RECURRENT SEQUENCES OF POLYNOMIALS IN 3-DIMENSIONAL TOPOLOGY

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Abstract. A sequence of polynomials in several variables is recurrent if it satisfies a linear recursion with fixed polynomial coefficients. The Newton polytope of a recurrent sequences of polynomials is quasi-linear. Our goal is to give examples of recurrent sequences of polynomials that appear in 3-dimensional topology, classical and quantum.

Contents

1. Introduction

1.1. **Recurrent sequences of polynomials.** A sequence of polynomials in several variables is recurrent if it satisfies a linear recursion with fixed polynomial coefficients. In other words, if $R = \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$, then a sequence $Q_n \in R$ (for $n = 0, 1, 2, \dots$) is *recurrent* if there exists a natural number d and $c_k \in R$ for $k = 0, \dots, d$ with $c_d \neq 0$, such that for all $n \in \mathbb{N}$ we have:

$$\sum_{k=0}^{d} c_k Q_{n+k} = 0 \tag{1}$$

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The Newton polytope of a polynomial is the convex hull of the exponents of its nonzero monomials. In [Gar13] it was shown that the Newton polytope of a recurrent sequence of polynomials is quasi-linear. Quasi-linear polytopes appear in the theory of stable-commutator length studied by Calegari-Walker [CW13]. The number of lattice points of quasi-linear polytopes is a quasi-polynomial as shown by Chen-Li-Sam [CLS12] generalizing work of Ehrhart [Ehr62]. In the present paper we will not discuss the important notion of quasi-linearity. Instead, our goal is to show that examples of recurrent sequences of polynomials (in one or several variables), appear naturally in 3-dimensional topology, classical and quantum. In all our examples, the variable n comes from Dehn filling.

1.2. **Dehn filling.** The result of $-1/n$ *Dehn filling* along an unknot C which bounds a disk D replaces a string that meets D with n full twists, right-handed if $n > 0$ and left-handed if $n < 0$; see Figure 1 and [Kir78].

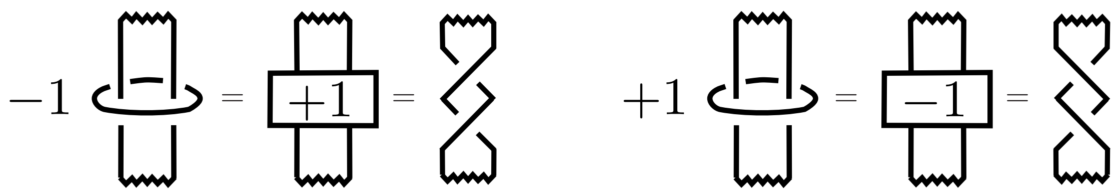


Figure 1. The effect of Dehn filling on a link.

Consider the 3-component seed link L of Figure 2, which contains a 2-component unlink $C = (C_1, C_2)$. For integers m_1, m_2, let $K(m_1, m_2)$ denote the knot obtained by $(-1/m_1, -1/m_2)$ filling on C. The 2-parameter family of (2-fusion) knots $K(m_1, m_2)$ was studied in [GvdV14] and [DG12]. It is easy to see that $K(m_1, m_2)$ is the closure of the 3-string braid β_{m_1,m_2}, where

$$\beta_{m_1,m_2} = ba^{2m_1+1}(ab)^{3m_2}$$

where $s_1 = a, s_2 = b$ are the standard generators of the braid group B_3 of 3-strands. There is a symmetry

$$K(m_1, m_2) = -K(1 - m_1, -1 - m_2) \tag{2}$$

where $-K$ denotes the mirror of K.

1.3. **The Alexander polynomial of a 2-parameter family of knots.** Let $\Delta_K(z) \in \mathbb{Z}[z^2]$ denote the Conway polynomial of a knot K [Kau87]. Note that $\Delta_K(t^{1/2} - t^{-1/2}) \in \mathbb{Z}[t^{\pm 1}]$ is the Alexander polynomial of a knot K. Let us abbreviate $\Delta(m_1, m_2) = \Delta_{K(m_1,m_2)}(z)$. We will explain the proof of the next proposition in Section 2.

Proposition 1.1. $\Delta(m_1, m_2)$ *satisfies the recursion relations*

$$\Delta(m_1 + 2, m_2) - (2 + z^2)\Delta(m_1 + 1, m_2) + \Delta(m_1, m_2) = 0 \tag{3a}$$

(3b)
$$\Delta(m_1, m_2+3)-(3+9z^2+6z^4+z^6)\Delta(m_1, m_2+2)+(3+9z^2+6z^4+z^6)\Delta(m_1, m_2+1)-\Delta(m_1, m_2) = 0$$

as well as

$$\Delta(m_1, m_2) - \Delta(1 - m_1, -1 - m_2) = 0 \tag{4}$$

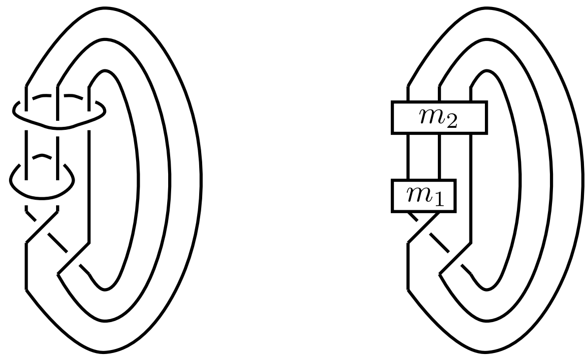


Figure 2. The seed link L (left) and the 2-fusion knot $K(m_1, m_2)$ (right).

with initial conditions

$$\begin{pmatrix} \Delta(0,0) & \Delta(0,1) \\ \Delta(1,0) & \Delta(1,1) \end{pmatrix} = \begin{pmatrix} 1 & z^6+5z^4+5z^2+1 \\ z^2+1 & z^8+7z^6+14z^4+8z^2+1 \end{pmatrix}. \tag{5}$$

1.4. **The Jones polynomial of a 2-parameter family of knots.** Let $J_K(q) \in \mathbb{Z}[q^{\pm 1}]$ denote the Jones polynomial of a knot K [Jon87]. Let us abbreviate $J(m_1, m_2) = J_{K(m_1,m_2)}(q)$. We will explain the proof of the next proposition in Section 2. Similar recursions hold for the colored Jones polynomial of $K(m_1, m_2)$ (for any fixed color) as well as for every quantum group invariant of $K(m_1, m_2)$.

Proposition 1.2. $J(m_1, m_2)$ *satisfies the recursion relations*

$$J(2+m_1, m_2) - (q+q^3)J(1+m_1, m_2) + q^4 J(m_1, m_2) = 0 \tag{6a}$$

$$J(m_1, 2+m_2) - (q^3+q^6)J(m_1, 1+m_2) + q^9 J(m_1, m_2) = 0 \tag{6b}$$

$$J(m_1, m_2)(q) - J(1-m_1, -1-m_2)(q^{-1}) = 0 \tag{6c}$$

with initial conditions

$$\begin{pmatrix} J(0,0) & J(0,1) \\ J(1,0) & J(1,1) \end{pmatrix} = \begin{pmatrix} 1 & -q^8+q^5+q^3 \\ -q^4+q^3+q & -q^{10}+q^6+q^4 \end{pmatrix}. \tag{7}$$

1.5. **The A-polynomial of some 1-parameter families of knots.** We now discuss recurrence relations of A-polynomials. The A-polynomial $A_M(m, l) \in \mathbb{Z}[m^{\pm 1}, l^{\pm 1}]$ of an oriented 3-manifold M with a torus boundary component equipped with a meridian and longitude was introduced in [CCG+94]. Roughly speaking, it parametrizes $\mathrm{SL}(2, \mathbb{C})$ representations of the fundamental group of M, restricted to the boundary torus, where a fixed meridian and longitude have eigenvalues m and l. An important example is the case when M is a hyperbolic manifold. In that case, there is a distinguished component of the character variety of $\mathrm{PSL}(2, \mathbb{C})$ representations which contains the diescrete faithful representation, [Thu77, NZ85]. This component lifts to several components of the $\mathrm{SL}(2, \mathbb{C})$ character variety (see [Cul86]) defined by the vanishing of a polynomial $A^{\mathrm{geom}}_M(m, l)$. In general, this polynomial has at most four factors of the form $p(\pm m, \pm l)$, discussed in detail in Champanerkar's thesis [Cha03, Sec.2.1.3]. Fixing an orientation on M, reduces the above factors to at most two of the form $p(\pm m, l)$. In the case of 2-bridge knots and $(-2, 3, 3+2n)$ pretzel knots, we further have $p(-m, l) = p(m, l)$.

Consider three seed links of Figure 3.

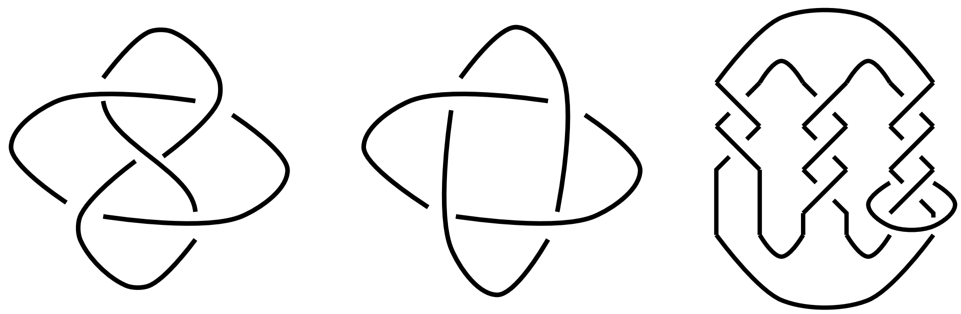


Figure 3. The Whitehead link (left), the twisted Whitehead link (middle) and the pretzel link (right).

Let K_n denote the *twist knot* obtained by $-1/n$ filling on a component of the Whitehead link. Hoste-Shanahan show that $A_{K_n}(m, l)$ is a recurrent sequence for $n > 0$ or $n < 0$; see [HS04, Thm.1]. Likewise, if K'_n denote the knot obtained by $-1/n$ surgery on a component of the twisted Whitehead link, Hoste-Shanahan shown that $A_{K'_n}(m, l)$ is recurrent when $n > 0$ or $n < 0$. Here, A_{K_n} and $A_{K'_n}$ denotes the A-polynomial of all non-abelian components, each with multiplicity one, and the recursion (one for $n > 0$ and another for $n < 0$) is of order 2.

Similarly, let $P_n = (-2, 3, 3+2n)$ denote the pretzel knot obtained by $-1/n$ surgery on the pretzel link. The author and Mattman show that A_{P_n} (i.e., all non-abelian components each with multiplicity one) is recurrent for $n > 0$ or $n < 0$; see [GM11, Thm.1.3]. The recursions are of order 4.

In Section 3 we will explain a general theorem regarding the behavior of the geometric component of the A-polynomial under filling.

2. The behavior of quantum invariants under filling

In this section, we explain how recurrent sequences of polynomials arise in Quantum Topology. Consider two endomorphisms A, B of a finite dimensional vector space V over the field $\mathbb{Q}(q)$. Let $\mathrm{tr}(D)$ denote the *trace* of an endomorphism D. The next lemma is an elementary application of the *Cayley-Hamilton* theorem.

Lemma 2.1. With the above assumptions, the sequence $\mathrm{tr}(AB^n) \in \mathbb{Q}(q)$ is recurrent. Moreover, a recursion depends only on the characteristic polynomial of B.

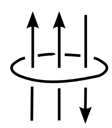
We now recall the relevant quantum invariants of links from [Jon87, Jan96, Tur88, Tur94]. Fix a simple Lie algebra $\mathfrak{g}$, a representation V of $\mathfrak{g}$, a knot K, and consider the *Quantum Group invariant* $Z^{\mathfrak{g}}_{V,K}(q) \in \mathbb{Z}[q^{\pm 1/d}]$, Here, $d \in \mathbb{N}$ depends on $\mathfrak{g}$, [Le00, Jan96] but not on V or K. In particular,

- When $\mathfrak{g} = \mathfrak{sl}_2$, and $V = \mathbb{C}^2$ is the defining representation, $Z^{\mathfrak{g}}_{V,K}(q)$ is the Jones polynomial of K.

- When $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $V = \mathbb{C}^2$, $Z^{\mathfrak{g}}_{V,K}(q)$ is the Alexander polynomial of K.

In what follows, we will not need the full formalism of quantum groups and ribbon caterogies. Instead, all we need to know is the fact that the quantum group invariant $Z^{\mathfrak{g}}_{V,K}(q)$ can be computed as the (quantum) trace of an operator associated to a tangle presentation of K.

Let L denote a 2-component link in S^3 with one unknotted component C_2, and let K_n denote the knot obtained by $-1/n$ filling on C_2. Since $S^3 \setminus C_2$ is a solid torus $S^1 \times D^2$ and L is a knot in $S^1 \times D^2$ it follows that L is the closure of an (r,r)-tangle α. Without loss of generality, we can assume that the writhe of α is zero. Choose an orientation on K. Let D denote a disk with boundary C_2. After isotopy, the intersection of L with D consists of r_+ positively oriented points and r_- negatively oriented ones, where $r_+ + r_- = r$. For example, for $(r_+, r_-) = (2,1)$ the intersection of L and D looks like



Let β_{r_+,r_-} denote the (r,r) tangle which is a 0-framed full twist on r strands. Kirby's calculus [Kir78] implies that the 0-framed knot K_n is obtained by the closure of the tangle $\alpha\beta^n_{r_+,r_-}$. If A and $B_{r,s}$ denote the endomorphism of $V^{\otimes r} \otimes (V^*)^{\otimes s}$ corresponding to α and $\beta_{r,s}$, then we have:

$$Z_{V,K_n}(q) = \mathrm{tr}(AB^n \mu^{\otimes r_+} \otimes \mu^{-\otimes r_-})$$

where $\mu = uv^{-1}$ and u is the Drinfeld element and v is the ribbon element of [Tur88, Sec.3]. The next theorem follows from the above discussion and Lemma 2.1.

Theorem 2.1. *Fix a simple Lie algebra $\mathfrak{g}$ and a representation V of $\mathfrak{g}$. With the above assumptions, the sequence $Z^{\mathfrak{g}}_{V,K_n}(q) \in \mathbb{Z}[q^{1/d}]$ is recurrent.*

Moreover, the minimal polynomial of β_{r_+,r_-}, gives a recurrence relation for Theorem 2.1. In practice, if we know the degree of the characteristic polynomial of β_{r_+,r_-}, and several values of the quantum group invariant, we can compute the recurrence of Theorem 2.1. This is how Equations (3a)-(3b) and (6a)-(6b) where obtained using $\beta_{2,0}$ and $\beta_{3,0}$. Equations (4) and (6c) follow from (2) and the fact that $Z^{\mathfrak{g}}_{V,-K}(q) = Z^{\mathfrak{g}}_{V,K}(q^{-1})$ for all $\mathfrak{g}$, V and K, where $-K$ denotes the mirror of K. Finally, the initial conditions (5) and (7) were obtained by a direct computation using the `KnotAtlas`; [BN05].

3. The behavior of the A-polynomial under filling

In this section we describe a general theorem regarding the behavior of the geometric component of the A-polynomial under filling.

Fix and oriented hyperbolic 3-manifold M which is the complement of a hyperbolic link with two components in a homology 3-sphere. Let (μ_1, λ_1) and (μ_2, λ_2) denote pairs of meridian-longitude curves along the two cusps C_1 and C_2 of M, and let M_n denote the result of $-1/n$ filling on C_2. Thurston proved that for all but finitely many n, M_n is hyperbolic; [Thu77, NZ85]. Let $A^{\mathrm{geom}}_n(m_1, l_1)$ denote the geometric component of the A-polynomial of M_n with the meridian-longitude pair (μ_1, λ_1) inherited from M.

Theorem 3.1. *With the above conventions, there exists a recurrent sequence $R_n(m_1, l_1) \in \mathbb{Z}[m_1, l_1]$ such that for all but finitely many integers n, $A_n^{\mathrm{geom}}(m_1, l_1)$ divides $R_n(m_1, l_1)$. In addition, a recursion for $R_n(m_1, l_1)$ can be computed explicitly via elimination given an ideal triangulation of M.*

Theorem 3.1 is general, but in favorable circumstances more is true. Namely, consider a family of knot complements K_n, obtained by $-1/n$ filling on a cusp of 2-component hyperbolic link L in S^3, with linking number f. Let $A_n^{\mathrm{geom}}(m, l)$ denote the geometric component of the A-polynomial of K_n with respect to the canonical meridian and longitude (μ, λ) of K_n.

Definition 3.1. We say that two component hyperbolic L link in S^3 with linking number f is *favorable* if $A_n^{\mathrm{geom}}(m, lm^{-f^2 n}) \in \mathbb{Q}[m^{\pm 1}, l^{\pm 1}]$ is recurrent, for all but finitely many values of n.

The shift $l \mapsto lm^{-f^2 n}$ accommodates the difference between the canonical meridian-longitude pair of K_n and the corresponding pair of the unfilled component of L.

In [Gar13] the author proved that the Newton polytope $N(R_n)$ of a recurrent sequence of polynomials $R_n \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ is *quasi-linear*, i.e., there exists a finite set J and periodic functions $s_{j,i} : \mathbb{N} \longrightarrow \mathbb{Q}^r$ for $j \in J$ and $i = 0, 1$ such that for all but finitely many n we have:

$$N(R_n) = \mathrm{conv}\{s_{j,1}(n)n + s_{j,0}(n) \,|\, j \in J\}$$

where $\mathrm{conv}(S)$ denotes the convex hull of a subset S of $\mathbb{R}^r$.

Corollary 3.2. If L is favorable, then $N(A_{K_n}^{\mathrm{geom}}(m, l))$ is quasi-quadratic.

Proof. If

$$N(A_{K_n}^{\mathrm{geom}}(m, lm^{-f^2 n})) = \mathrm{conv}\left\{ \begin{pmatrix} u_{j,1}(n)n + u_{j,0}(n) \\ v_{j,1}(n)n + v_{j,0}(n) \end{pmatrix} \,|\, j \in J \right\}$$

for periodic functions $u_{j,i}, v_{j,i} : \mathbb{N} \longrightarrow \mathbb{Q}$, then

$$N(A_{K_n}^{\mathrm{geom}}(m, l)) = \mathrm{conv}\left\{ \begin{pmatrix} u_{j,1}(n)n + u_{j,0}(n) \\ f^2 n^2 u_{j,1}(n) + (f^2 u_{j,0}(n) + v_{j,1}(n))n + v_{j,0}(n) \end{pmatrix} \,|\, j \in J \right\}$$

$\square$

Remark 3.3. The Whitehead link, the twisted Whitehead link and the pretzel link of Figure 3 are favorable; see [HS04, GM11]. The corresponding Newton polygons are indeed quadratic: generically hexagons the twist knots [HS04, Fig.3] and for the pretzel knots [GM11, Thm.1.3,Fig.2].

4. Proof of Theorem 3.1

Fix an oriented hyperbolic 3-manifold M with two cusps C_1 and C_2 and choice of meridian-longitude (μ_i, λ_i) on each cusp for $i = 1, 2$. Let K_n denote the result of $-1/n$ filling on C_2, a hyperbolic manifold for all but finitely many n; [Thu77, NZ85]. Let $A_n^{\mathrm{geom}}(m_1, l_1)$ denote the A-polynomial of K_n with the conventions of Section 1.5.

We consider two cases: M has strongly geometrically isolated cusps, or not. For a definition of *strong geometric isolation*, see [NR93] and also [Cal01, CW13].

When M is strongly geometrically isolated, Dehn filling on one cusp does not change the shape of the other. This implies that $A_n^{\mathrm{geom}}(m_1, l_1)$ is independent of n (for all but finitely many n) and certainly recurrent.

If M does not have strongly geometrically isolated cusps, consider the geometric component of the $\mathrm{PSL}(2,\mathbb{C})$ character variety of M, which lifts to a union X' of finitely many components of $\mathrm{SL}(2,\mathbb{C})$ character variety of M. Consider a finite covering X'' of X' such that the eigenvalues of the meridians and longitudes are rational functions on X. The *hyperbolic Dehn filling* theorem of Thurston implies that X is a complex affine surface; see [Thu77] and also [NZ85]. We will work with each component X of X'' separately. So, the field F of rational functions on X has transendence degree 2. Now X has four nonconstant rational functions: the eigenvalues of the meridians m_1, m_2 and the longitudes l_1, l_2 around each cusp. So, each triple $\{m_1, l_1, m_2\}$ and $\{m_1, l_1, l_2\}$ of elements of F is polynomially dependent i.e., satisfies a polynomial equation

$$P(m_1, l_1, m_2) = 0 \qquad Q(m_1, l_1, l_2) = 0 \tag{8}$$

where $P(m_1, l_1, m_2) \in \mathbb{Q}(m_1, l_1)[m_2]$ and $Q(m_1, l_1, l_2) \in \mathbb{Q}(m_1, l_1)[l_2]$ are polynomials of strictly positive (by hypothesis) degrees d_P and d_Q with respect to m_2 and l_2. The union X_n of the geometric components of the $\mathrm{SL}(2,\mathbb{C})$ character variety of K_n is the intersection of X with the Dehn-filling equation $m_2 l_2^{-n} = 1$ [Thu77]. This is a surprising fact since Dehn filling imposes an $\mathrm{SL}(2,\mathbb{C})$ matrix condition which a priori involves 3 polynomial equations and not one as stated above. The Dehn filling equation $m_2 l_2^{-n} = 1$ is necessary, but not (in general) sufficient to cut out nongeometric components of the $\mathrm{SL}(2,\mathbb{C})$ character variety of K_n from those of the character variety of M.

So, on X_n we have $P(m_1, l_1, l_2^n) = 0$. Let $p(m_1, l_1)$ and $q(m_1, l_1)$ denote the coefficient of $m_2^{d_P}$ and $l_2^{d_Q}$ in $P(m_1, l_1, m_2)$ and $Q(m_1, l_1, l_2)$ respectively. Let $R_n(m_1, l_1) \in \mathbb{Q}(m_1, l_1)$ denote the *resultant* of $P(m_1, l_1, l_2^n)$ and $Q(m_1, l_1, l_2)$ (both are elements of $\mathbb{Q}(m_1, l_1)[l_2]$) with respect to l_2; see [Lan02, Sec.IV.8]. It follows that

$$R_n(m_1, l_1) = p(m_1, l_1)^{d_Q} \prod_{l_2 : Q(m_1, l_1, l_2) = 0} P(m_1, l_1, l_2^n) \in \mathbb{Q}(m_1, l_1)$$

Since $R_n(m_1, l_1)$ is a $\mathbb{Q}(m_1, l_1)$-linear combination of $P(m_1, l_1, l_2^n)$ and $Q(m_1, l_1, l_2)$ (see [Lan02, Sec.IV.8]) and since $P(m_1, l_1, l_2^n)$ and $Q(m_1, l_1, l_2)$ vanish on the curve X_n, it follows that $A_n^{\mathrm{geom}}(m_1, l_1)$ divides the numerator of $R_n(m_1, l_1)$. Moreover, by the above equation, $R_n(m_1, l_1)$ is a $\mathbb{Q}(m_1, l_1)$-linear combination of the n-th powers of a finite set of elements l_2 algebraic over $\mathbb{Q}(m_1, l_1)$. It follows that $R_n(m_1, l_1)$ satisfies a linear recursion with constant coefficients in $\mathbb{Q}[m_1, l_1]$. Lemma 4.1 below implies that there exists $r(m_1, l_1), s(m_1, l_1) \in \mathbb{Q}[m_1, l_1]$ such that $rs^n R_n \in \mathbb{Q}[m_1, l_1]$ is recurrent. Since $R_n = (rs^n R_n)/(rs^n)$, it follows that the numerator of R_n is a divisor of $rs^n R_n \in \mathbb{Q}[m_1, l_1]$, a recurrent sequence. And A_n^{geom} divides the numerator of R_n, hence divides $rs^n R_n$. Theorem 3.1 follows. $\square$

Lemma 4.1. If $R_n \in \mathbb{Q}(x)$ is recurrent, $x = (x_1, \dots, x_r)$ then there exist $r, s \in \mathbb{Q}[x]$ such that $sr^n R_n \in \mathbb{Q}[x]$ is recurrent.

Proof. R_n satisfies a linear recursion

$$\sum_{k=0}^{d} c_k R_{n+k} = 0$$

for some $d \in \mathbb{N}$ and $c_k \in \mathbb{Q}[x]$ with $c_d \neq 0$. Let $r = c_d$ and define $\widetilde{R}_n = r^n R_n$. It follows that $\widetilde{R}_n$ satisfies the linear recursion

$$\sum_{k=0}^{d} c_k r^{d-1-k} \widetilde{R}_{n+k} = 0$$

The above recursion is monic (since $c_d r = 1$) and has coefficients in $\mathbb{Q}[x]$. Hence $\widetilde{R}_n \in \mathbb{Q}[x][\widetilde{R}_0, \dots, \widetilde{R}_{d-1}]$. Choose $s \in \mathbb{Q}[x]$ such that $s\widetilde{R}_k \in \mathbb{Q}[x]$ for $k = 0, \dots, d-1$. Then $s\widetilde{R}_n \in \mathbb{Q}[x]$ is recurrent. □

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