Max-Planck-Institut für Mathematik

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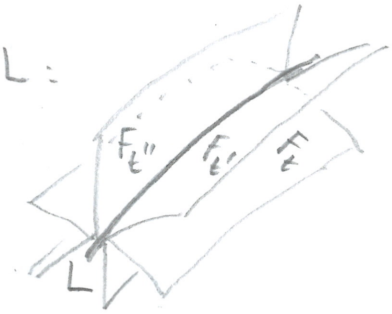
Fibred knots

Recall Thurston's virtual fibering conjecture, since ~5 years a theorem by Agol: „Up to taking a finite degree covering, (closed) hyperbolic 3-manifolds are surface bundles over S^1", an essential piece of the geometrization program. Today's goal is to take a closer look at knot complements fibering over S^1.

- Definition and first examples (links of singularities)
- Infinite cyclic covers and π_1
- The monodromy
- „Geometrization" of fibred knot complements
- Constructions of fibred knots and links
- (An application: Casson and Gordon's theorem on homotopically ribbon fibred knots)

Def.: A link $L \subset S^3$ is fibred if there exists a locally trivial fibre bundle $p: S^3 \setminus L \longrightarrow S^1$ such that $F_t := p^{-1}(t) \cup L$ are Seifert surfaces for L. $(t \in S^1)$

($\Rightarrow$ locally near L:



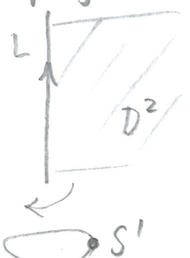
"pages of an open book")

First examples:

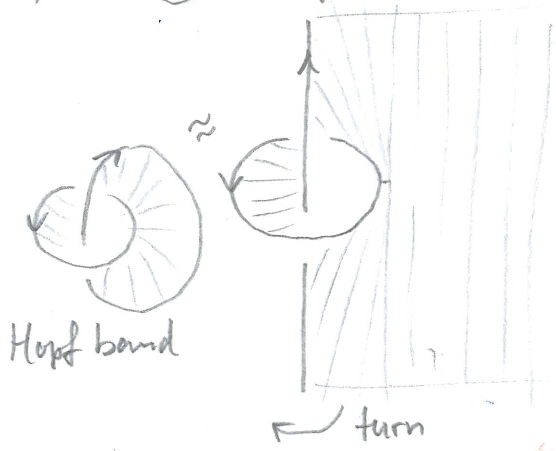
1) $L =$  unknot

$$S^3 \setminus L \cong S^1 \times \mathring{D}^2 \xrightarrow{p = pr_{S^1}} S^1$$

stereographic proj. to $\mathbb{R}^3$:



2) $L =$  Hopf link



3) <u>Links of singularities</u> Take $f \in \mathbb{C}[x,y] \setminus \mathbb{C}$ and consider

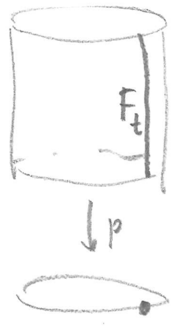
$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C} \\ \cup & & \cup \\ S_r^3 \setminus f^{-1}(0) & \xrightarrow{\frac{f}{|f|}} & S^1 \end{array}$$

<u>Milnor '62</u>: For $r>0$ small enough:
- $L_f := S_r^3 \cap f^{-1}(0)$ is a link which does not depend on r (small)
- $p := \frac{f}{|f|}$ makes L_f a fibred link.

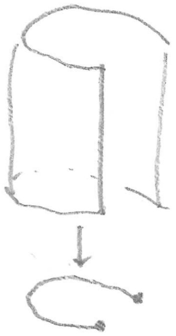
e.g.

f	L_f
x	
xy	
$x^n - y^m$	$T(n,m)$ torus link
$xy^2 - x^4$	
$\vdots$	

<u>Infinite cyclic covers</u> $K = \partial F$ fibred knot in S^3.



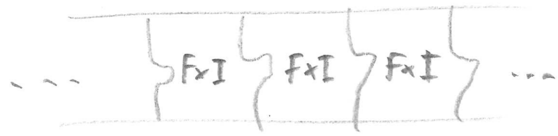
$S^3 \setminus K$ $\longrightarrow$ cut along F



trivial bundle (internal contractible)

$F \times [0,1]$
$\downarrow$
$[0,1]$

$F \times \mathbb{R}$



(cf. Aru's lecture on Alexander modules)

$$F \times \mathbb{R} \xrightarrow{\mathbb{Z}\text{-cover}} S^3 \setminus K \xrightarrow{p} S^1$$

$$\pi_1(F) \longrightarrow \pi_1(S^3 \setminus K) \xrightarrow{p_*} \mathbb{Z}$$
$$\mu_K \longmapsto 1$$

p_* is the abelianization!

$\Rightarrow \pi_1(F) = [G,G] \triangleleft G := \pi_1(S^3 \setminus K)$

In fact:

<u>Stallings' fibration theorem '62</u>: For M^3 irred., $1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$ exact, the following statements are equivalent:

1) K is finitely generated
2) $K = \pi_1(F)$, F a surface (hence K is free if $\partial F \neq \emptyset$!)
3) $F \longrightarrow M \xrightarrow{\exists p} S^1$ fibre bundle inducing $1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$.

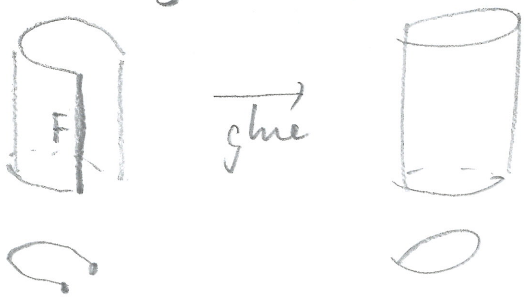
(cf. Lück's lecture)

Other important fact related to $\mathbb{Z}$-covering:

- fibre surfaces are minimal genus Seifert surfaces
- fibred knots in S^3 have unique (up to isotopy) Seifert surfaces of minimal genus.

Therefore: fibred knots $\overset{1:1}{\longleftrightarrow}$ fibre surfaces

Monodromy:



The required glueing map $\varphi: F \to F$ is called the monodromy (defined up to isotopy in F)
Really, $\varphi = \Phi_{2\pi}$, $\Phi_t: S^3 \to S^3$ flow of a vector field s.t. $\Phi_t(F_s) = F_{s+t}$, fixed on $L = \partial F \subset S^3$.

$$\Phi_\pi\big|_{F_0} : F_0 \to S^3 \setminus \mathring{F}_0, \quad \alpha \mapsto \alpha^+ = \Phi_\pi(\alpha)$$

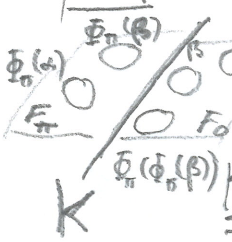
$\longleftarrow$ is the normal push-off used in the def. of Seifert form!

Lemma: $K \subset S^3$ fibred with monodromy $\varphi: F \to F$, $\varphi_*: H_1(F) \to H_1(F)$, then $\Delta_K(t) \doteq \chi_{\varphi_*}(t)$ (characteristic polynomial).

proof: $x, y \in H_1(F) \Rightarrow S_F(x,y) = lk(x, \Phi_\pi(y)) \overset{\Phi_t \text{ flow (isotopy)}}{=} lk(\Phi_\pi(x), \Phi_{2\pi}(y))$

$$= lk(\varphi_*(y), \Phi_\pi(x)) = S_F(\varphi_*(y), x)$$

$\uparrow$ lk symmetric



In terms of matrices:

$\left.\begin{array}{ll} S & \text{matrix of } S_F \\ M & \text{—"— } \varphi_* \end{array}\right\}$ w.r.t. a fixed basis of $H_1(F)$

$$x^T S y = (My)^T S x = x^T S^T M y \quad \forall x,y \Rightarrow S = S^T M.$$

$S - S^T = S^T M - S^T = S^T(M - \mathbb{1})$ non-singular ($S - S^T$ is the intersection form on F)

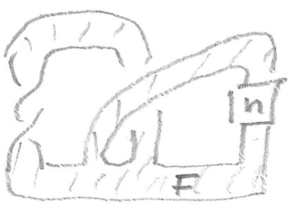
$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

$\Rightarrow \det S = \pm 1$ and $M = (S^T)^{-1} \cdot S$.

$$\Delta_K(t) := \det(tS^T - S) = \det(S^T) \cdot \det(t \cdot \mathbb{1} - M) = \chi_{\varphi_*}(t) \quad \blacksquare$$

<u>Corollary</u>: For $K \subset S^3$ fibred, Δ_K is monic and $2g(K) = \deg \Delta_K$, $\det S_F = \pm 1$.

<u>Example</u>: • Twist knots K_n

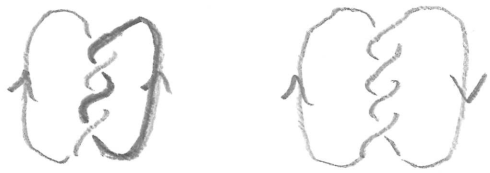


$S_F = \begin{pmatrix} 1 & * \\ 0 & n \end{pmatrix}$ $\left.\begin{array}{l} K_n \text{ fibred} \\ \text{genus } 1 \end{array}\right\} \Rightarrow n = \pm 1$

$K_1 = T(2,3) =$ trefoil is fibred

$K_{-1} =$  $=$ figure eight is fibred (see later)

• orientations are important!



fibred
$T(2,4)$

<u>not</u> fibred!
$(\exists F \cong S^1 \times I,\ S_F = (2))$

<u>Geometrization of fibred knot complements</u>: (Thurston)

$K \subset S^3$ fibred, $\varphi : F \to F$ monodromy

K hyperbolic $\Longleftrightarrow$ φ pseudo-Anosov

K torus knot ($S^3 \setminus K$ seifert fibred) $\Longleftrightarrow$ φ periodic ($\varphi^n \underset{\text{isot. free on } \partial F}{\sim} \mathrm{id}$ for some $n > 0$)

($\triangle$ $\Leftarrow$ only holds for <u>knots</u> rather than links, also $\varphi_* : H_1(F) \circlearrowleft$ periodic is <u>not</u> sufficient!)

K satellite (incompressible torus) $\Longleftrightarrow$ φ reducible

Remark: K link of singularity, φ monodromy $\Rightarrow$ φ is reducible and all components are periodic (no p.A. components)

$\leadsto$ There are many more fibred links than just the links of singularities:

Constructions of fibred links (Stallings '78, Gabai '83, Harer '82)

{fibred links in S^3}, resp. {fibre surfaces in S^3} are closed under the following operations:

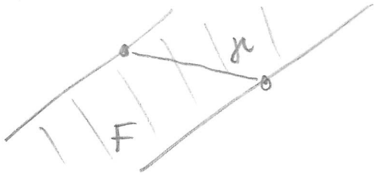
- connected sum
- nontrivial cabling
- plumbing, deplumbing, Murasugi sum
- Stallings twist

Hopf band

H

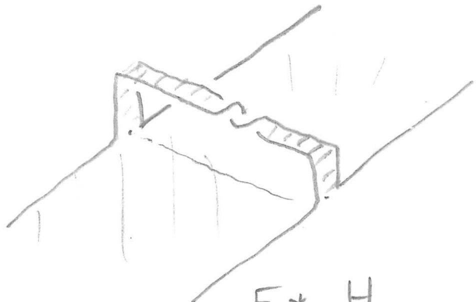
Hopf plumbing

deplumbing



any Seifert surface

$\gamma \subset F$ properly embedded interval



$F *_\gamma H$

(on abstract level: attach 1-handle to F)

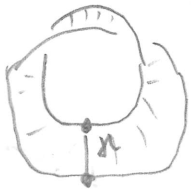
Thm (Stallings 78 $\Rightarrow$, Gabai 83 $\Leftarrow$)

F fibre surface $\Longleftrightarrow$ $F *_\gamma H$ fibre surface

Example: 1)

Seifert algo.

$=$



$*_\gamma H$

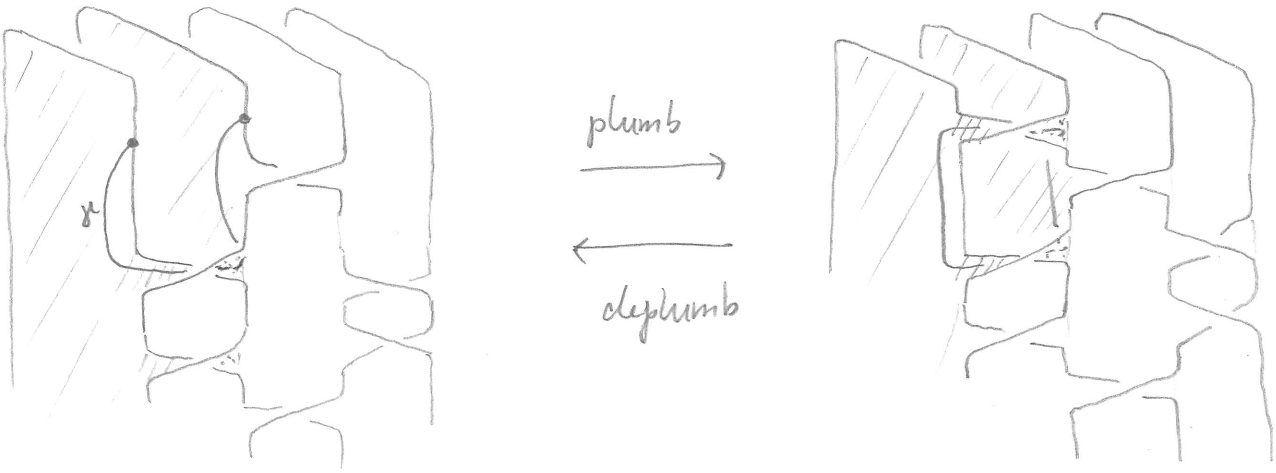
figure eight $= K_{-1}$ twist knot

plumbing of a positive and a negative Hopf band

(trefoil: two positive Hopf bands)

2) $L = \hat{\beta}$, β homogeneous braid (s.t. every generator appears at least once) $\Longrightarrow$ L is fibred

proof: Induction on word length of β and braid index.
Use the standard braided surface: Removing top-left-most crossing corresponds to deplumbing a Hopf band.
(picture)



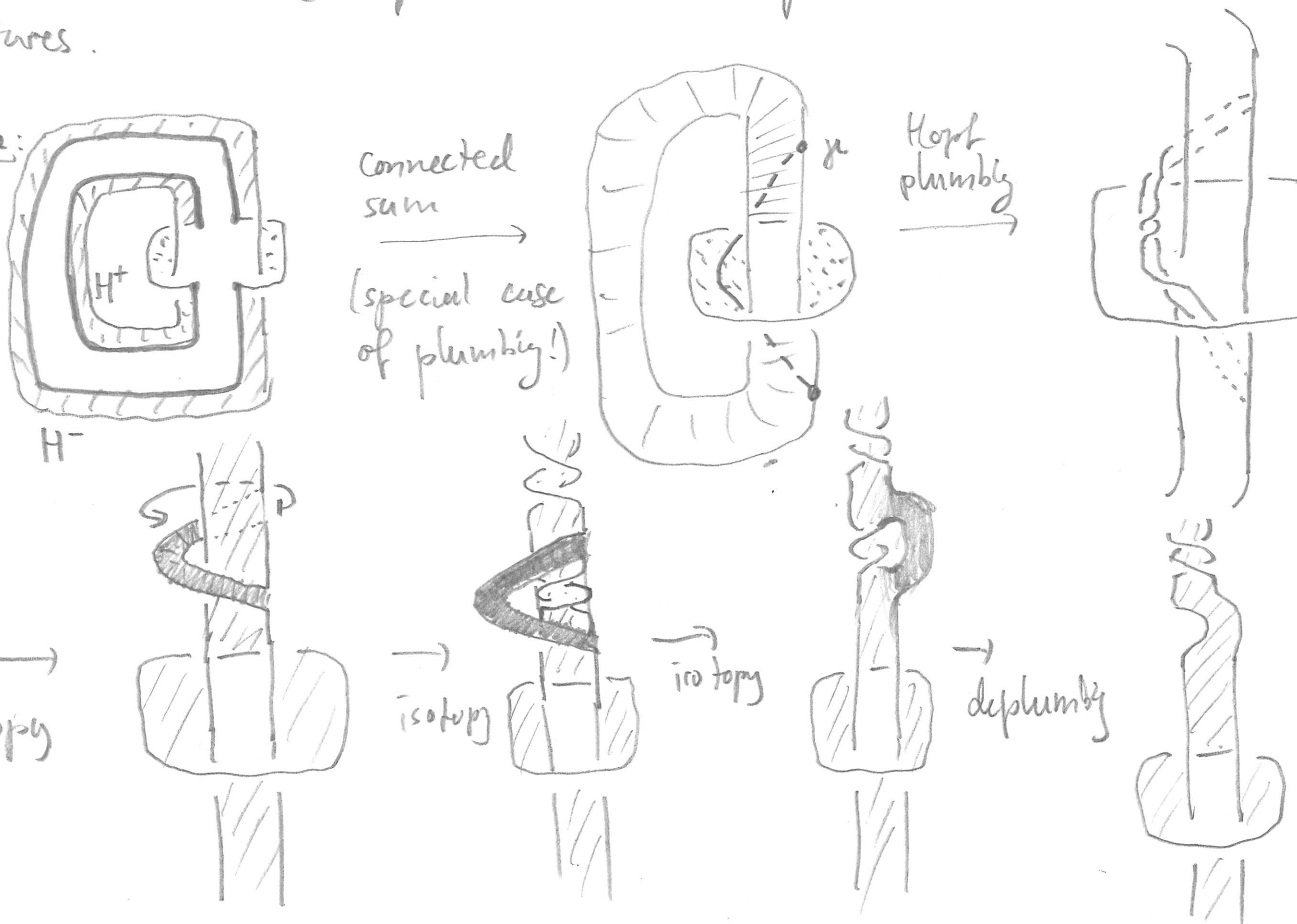
⋮

homogeneous braid: all crossings between two given adjacent strands are same sign.

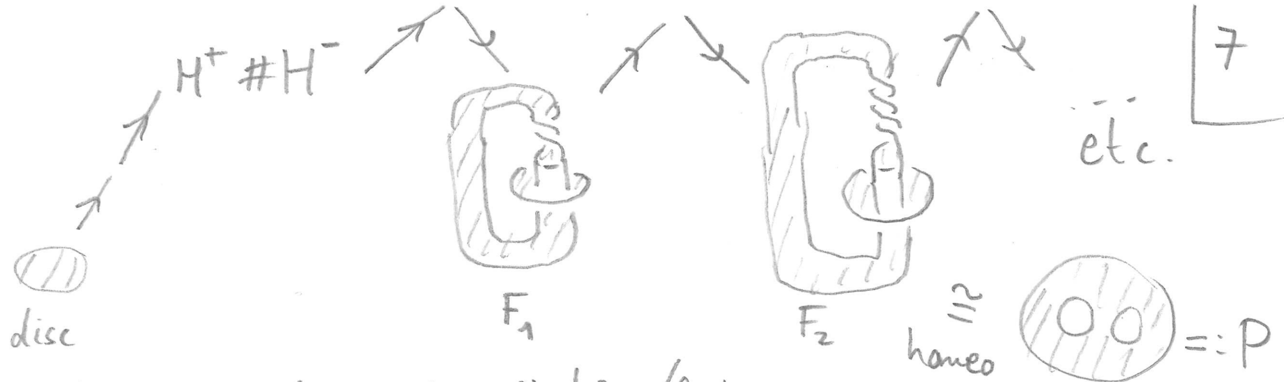
<u>Thm</u> (Giroux-Goodman '06): Let $F_1, F_2 \subset S^3$ be fibre surfaces. Then there exists a fibre surface F s.t.

proof uses Giroux' correspondence between open books and contact structures.

<u>Example</u>:



So,



all of these are fibred by Stallings/Gabai,
however, if F_n were a plumbing of Hopf bands (without deplumbing), we could deplumb the last band. This corresponds to cutting F_n along a properly embedded interval. (non-separating).
There are 3 candidates: 

. Each of these cuts yields an annulus with 0, $n-1$ or $n+1$ full twists, which has to be fibred. This is impossible unless $n \in \{0, 2, -2\}$.

$\Longrightarrow$ deplumbing is necessary in Giroux-Goodman's theorem!

Example:

is a fibre surface. I do not know how to construct it by Hopf plumbing and deplumbing. (Numbers indicate powers of Dehn twists in the monodromy)

Lemma: Let $K \subset S^3$ be a fibred knot (1 component), monodromy $\varphi : F \to F$.
Suppose $\varphi = \prod_{i=1}^{N} D_{c_i}^{\pm n_i}$, D_{c_i} Dehn twist on a curve $c_i \subset F$.

Then $N \geq b_1(F) = 2g(K)$. (sharp if F is a plumbing of Hopf bands)

proof: Consider $V := \operatorname{span}\{c_1, \dots, c_N\} < H_1(F)$ and let
$$V^\perp := \{ v \in H_1(F) \mid i(v, c_i) = 0 \ \forall i \}, \quad V \oplus V^\perp = H_1(F)$$
(i: intersection form on F.)

- If $N < b_1(F)$, then $\exists\, v \in V^\perp \setminus \{0\}$, and $\varphi_*(v) = v \Rightarrow \chi_{\varphi_*}(1) = \Delta_K(1) = 0$, contradiction! $\square$

Homotopy ribbon concordance of fibred knots

M^3 homology-S^3, $K \subset M^3$ knot.

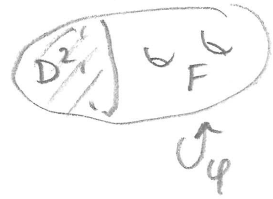
Def: K is <u>homotopically ribbon</u>, if $\exists V^4$ homology-B^4, $\partial V = M$ and a smooth disc $D^2 \subset V^4$, $\partial D = K$ such that
$$\pi_1(M \setminus K) \overset{i_*}{\twoheadrightarrow} \pi_1(V \setminus D) \text{ surjective.}$$

Thm (Casson-Gordon '83) A fibred knot $K \subset M$ is homotopically ribbon if and only if its monodromy $\varphi : F \to F$ extends over a handlebody H, i.e. $\partial H = \bar{F} := F \cup_{\partial F} D^2$, $\Phi : H \to H$ s.t. $\Phi|_F = \varphi$

"$\Leftarrow$" (idea)

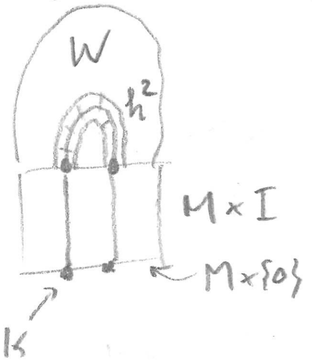
If φ extends, consider $W^4 := H \times I/_{\Phi}$ mapping torus,

$\partial W = \bar{F} \times I /_{\bar{\varphi}} \cong M_0(K)$ 0-surgery on K



$V := (M \times I \cup h^2) \cup_{M_0(K)} W$

2-handle attached along $K \times \{1\}$ with 0-framing



$$K \times \{0\} = \partial(\underbrace{K \times I \cup (\text{core of } h^2)}_{=: D^2})$$

- Show that V is a homology-B^4.
- Surjectivity on π_1 follows from $\pi_1(F) \twoheadrightarrow \pi_1(H)$