Z}^w) & \to & \Gamma(\pi_2(K)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \to & \pi_3(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \to & \mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \to & 0 \end{array}$$

The top row shows that $\operatorname{Tor}_1^{\mathbb{Z}\pi}(\mathbb{Z}\pi/N^w,\mathbb{Z}^w) = 0$ and $\mathbb{Z}\pi/N^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w = \mathbb{Z}/|\pi|\mathbb{Z}$. Using that $\pi_3 K \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \cong H_4^w(B)$ and $\pi_2(B) \cong \pi_2(K)$, the bottom row can be identified with the short exact sequence:

$$0 \to \Gamma(\pi_2(B)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \xrightarrow{i} H_4^w(B) \xrightarrow{q} \mathbb{Z}/|\pi| \to 0 \qquad (3.3)$$

with $i(\lambda \otimes 1) = |\pi| f_*[X]$ and $q(f_*[X]) = 1 + |\pi|\mathbb{Z}$, exactly as in the oriented case [HK88, page 89]. From here Theorem 1.3 follows as in the proof of [HK88, Theorem 1.1].

Now we are nearly ready to prove the following theorem which completes the proof of Theorem 1.4. Recall that $(Y,g) \mapsto g_*[Y] - f_*[X]$ defines an injection $\mathcal{S}_4^{\mathrm{PD}}(B,w,f_*\lambda_X) \to H_4^w(B)$ with image inside the torsion subgroup.

Theorem 3.4. *For all $(Y,g) \in \mathcal{S}_4^{\mathrm{PD}}(B,w,\lambda)$ we have $q(g_*[Y] - f_*[X]) = 0$ and $(Y,g) \mapsto i^{-1}(g_*[Y] - f_*[X])$ defines a bijection*

$$\mathcal{S}_4^{\mathrm{PD}}(B,w,\lambda) \leftrightarrow \operatorname{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w).$$

Before proving Theorem 3.4 we will first show how it implies Theorem 1.1.

Proof of Theorem 1.1. It remains to show that for a given closed, connected 4-manifold M with fundamental group π and orientation character w the element $c_*[M] \in H_4(\pi;\mathbb{Z}^w)$ is determined by the stable quadratic 2-type of M. Let B denote the Postnikov 2-type of M and consider the sequence $M \xrightarrow{f} B \xrightarrow{c} B\pi$. Considering the following diagram and following the construction of (3.3), we see that the map

i in (3.3) can be identified with the canonical map $H_4(\widetilde{B}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \to H_4^w(B)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(\pi_2(\widetilde{K})) & \xrightarrow{\widehat{\eta}} & \pi_3(\widetilde{K}) & \longrightarrow & H_3(\widetilde{K}) & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\cong} & & \uparrow & & \\
 & & & & \pi_4(\widetilde{B},\widetilde{K}) & \xrightarrow{\cong} & H_4(\widetilde{B},\widetilde{K}) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & \longrightarrow & H_4(\widetilde{B}) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

Now given another closed, connected 4-manifold N with orientation character w and a 3-equivalence $g\colon N \to B$ such that $g^*\lambda_N = f^*\lambda_M$, then $f_*[M] - g_*[N]$ is contained in $\operatorname{Tors}(\Gamma(\pi_2(B)) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w) \cong \operatorname{Tors}(H_4(\widetilde{B}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ by Theorem 3.4. Since the composition $\widetilde{B} \to B \xrightarrow{c} B\pi$ is trivial, we thus have $c_*f_*[M] = c_*g_*[N] \in H_4^w(B\pi)$ as claimed. □

For the proof of Theorem 3.4 we need the following lemma.

Lemma 3.5. *If $w \equiv 1$ or if the 2-Sylow subgroup of π has order bigger than 2, there exists a homomorphism*

$$\kappa\colon \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \to \mathbb{Z} \text{ with } \lambda \otimes 1 \mapsto 1.$$

We will first prove Theorem 3.4 assuming Lemma 3.5, and then we will prove Lemma 3.5.

Proof of Theorem 3.4. We will first show that $q(g_*[Y] - f_*[X]) = 0$.

In the case that $w \equiv 1$ or the 2-Sylow subgroup of π has order more than 2, we will show that $i(\operatorname{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B))) = \operatorname{Tors}(H_4^w(B))$. This proves $q(g_*[Y] - f_*[X]) = 0$ in this case. The image of $\operatorname{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 B))$ is obviously contained in $\operatorname{Tors}(H_4^w(B))$ and we have to show the other inclusion.

Let $x \in \operatorname{Tors}(H_4^w(B))$ be given. Because $H_4(\widetilde{B}) \cong \Gamma(\pi_2 B)$ is torsion free, we see that $|\pi|x = p_* \operatorname{tr}_*(x) = 0$. From (3.3) and $q(f_*[X]) = 1 \in \mathbb{Z}/|\pi|$ we see that there exists $k \in \mathbb{N}$ and $y \in \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w$ such that $x + kf_*[X] = i(y)$. It follows that

$$|\pi| i(y) = |\pi| x + k|\pi| f_*[X] = ki(\lambda \otimes 1)$$

Thus, $|\pi| y = k(\lambda \otimes 1)$ and by Lemma 3.5 we have $|\pi|\kappa(y) = k\kappa(\lambda \otimes 1) = k$. Therefore, $|\pi|$ divides k and

$$x = i(y - \frac{k}{|\pi|}(\lambda \otimes 1)) \in \operatorname{im}(i).$$

We now consider the case $\pi = Q \oplus \mathbb{Z}/2$ with $|Q|$ odd and $w = p_2$ the projection onto $\mathbb{Z}/2 \cong \{\pm 1\}$. Since $f_*[X] \in H_4^w(B)$ maps to a generator of $\mathbb{Z}/|\pi| \cong \mathbb{Z}/|Q| \oplus \mathbb{Z}/2$ under q there is $x \in \mathbb{Z}/|Q|$ with $q(f_*[X]) = (x, 1 + 2\mathbb{Z}) \in \mathbb{Z}/|Q| \oplus \mathbb{Z}/2$. For any

$(Y, g) \in \mathcal{S}_4^{\mathrm{PD}}(B, w, \lambda)$ there is in the same way $y \in \mathbb{Z}/Q$ with $q(g_*[Y]) = (y, 1+2\mathbb{Z})$ and hence $q(g_*[Y] - f_*[X]) = (y - x, 0)$.

Taking double coverings with respect to the subgroup $Q \leq \pi$ we get a commutative square

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{B} \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

where $p_* \widehat{f}_*[\widehat{X}] = 2 f_*[X]$. So we may conclude from the following commutative diagram that $\widehat{q}(g_*[\widehat{Y}]) = y$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\pi_2 \widehat{B}) \otimes_{\mathbb{Z}Q} \mathbb{Z} & \xrightarrow{\widehat{i}} & H_4(\widehat{B}) & \xrightarrow{\widehat{q}} & \mathbb{Z}/|Q| & \longrightarrow & 0 \\ & & \downarrow & & \downarrow p_* & & \downarrow \cdot 2 & & \\ 0 & \longrightarrow & \Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w & \xrightarrow{i} & H_4^w(B) & \xrightarrow{q} & \mathbb{Z}/|\pi| & \longrightarrow & 0 \end{array}$$

Since $\widehat{X}$ is an orientable Poincaré complex we already know that Theorem 3.4 holds for $\widehat{X}$. Therefore, $0 = \widehat{q}(\widehat{g}_*[\widehat{Y}] - \widehat{f}_*[\widehat{X}]) = y - x \in \mathbb{Z}/|Q|$ and thus also $q(g_*[Y] - f_*[X]) = 0$.

By the above, the map $\mathcal{S}_4^{\mathrm{PD}}(B, w, \lambda) \to \mathrm{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ is well defined and we already now that it is injective. Showing surjectivity is easy since we can use [HK88, page 89] to reattach the top cell of X as follows. Recall that $X = K \cup_\alpha D^4$. For a given element $\beta \in \mathrm{Tors}(\Gamma(\pi_2 B) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w)$ pick a preimage $\widehat{\beta} \in \Gamma(\pi_2 B) \subseteq \pi_3(K)$ and attach a 4-cell to K along a representative of $\widehat{\beta} + \alpha$. The resulting complex X_β has the same intersection form as X and also $H_3(\widetilde{X_\beta}) = 0$ because $\widehat{\beta}$ maps trivially to $H_3(\widetilde{K})$. As in [HK88], the map $f|_K$ extends to a 3-equivalence $f_\beta \colon X_\beta \to B$ such that $f_{\beta *}[X_\beta] = f_*[X] + \beta$. □

For the proof of Lemma 3.5 we need the following constructions of maps from $\Gamma(\pi_2(X)) = \Gamma(\pi_2(B))$ to $\mathbb{Z}^w$. Consider the isomorphism

$$\zeta^{-1} \colon \Gamma(\pi_2(X)) \to \mathrm{Sym}_{\mathbb{Z}}(H^2(\widetilde{X}), H_2(\widetilde{X}))$$

from Theorem 2.4. Precomposing with the inverse of $\mathrm{PD} \colon H^2(\widetilde{X}) \xrightarrow{-\cap[\widetilde{X}]} H_2(\widetilde{X})$, we get a map

$$\kappa' \colon \Gamma(\pi_2(X)) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(\widetilde{X}), H_2(\widetilde{X})). \tag{3.6}$$

The construction of κ' and the following two lemmas below are due to Stefan Bauer [Bau88].

Lemma 3.7. *If we let π act on $\mathrm{Hom}_{\mathbb{Z}}(H_2(\widetilde{X}), H_2(\widetilde{X}))$ by $g\varphi = w(g)(g_* \circ \varphi \circ g_*^{-1})$, then κ' is equivariant. In particular,*

$$\mathrm{trace} \circ \kappa' \colon \Gamma(\pi_2(X)) \to \mathbb{Z}^w$$

is equivariant.

Proof. If we let π act on $\mathrm{Sym}(H^2(\widetilde{X}), H_2(\widetilde{X}))$ by $g\psi = g_* \circ \psi \circ g^*$, then by naturality ζ is equivariant. We have

$$g_*(g^*\alpha \cap [\widetilde{X}]) = \alpha \cap g_*[\widetilde{X}] = w(g)(\alpha \cap [\widetilde{X}])$$

and thus $g^* \circ \mathrm{PD}^{-1} = w(g)(\mathrm{PD}^{-1} \circ g_*^{-1})$. This proves the lemma since $\kappa'(x) = \zeta(x) \circ \mathrm{PD}^{-1}$. □

Lemma 3.8. *When we view λ_X as an element of $\Gamma(\pi_2(X))$ as in Definition 2.5, then*

$$\kappa'(\lambda_X) = \mathrm{Id}_{H_2(\widetilde{X})}.$$

Proof. Recall that λ_X is the image of $- \cap [\widetilde{X}]$ under ζ. Thus, precomposing with the inverse of $\mathrm{PD} = - \cap [\widetilde{X}]$ gives the identity on $H_2(\widetilde{X})$. □

A central involution in a group G is an element of order 2 in the center of G.

Lemma 3.9. *Let G be a 2-group, $w\colon G \to \mathbb{Z}/2$. If $w \equiv 1$ or $|G| > 2$, there exists a central involution $\tau \in G$ with $w(\tau) = 1 \in \{\pm 1\}$.*

Proof. Since G is a 2-group there exists a central involution. We are done if $w(\tau) = 1$. Otherwise the map w splits and since τ is central, we have $G = \langle \tau \rangle \times \ker(w)$. Since $|G| > 2$ there again exists a central involution $\tau' \in \ker(w)$ and this is still central in $G = \langle \tau \rangle \times \ker(w)$. □

Let $\tau_*\colon \mathrm{Hom}_Z(H_2(\widetilde{X}), H_2(\widetilde{X})) \to \mathrm{Hom}_Z(H_2(\widetilde{X}), H_2(\widetilde{X}))$ be given by precomposition with the action of τ on $\widetilde{X}$.

Lemma 3.10. *The map*

$$\mathrm{trace} \circ \tau_* \circ \kappa'\colon \Gamma(\pi_2(X)) \to \mathbb{Z}^w$$

is equivariant and the element λ_X is mapped to -2.

Proof. Since κ' and the trace are equivariant, we have to show that τ_* is equivariant. This is the case, because the group element τ is central. Since $\kappa'(\lambda_X) = \mathrm{Id}_{H_2(\widetilde{X})}$ by Lemma 3.8, we have $\tau_*(\kappa'(\lambda_X)) = \tau\colon H_2(\widetilde{X}) \to H_2(\widetilde{X}; \mathbb{Z})$. Since $w(\tau) = 1$ and τ acts fixed point free on $\widetilde{X}$, the Lefschetz formula yields

$$0 = \mathrm{trace}(\tau_{H_0(\widetilde{X})}) + \mathrm{trace}(\tau_{H_4(\widetilde{X})}) + \mathrm{trace}(\tau_{H_2(\widetilde{X})}) = 2 + \mathrm{trace}(\tau_*(\kappa'(\lambda_X)))$$

and therefore $\mathrm{trace}(\tau_*(\kappa'(\lambda_X))) = -2$. □

We want to show that the image of $\mathrm{trace} \circ \tau_* \circ \kappa'$ is always even. For this we need the following lemma.

Lemma 3.11. *Let $\tau \in \pi$ have order 2. Then the symmetric bilinear form*

$$H^2(\widetilde{X}; \mathbb{Z}/2) \otimes H^2(\widetilde{X}; \mathbb{Z}/2) \to \mathbb{Z}/2, \quad x \otimes y \mapsto \langle \tau x \cup y, [\widetilde{X}] \rangle$$

is even, i.e. $\langle \tau x, x \rangle = 0$ for all $x \in H^2(\widetilde{X}; \mathbb{Z}/2)$.

Proof. Let $q\colon \widetilde{X} \to \widehat{X} := \widetilde{X}/\tau$ be the double covering. Then $q^* \circ \mathrm{tr}^*(x) = x + \tau x$ and

$$\begin{aligned}
\langle q^* \circ \mathrm{tr}^*(x) \cup x, [\widetilde{X}] \rangle &= \langle \mathrm{tr}^*(q^* \circ \mathrm{tr}^*(x) \cup x), [\widehat{X}] \rangle \\
&= \langle \mathrm{tr}^*(x) \cup \mathrm{tr}^*(x), [\widehat{X}] \rangle \\
&= \langle Sq^2(\mathrm{tr}^*(x)), [\widehat{X}] \rangle \\
&= \langle Sq^2(x), \mathrm{tr}_*[\widehat{X}] \rangle \\
&= \langle x \cup x, [\widetilde{X}] \rangle.
\end{aligned}$$

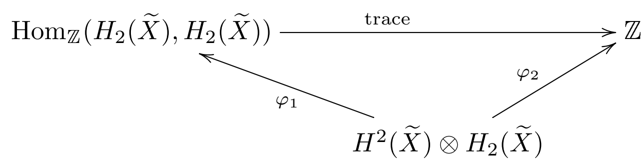
Here we used the identities from Lemma 2.8 (with $p = q$). It follows that

$$\langle x \cup \tau x, [\widetilde{X}]\rangle = \langle x \cup (q^* \circ \mathrm{tr}^*(x) + x), [\widetilde{X}]\rangle = 2\langle x \cup x, [\widetilde{X}]\rangle = 0$$

for all $x \in H^2(\widetilde{X}; \mathbb{Z}/2)$. □

Lemma 3.12. *The image of* $\mathrm{trace} \circ \tau_* \circ \kappa'$ *is contained in* $2\mathbb{Z}^w$.

Proof. The trace fits into the following commutative diagram:



where φ_1 is the slant map and φ_2 the evaluation, i.e. $\varphi_1(\alpha \otimes x) := (y \mapsto (\alpha \cap y)x)$ and $\varphi_2(\alpha \otimes x) := \alpha \cap x$.

Let $\{\beta_i\}$ be a $\mathbb{Z}$-basis of $H_2(\widetilde{X})$. Every element in $H^2(\widetilde{X}) \otimes H_2(\widetilde{X})$ can be written as $\sum_{i,j} \lambda_{i,j}\beta_i \otimes (\beta_j \cap [\widetilde{X}])$ for some $\lambda_{i,j} \in \mathbb{Z}$. The map

$$\Psi := \varphi_1(\sum_{i,j} \lambda_{i,j}\beta_i \otimes (\beta_j \cap [\widetilde{X}])) \circ \mathrm{PD}\colon H^2(\widetilde{X}) \to H_2(\widetilde{X})$$

is given by $\Psi(\alpha) = \sum_{i,j} \lambda_{i,j}((\alpha \cup \beta_i) \cap [\widetilde{X}])(\beta_j \cap [\widetilde{X}])$. Recall that it is symmetric if $\alpha' \cap \Psi(\alpha) = \alpha \cap \Psi(\alpha')$. Let $\{\beta_i^*\}$ be the basis dual to $\{\beta_i\}$ in the sense that $(\beta_i^* \cup \beta_j) \cap [\widetilde{X}] = \delta_{ij}$. We see that if Ψ is symmetric, then

$$\begin{aligned}
\lambda_{i',j'} &= \sum_{i,j} \lambda_{i,j}((\beta_{i'}^* \cup \beta_i) \cap [\widetilde{X}])((\beta_{j'}^* \cup \beta_j) \cap [\widetilde{X}]) \\
&= \beta_{j'}^* \cap \Psi(\beta_{i'}^*) \\
&= \beta_{i'}^* \cap \Psi(\beta_{j'}^*) \\
&= \sum_{i,j} \lambda_{i,j}((\beta_{j'}^* \cup \beta_i) \cap [\widetilde{X}])((\beta_{i'}^* \cup \beta_j) \cap [\widetilde{X}]) \\
&= \lambda_{j',i'}
\end{aligned}$$

One easily computes

$$\tau_*(\varphi_1(\sum_{i,j} \lambda_{i,j}\beta_i \otimes (\beta_j \cap [\widetilde{X}]))) = \varphi_1(\sum_{i,j} \lambda_{i,j}\tau\beta_i \otimes (\beta_j \cap [\widetilde{X}]))$$

and using that τ has order two and $w(\tau) = 1$, we get

$$(\tau\beta_i \cup \beta_j) \cap [\widetilde{X}] = (\tau^2\beta_i \cup \tau\beta_j) \cap w(\tau)[\widetilde{X}] = (\beta_i \cup \tau\beta_j) \cap [\widetilde{X}].$$

Let Ψ be symmetric. By Lemma 3.11 we then get

$$\begin{aligned}
\varphi_2(\sum_{i,j} \lambda_{i,j}\tau\beta_i \otimes (\beta_j \cap [\widetilde{X}])) &= \sum_{i,j} \lambda_{i,j}(\tau\beta_i \cup \beta_j) \cap [\widetilde{X}] \\
&= \sum_{i} (\tau\beta_i \cup \beta_i) \cap [\widetilde{X}] + 2\sum_{i<j} \lambda_{i,j}(\tau\beta_i \cup \beta_j) \cap [\widetilde{X}] \\
&= 2\sum_{i} (\beta_i \cup \beta_i) \cap [\widetilde{X}] + 2\sum_{i<j} \lambda_{i,j}(\tau\beta_i \cup \beta_j) \cap [\widetilde{X}]
\end{aligned}$$

This implies the lemma since by definition of κ' every element in its image is coming from $\mathrm{Sym}_{\mathbb{Z}}(H^2(\widetilde{X}), H_2(\widetilde{X}))$. □

Proof of Lemma 3.5. Since $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w = \mathbb{Z}$, $f_*\colon \pi_2(X) \to \pi_2(B)$ is an isomorphism and $f^*\lambda = \lambda_X$, it suffices to show that there exists a $\mathbb{Z}\pi$-homomorphism

$$\kappa\colon \Gamma(\pi_2(X)) \to \mathbb{Z}^w$$

with $\kappa(\lambda_X) = 1$.

We have $\mathrm{rank}(H_2(\widetilde{X})) = \chi(\widetilde{X}) - 2 = |\pi|\chi(X) - 2$, where χ denotes the Euler characteristic. Thus it suffices to show that there exist homomorphisms $\kappa_i\colon \Gamma(\pi_2(X)) \to \mathbb{Z}^w$ such that:

(1) $\kappa_1(\lambda_X) = |\pi|$,
(2) $\kappa_2(\lambda_X) = \mathrm{rank}(H_2(\widetilde{X}))$ and
(3) $\kappa_3(\lambda_X)$ is odd,

because if $\kappa_3(\lambda_X) = 2n+1$, then for $\kappa = \kappa_3 + n\kappa_2 - n\chi(X)\kappa_1$ we have $\kappa(\lambda_X) = 1$. **(1):** For a $\mathbb{Z}\pi$-module M let M^{π_w} denote the submodule consisting of those objects m with $gm = w(g)m$ for all $g \in \pi$. We have $\lambda_X \in \Gamma(\pi_2(X))^{\pi_w}$. Any $\mathbb{Z}$-homomorphism $\Gamma(\pi_2(X))^{\pi_w} \to \mathbb{Z}^w$ is already a $\mathbb{Z}\pi$-homomorphism, since π acts on both sides via w. Since X is a finite 4-dimensional Poincaré complex with finite fundamental group, $\pi_2(X)$ is free abelian. By Lemma 2.2, $\Gamma(\pi_2(X))$ is therefore free abelian. Capping with $[\widetilde{X}]$ gives an isomorphism $H^2(\widetilde{X}) \to H_2(\widetilde{X})$ and hence $\lambda_X \in \Gamma(\pi_2(X))^{\pi_w}$ is primitive. Therefore, there exists a $\mathbb{Z}$- and thus $\mathbb{Z}\pi$-linear map

$$d\colon \Gamma(\pi_2(X))^{\pi_w} \to \mathbb{Z}^w$$

with $d(\lambda_X) = 1$. We define κ_1 as the composition

$$\kappa_1\colon \Gamma(\pi_2(X)) \xrightarrow{\cdot N^w} (\Gamma(\pi_2(X))^w)^\pi \xrightarrow{d} \mathbb{Z}^w.$$

(2): Define $\kappa_2 := \mathrm{trace} \circ \kappa'$. By Lemma 3.8,

$$\kappa_2(\lambda_X) = \mathrm{trace}(\mathrm{Id}_{H_2(\widetilde{X})}) = \mathrm{rank}(H_2(\widetilde{X})).$$

(3): Let $G \leq \pi$ be a 2-Sylow subgroup and $p\colon X' \to X$ the corresponding covering. Then the map

$$tr^\pi_G\colon \Gamma(\pi_2 X) \otimes_{\mathbb{Z}\pi} \mathbb{Z}^w \to \Gamma(\pi_2 X) \otimes_{\mathbb{Z}G} \mathbb{Z}^{p^* w}$$

has the property that

$$tr^\pi_G(\lambda_X \otimes 1) = \sum_{g \in G\backslash\pi} (g(\lambda_X) \otimes g \cdot 1) = [\pi : G](\lambda_X \otimes 1).$$

Since $[\pi : G]$ is odd and $\lambda_X = \lambda_{X'}$, we can therefore assume that π is a 2-group by composing κ_3 for G with tr^π_G. Note that $p^*\colon H^1(X;\mathbb{Z}/2) \to H^1(X';\mathbb{Z}/2)$ is injective and thus p^*w is non trivial if and only if w is non trivial.

Let κ' be the homomorphism from (3.6). By Lemma 3.12, the image of $\mathrm{trace} \circ \tau_* \circ \kappa'$ is always even and we can define $\kappa_3 := -\frac{1}{2}(\mathrm{trace} \circ \tau_* \circ \kappa')$. By Lemma 3.10, $\kappa_3(\lambda_X) = 1$. □

Proof of Proposition 1.6. By [HK88, Theorem 2.1] we have

$$\Gamma(\mathbb{Z}\pi) \cong F \oplus \bigoplus_{g \in \pi\setminus\{1\}, g^2=1} \mathbb{Z}\pi/\mathbb{Z}\pi(1-g),$$

for some free module F. It is an easy calculation that this implies

$$\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi)) \cong (\mathbb{Z}/2)^r,$$

where r is the number of $g \in \pi \setminus \{1\}$ with $g^2 = 1$ and $w(g) = -1$. Since

$$\Gamma(\pi_2 X) \cong \Gamma(\mathbb{Z}\pi) \oplus \Gamma(M) \oplus \mathbb{Z}\pi \otimes_{\mathbb{Z}} M,$$

$\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi))$ is a summand of $\mathrm{Tors}(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 X))$. □

Proposition 3.13. *There is a Poincaré complex with the same quadratic* 2*-type as* $\mathbb{RP}^4 \# \mathbb{CP}^2$ *which is not homotopy equivalent to* $\mathbb{RP}^4 \# \mathbb{CP}^2$.

Proof. Let B be the Postnikov 2-type of $\mathbb{RP}^4 \# \mathbb{CP}^2$. By Lemma 3.1, B-polarized Poincaré complexes up to homotopy equivalence over B are classified by the image of their fundamental class in $H_4(B; \mathbb{Z}^w)$. Hence to understand them up to homotopy equivalence, we have to study the action of all self-homotopy equivalences of B on $H_4(B; \mathbb{Z}^w)$. As $\pi_2(B)$ is free, the k-invariant of B is trivial and the map $B \to B\mathbb{Z}/2$ admits a splitting s. Using the Serre spectral sequence and the proof of Proposition 1.6, one computes

$$\begin{aligned} H_4(B; \mathbb{Z}^w) &\cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} H_4(\widetilde{B}; \mathbb{Z}) \oplus H_4(B\mathbb{Z}/2; \mathbb{Z}^w) \\ &\cong \mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2]) \oplus \mathbb{Z}/2 \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \end{aligned}$$

By obstruction theory, for every self-homotopy equivalence φ of B we have $\varphi \circ s \simeq s$. Hence φ acts diagonally on $H_4(B; \mathbb{Z}^w)$ with the identity on $H_4(B\mathbb{Z}/2; \mathbb{Z}^w)$ and on $\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} H_4(\widetilde{B}; \mathbb{Z})$ by the map on $\pi_2(B)$ induced by φ. The automorphisms of $\pi_2(B) \cong \mathbb{Z}[\mathbb{Z}/2]$ are given by multiplicaton with $\{\pm 1, \pm g\}$ where $g \in \mathbb{Z}/2$ is the generator. It follows that the action of φ on $H_4(B; \mathbb{Z}^w)$ is given by ± 1 and thus is the identity on the torsion subgroup. In particular, it is the identity on the torsion subgroup $\mathbb{Z}/2$ of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \Gamma(\mathbb{Z}[\mathbb{Z}/2])$. It thus follows from Theorem 1.4 that there is precisely one Poincaré complex which has the same quadratic 2-type as $\mathbb{RP}^4 \# \mathbb{CP}^2$ but is not homotopy equivalent to it. □

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