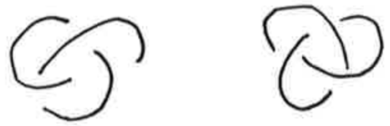
June 7, 2018
MPIM.

Alexander polynomial

1. Alexander module

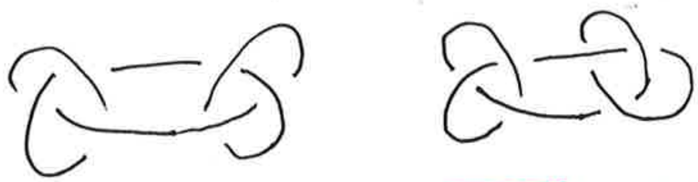
Goal: distinguish knots i.e. come up with good invariants

<u>Theorem</u> [Gordon-Luecke 1989]: $S^3 \setminus K_1 \cong S^3 \setminus K_2$
$\iff K_1 = K_2$ or $K_1 = mK_2$ with some choice of orientation.

E.g.  have homeo. exteriors.
[$S^3\setminus K_1 \underset{o.p}{\cong} S^3 \setminus K_2 \iff K_1 = K_2$ up to orientation]

~~[illegible]~~ <u>Def$^{\underline{n}}$</u>: The <u>knot group</u> of K is $\pi_1(S^3 \setminus K)$.

Consequences: <u>Prime</u> knots K_1, K_2 are equivalent (up to orientation & mirror)
$\iff \pi_1(S^3\setminus K_1) \cong \pi_1(S^3 \setminus K_2)$

 have isomorphic π_1 but are not equiv. (even up to mirror & orient.)

The triple $(\pi_1(S^3\setminus K) \boxed{, \mu, \lambda})$ peripheral str $\rightsquigarrow$ quandles. (complete invt)
oriented meridian longitude
determines the class of K completely.

Point: knot group is a pretty good invt, want to study it.

Let $G = \pi_1(S^3 \setminus K)$. Normally generated by meridian of K.
hard to understand directly

$G/[G,G] = H_1(S^3\setminus K) \cong \mathbb{Z}\langle\mu\rangle \quad \forall K.$
$\rightarrow$ lost all the information.

Goal: study other quotients.

$$1 \longrightarrow [G,G] \longrightarrow G \longrightarrow \underset{\substack{\mathbin{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}\langle\mu\rangle}}{G/[G,G]} \longrightarrow 1$$

Let $X := S^3 \setminus K$.
$\tilde{X}$ the <u>infinite cyclic cover</u> (corr. to map $G \twoheadrightarrow \mathbb{Z}\langle\mu\rangle$) with deck gp $[G,G]$)

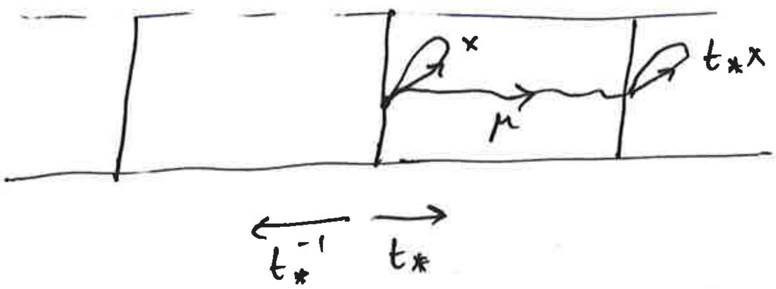
$\rightsquigarrow$ chains of $\tilde{X}$ have a $\mathbb{Z}[t,t^{-1}]$-module structure, where t acts by μ.

Alexander module. $\mathcal{A}(K) := H_1(\tilde{X};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$module
$(\cong H_1(X;\mathbb{Z}[t,t^{\pm 1}]))$.

$$\mathcal{A}(K) \cong [G,G] \big/ [[G,G],[G,G]]$$

$\uparrow$ given by $p_*: \pi_1\tilde{X} \to \pi_1 X$

if $x \in [G,G]$, $t_*[x] = \mu x \mu^{-1}$



Note: the orientation K tells us which is the action of t and which is by t^{-1}. [This can in general change the isotype of a module]

---

2. Presentation using knot gp:

$K =$  (left-handed) trefoil

$G \cong \langle x,y \mid xyx = yxy \rangle$

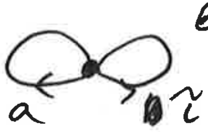
$G_{ab} \cong \mathbb{Z}$

Step 1: change basis s.t. one gen $\mapsto 1$, others $\mapsto 0$ in the abelianisation.

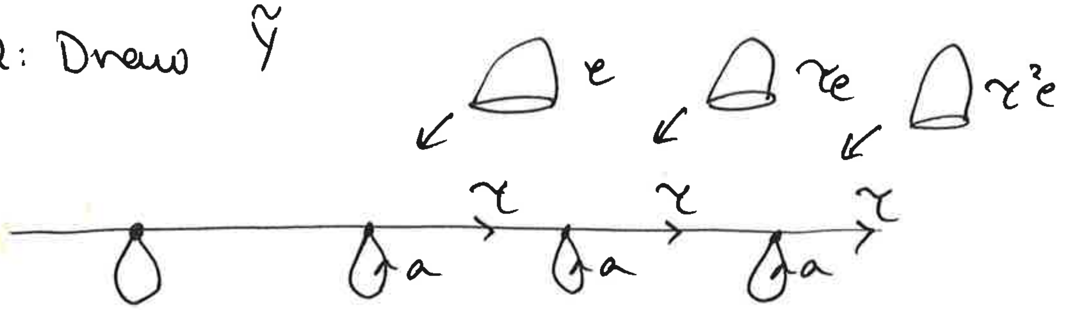
$$\langle x,y \mid xyx = yxy \rangle \xrightarrow[\tau = x]{a = yx^{-1}} \langle \tau, a \mid \tau a \tau^2 = a\tau^2 a \tau \rangle$$

i.e. $y = a\tau$

 e 2-cell along ~~[illegible]~~ $\tau a \tau^2 \tau^{-1} a^{-1} \tau^{-2} a^{-1}$

$Y =$ 

Step 2: Draw $\tilde{Y}$



$$\pi_1 \tilde{Y} = [G,G] = \langle \underbrace{\tau^i a \tau^{-i}}_{i \in \mathbb{Z}} \mid \tau^i \left(\begin{smallmatrix}\text{relation}\\ \text{from 2-cell}\end{smallmatrix} \right) \tau^{-i} \rangle$$

$$\tau a \tau^2 \tau^{-1} a^{-1} \tau^{-2} a^{-1} = (\tau a \tau^{-1})(\tau^2 a^{-1} \tau^{-2}) a^{-1}$$
$$= (\tau a \tau^{-1})(\tau^2 a \tau^{-2})^{-1} a^{-1}$$

If $\alpha_i = \tau^i a \tau^{-i}$, rel is. $\alpha_1 \alpha_2^{-1} \alpha_0^{-1}$
$\alpha_2 \alpha_3^{-1} \alpha_1^{-1}$
$\vdots$

$$\mathcal{A}(K) \cong \langle \alpha_0 \mid t\alpha_0 - t^2\alpha_0 - \alpha_0 \rangle \quad \text{as a } \mathbb{Z}[t,t^{-1}]\text{-module}$$

$$\cong \langle \alpha_0 \mid \alpha_0(t^2 - t + 1) \rangle$$

$$\cong \mathbb{Z}[t,t^{-1}] / (t^2 - t + 1)$$

↑ Alexander module of .

[this is basically Fox calculus
i.e. why Fox calc. works]

~~3. Computation / [illegible] / [illegible] / [illegible]~~

~~[illegible]~~

3. Properties of Alexander modules | polynomials

$\mathcal{A}(K)$ is a finitely presented, torsion module over $\mathbb{Z}[t,t^{-1}]$ with a square presentation matrix M.

$\Delta_K(t) = \det M$.

$\Delta_K(t) \doteq \Delta_K(t^{-1})$

↑ up to units in $\mathbb{Z}[t^{\pm 1}]$
i.e. mult by $t^{\pm n}$

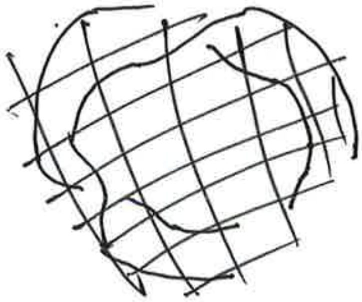
$\Delta_K(1) = \pm 1$.

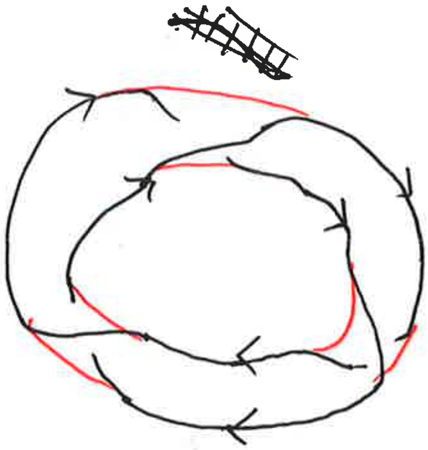
4. Computation using Seifert surfaces.

Any knot in S^3 bounds an embedded, compact, oriented surface in S^3.

Such a surface is called a <u>Seifert surface</u>.

$\exists$ algorithm to construct a seifert surface given a knot diagram

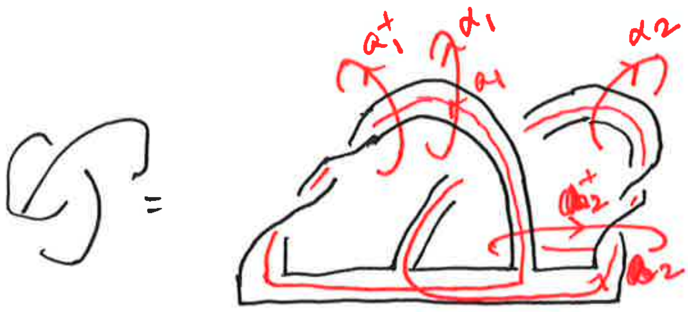




Σ is oriented since K is.

alternatively $K = 0$ in H_1 $\Rightarrow$ K bounds $C \in H_2(S^3 \setminus K, \partial)$
$\cong H^1(S^3 \setminus K)$
$\rightsquigarrow$ map $f : S^3 \setminus K \to S^1$
take preimage of reg value

Sometimes easy to visualise



[draw big]

Seifert form: Let Σ be a Seifert surface for a knot K.

$$\beta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$$
$$(x, y) \longmapsto lk(x, y^+)$$

where y^+ is the positive normal pushoff.

- bilinear form.
- NOT an invariant of K.

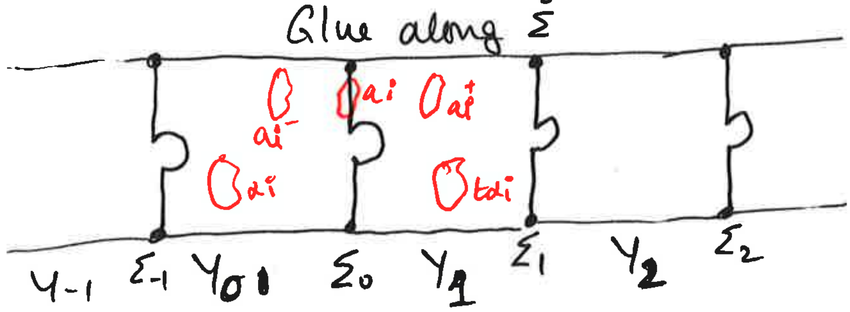
E.g. $\begin{array}{cc} & \begin{array}{cc} a_1^+ & a_2^+ \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$ <u>Seifert matrix</u>

Theorem: $\mathcal{A}(K)$ is presented by $V - tV^T$, where V is any Seifert matrix for K and Seifert surface Σ, with respect to a basis $\{a_i\}_{i \le 2g}$ for $H_1(\Sigma)$ and generators $\{\alpha_i\}_{i \le 2g}$ for $\mathcal{A}(K)$ where $lk(\alpha_i, a_j) = \delta_{ij}$

[relations are rows]

Proof: Cut S^3 along Σ. $Y = S^3 \setminus \Sigma$ $X = S^3 \setminus K$

Glue along $\mathring{\Sigma}$



$\tilde{X}$ infinite cyclic cover of X.

Calculate $H_1(Y) \cong \mathbb{Z}^{2g}\langle \alpha_1, \ldots \alpha_{2g}\rangle$ [Alexander duality]

Calculate $H_1(\tilde{X})$ by Mayer-Vietoris.

using $U = \coprod Y_{even}$ $V = \coprod Y_{odd}$.

$$U \cap V = \coprod_{i=-\infty}^{\infty} \mathring{\Sigma}_i$$

$$H_1(U) \oplus H_1(V) \cong \bigoplus_{i=-\infty}^{\infty} H_1(Y_i) \quad \text{as abelian gp}$$

$$\cong H_1(Y_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \quad \text{as a } \mathbb{Z}[t,t^{-1}]\text{-module}$$

$$H_1(U \cap V) \cong \bigoplus_{i=-\infty}^{\infty} H_1(\Sigma_i) \cong H_1(\Sigma_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}] \quad \text{as a } \mathbb{Z}[t,t^{-1}]\text{-module}$$

M.V. $H_1(U \cap V) \xrightarrow{\iota_1 - \iota_2} H_1(U) \oplus H_1(V) \longrightarrow H_1(\tilde{X}) \longrightarrow H_0(U \cap V) \rightarrow H_0(U) \oplus H_0(V)$

$\qquad \cong \Lambda^{2g}\langle a_1, \ldots a_{2g}\rangle \qquad \cong \Lambda^{2g}\langle \alpha_1, \ldots \alpha_{2g}\rangle$

check that it's monic

where ι_i are inclusions. $\Lambda = \mathbb{Z}[t, t^{-1}]$.

So relations are the images of $\{a_i\}$ under $\iota_1 - \iota_2$

$$a_i \xmapsto{\iota_1 - \iota_2} a_i^- - a_i^+$$

Note: $a_i^- = \sum_{j=1}^{2g} lk(a_i^-, a_j)\alpha_j$ $\qquad -a_i^+ = -\sum_{j=1}^{2g} lk(a_i^+, a_j) t\alpha_j$

$= \sum_{j=1}^{2g} lk(a_i, a_j^+)\alpha_j$ $\qquad = \sum_{j=1}^{2g} -V_{ji} t\alpha_j$

$= \sum_{j=1}^{2g} V_{ij}\alpha_j$

So the relation given by a_i is $\sum_{j=1}^{2g} (V_{ij} - tV_{ji})\alpha_j$] ith row of $V - tV^T$.

$\Delta_K(t) := \det(V - tV^T)$ is the Alexander polynomial.

$\Delta_K(t^{-1}) = t^{-2g}\Delta_K(t) \Rightarrow \Delta_K(t) \doteq \Delta_K(t^{-1})$

$\Delta_K(1) = \pm 1$ since $\det(V - V^T) = \pm 1$ (int matrix)

tensor with $\mathbb{Q}$ $\longrightarrow$ $\mathbb{Q}[t, t^{-1}]$ is a PID $\Rightarrow$ A is product of cyclic modules

$\Rightarrow \Delta_K$ is the product of orders of each cyclic comp.

~~Seifert~~: More from a Seifert matrix:

$V + V^T$ is symmetric.

$\sigma(K) := \sigma(V + V^T)$ ordinary signature of K.

e.g. $\sigma($  $) = -2$ $\qquad \sigma($  $) = 2.$ (but have isom π_1)

More generally, let $w \in S^1 \subseteq \mathbb{C}$

$(1-w)V + (1-\bar{w})V^T$ is Hermitian $\qquad$ [A Herm := $A = \bar{A}^t$ ← conj. trans ⇒ has real eigens]

$\sigma_w(K) := \sigma((1-w)V + (1-\bar{w})V^T)$

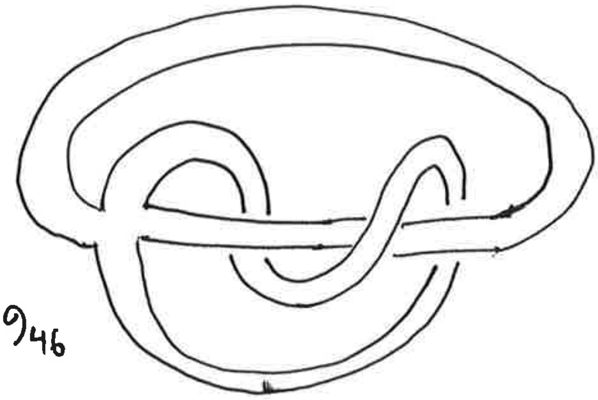
Levine-Tristram signature of K.

↳ piecewise constant function on S^1, jumps possible at roots of Alexander poly

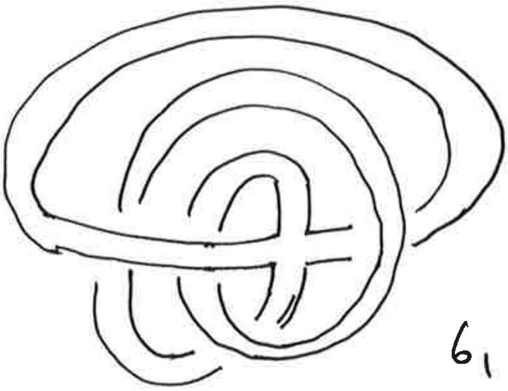
$\rightarrow \Delta_K(w) = \det\left(\frac{w}{1-w} M_w\right)$

Remarks:

1. Module has more info than poly:



9_{46}



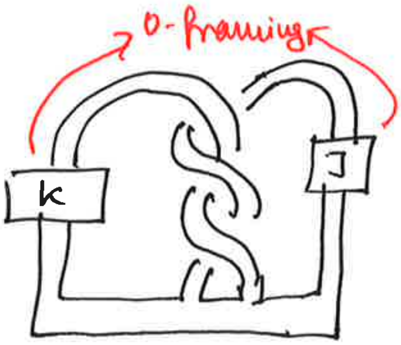
6_1

	9_{46}	6_1
V	$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$
$V - tV^T$	$\begin{bmatrix} 0 & 2-t \\ 1-t & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2-t \\ 1-t & 1-t \end{bmatrix}$
$\Delta_K(t)$	$2-5t+2t^2$	

However Alex modules not iso: $\mathcal{A}(6_1)$ cyclic
$\mathcal{A}(9_{46})$ has 2 gens.

2. $\exists$ knots with same Seifert matrix.

E.g.



Seifert matrix doesn't see K, J.
— No invariant from Seifert matrix can distinguish.

need more powerful invariants!