The quantum MacMahon Master Theorem

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We state and prove a quantum generalization of MacMahon's celebrated Master Theorem and relate it to a quantum generalization of the boson–fermion correspondence of physics.

In this article we state and prove a quantum generalization of MacMahon's celebrated Master Theorem conjectured by S.G. and T.T.Q.L. Our result was motivated by quantum topology. In addition to its potential importance in knot theory and quantum topology (explained in brief in ref. 4), this article answers George Andrews's long-standing open problem (1) of finding a natural q-analog of MacMahon's Master Theorem.

MacMahon's Master Theorem

Let us recall the original form of MacMahon's Master Theorem and some of its modern interpretations.

Consider a square matrix $A = (a_{ij})$ of size r with entries in some commutative ring. For $1 \le i \le r$, let $X_i := \Sigma_{j=1}^r a_{ij}x_j$, (where x_i are commuting variables) and for any vector $(m_1, \dots, m_r)$ of nonnegative integers let $G(m_1, \dots, m_r)$ be the coefficient of $x_1^{m_1}x_2^{m_r} \dots x_r^{m_r}$ in $\Pi_{i=1}^r X_i^{m_i}$. MacMahon's Master Theorem is the following identity (see ref. 2):

$$\sum_{m_1,m_2,\dots,m_r=0}^{\infty} G(m_1, \dots, m_r) = 1/\det(I - A). \qquad \textbf{[1]}$$

There are several equivalent reformulations of MacMahon's Master Theorem (see, for example, ref. 3 and references therein). Let us mention one of these studies, which is of importance to physics.

Given a matrix $A = (a_{ij})$ of size r with commuting entries that lie in a ring $\mathcal{R}$, and a nonnegative integer n, we can consider its symmetric and exterior powers $S^n(A)$ and $\Lambda^n(A)$, and their traces tr $S^n(A)$, and tr $\Lambda^n(A)$, respectively. Because

$$\mathrm{tr}S^n(A) = \sum_{m_1+\dots m_r=n} G(m_1, \dots, m_r)$$

$$\det(I - tA) = \sum_{n=0}^{\infty} (-1)^n \mathrm{tr}\Lambda^n(A)t^n,$$

the following identity

$$\frac{1}{\Sigma_{n=0}^{\infty}(-1)^n \mathrm{tr}\Lambda^n(A)t^n} = \sum_{n=0}^{\infty} \mathrm{tr}S^n(A)t^n \qquad \textbf{[2]}$$

in $\mathcal{R}[[t]]$ is equivalent to Eq. **1**. In physics, Eq. **2** is called the boson–fermion correspondence, where bosons (fermions) are commuting (skew-commuting) particles corresponding to symmetric (exterior) powers.

Quantum Algebra, Right-Quantum Matrices, and Quantum Determinants

In r-dimensional quantum algebra we have r indeterminate variables x_i $(1 \le i \le r)$, satisfying the commutation relations $x_jx_i = qx_ix_j$ for all $1 \le i < j \le r$. We also consider matrices $A = (a_{ij})$ of r^2 indeterminates a_{ij}, $1 \le i, j \le r$, which commute with the x_i and such that for any 2-by-2 minor of (a_{ij}), consisting of rows i and i', and columns j and j' (where $1 \le i < i' \le r$, and $1 \le j < j' \le r$), writing $a := a_{ij}$, $b := a_{ij'}$, $c := a_{i'j}$, $d := a_{i'j'}$, we have the *commutation relations*:

$$ca = qac, \quad (q\text{-commutation of the entries in a column}) \qquad \textbf{[3]}$$

$$db = qbd, \quad (q\text{-commutation of the entries in a column}) \qquad \textbf{[4]}$$

$$ad = da + q^{-1}cb - qbc \quad (\text{cross commutation relation}). \qquad \textbf{[5]}$$

We will call such matrices A *right-quantum matrices*.

The *quantum determinant*, (first introduced in ref. 3) of any (not necessarily right-quantum) r by r matrix $B = (b_{ij})$ may be defined by

$$\det_q(B) := \sum_{\pi \in S_r} (-q)^{-\mathrm{inv}(\pi)} b_{\pi_1 1} b_{\pi_2 2} \cdots b_{\pi_r r},$$

where the sum ranges over the set of permutations, S_r, of $\{1, \dots, r\}$, and for any of its members, π, inv(π) denotes the number of pairs $1 \le i < j \le r$ for which $\pi_i > \pi_j$.

A *q*-Version of MacMahon's Master Theorem

We are now ready to state our quantum version of MacMahon's Master Theorem.

Theorem 1 (Quantum MacMahon Master Theorem). *Fix a right-quantum matrix A of size r. For $1 \le i \le r$, let $X_i := \Sigma_{j=1}^r a_{ij}x_j$, and for any vector $(m_1, \dots, m_r)$ of nonnegative integers let $G(m_1, \dots, m_r)$ be the coefficient of $x_1^{m_1}x_2^{m_r} \dots x_r^{m_r}$ in $\Pi_{i=1}^r X_i^{m_i}$. Let*

$$\mathrm{Ferm}(A) = \sum_{J \subset \{1,\dots,r\}} (-1)^{|J|} \det_q(A_J)$$

where the summation is over the set of all *subsets J of $\{1, \dots, r\}$, and A_J is the J by J submatrix of A, and*

$$\mathrm{Bos}(A) = \sum_{m_1,\dots,m_r=0}^{\infty} G(m_1, \dots, m_r).$$

Then

$$\mathrm{Bos}(A) = 1/\mathrm{Ferm}(A).$$

When we specialize to $q = 1$, *Theorem 1* recovers Eq. **2**, which explains why our result is a q-version of the MacMahon Master Theorem. For a motivation of *Theorem 1*, see *Some Remarks on the Boson–Fermion Correspondence.*

The above result is not only interesting from the combinatorial point of view, but it is also a key ingredient in a finite noncom-

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mutative formula for the colored Jones function of a knot (see ref. 4).

Computer Code

The results of the article have been verified by computer code (written by D.Z.). Maple programs QuantumMACMAHON and qMM are available at www.math.rutgers.edu/~zeilberg/programs.html. QuantumMACMAHON rigorously proves *Theorem 1* for any fixed r.

Proof

Some Lemmas on Operators. The proof will make crucial use of a *calculus of difference operators* developed by D.Z. (5). This calculus of difference operators predates the more advanced *calculus of holonomic functions* developed by D.Z. (6).

Difference operators act on *discrete functions* F, that is functions whose domain is $\mathbb{N}^r$. For example, consider the shift-operators M_i and the multiplication operator Q_i, which act on a discrete function $F(m_1, \ldots, m_r)$ by

$$(M_iF)(m_1, \ldots, m_r) := F(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_r)$$

$$(Q_iF)(m_1, \ldots, m_r) := q^{m_i}F(m_1, \ldots, m_r).$$

It is easily seen that

$$M_iQ_i = qQ_iM_i.$$

Abbreviating Q_i as q^{m_i}, we obtain that

$$M_iq^{m_i} = q^{m_i+1}M_i \quad M_iq^{m_j} = q^{m_j}M_i \quad \text{for } i \neq j. \qquad [6]$$

Another example is the operator $\hat{x}_i$, which left-multiplies F by x_i. Notice that $\hat{x}_j\hat{x}_i = q\hat{x}_i\hat{x}_j$ for $j > i$. In the proof below, we will denote $\hat{x}_i$ as x_i. In that case, the identity $x_jx_i = qx_ix_j$ for $j > i$ holds in the quantum algebra in the algebra of operators.

Before embarking on the proof, we need the following readily and verified lemmas.

Lemma 1. (*Commuting X_i with X_j*): *For $1 \leq i < j \leq r$, $X_jX_i = qX_iX_j$.*

Lemma 2. (*Commuting x_i with X_j*): *For each of the a_{ij}, define the operator Q_{ij} acting on expressions P* involving *a_{ij} by $Q_{ij}P(a_{ij}) := P\ (qa_{ij})$. Then, for any $1 \leq i, j \leq r$, and integer m_i and any expression F*

$$x_i^{-m_i}X_jF = [(Q_{j1}^{-1}Q_{j2}^{-1}\cdots Q_{j,i-1}^{-1}Q_{j,i+1}\cdots Q_{jr})^{m_i}X_j]x_i^{-m_i}F.$$

Lemma 3. (*Column expansion with respect to the last column*): *Given an r by r matrix (a_{ij}) (not necessarily quantum), let A_i be the minor of the entry a_{ir}, i.e., the $r - 1$ by $r - 1$ matrix obtained by deleting the i^{th} row and r^{th} column. Then*

$$\det_q(A) = \sum_{i=1}^{r} (-q)^{i-r}(\det_qA_i)a_{ir}.$$

Lemma 4. *If A is a matrix that satisfies Eq.* **5** *and A' denotes a matrix obtained by interchanging the i and j columns of A, then $\det_q(A') = (-q)^{-\mathrm{inv}(ij)}\det_q(A)$.*

Proof. Suppose first that we interchange two adjacent columns i and $j := i + 1$. Consider the involution of S_r that sends a permutation π to $\pi' = \pi(ij)$. Given $\pi \in S_r$, let $(A; \pi) = (-1)^{-\mathrm{inv}(\pi)}a_{\pi_1 1} \ldots a_{\pi_r r}$ denote the contribution of π in $\det_q(A)$. Then, $\det_q(A) = \Sigma_\pi\ (A; \pi)$. Eq. **5** implies that

$$(A; \pi) + (A; \pi') = (-q)((A'; \pi) + (A'; \pi')).$$

Summing over all permutations proves the result when $j = i + 1$.

Observe that when $j = i + 1$, the matrix A' is no longer right-quantum since it does not satisfy Eq. **5**. However, the proof used only the fact that Eq. **5** holds for the i and $i + 1$ columns of A.

Thus, the proof can be iterated $\mathrm{inv}(ij)$ times to commute the i and $j > i$ columns of A. The result follows.

Lemma 5. (*Equal columns imply that* $\det_q$ *vanishes*): *Let A be a right-quantum matrix. In the notation of Lemma 3, for all $j \neq r$,*

$$\sum_{i=1}^{r} (-q)^{i-r}(\det_qA_i)a_{ij} = 0.$$

Proof. If $j = r - 1$, it is easy to see that q-commutation along the entries in every column of A imply that the sum vanishes.

If $j < r - 1$, use *Lemma 4* to reduce it to the case of $j = r - 1$.

Remark. One can give an alternative proof of *Lemmas 4* and *5* from the trivial 2-by-2 case and, by induction using the q-Laplace expansion of a q-determinant that is completely analogous to the classical case.

Proof of *Theorem 1*. The proof is a quantum-adaptation of the "operator-elimination" proof of MacMahon's Master Theorem given in ref. 5. Fix a right-quantum matrix A.

Observe that $G(m_1, \ldots, m_r)$ is the coefficient of $x_1^0 \ldots x_r^0$ in

$$H(m_1, \ldots, m_r; x_1, \ldots, x_r) := x_r^{-m_r}\cdots x_2^{-m_2}x_1^{-m_1} \prod_{i=1}^{r} X_i^{m_i}.$$

We will think of H as a *discrete function* that is as a function of $(m_1, \ldots, m_r) \in \mathbb{N}^r$. H takes values in the ring of noncommutative Laurrent polynomials in the x_is, with coefficients in the ring generated by the entries of A, modulo the ideal given by Eqs. **3–5**.

Let us see how the shift operators M_i acts on H. By definition,

$$M_iH(m_1, \ldots, m_r; x_1, \ldots, x_r) =$$

$$x_r^{-m_r}\cdots x_{i+1}^{-m_{i+1}}x_i^{-m_i-1}x_{i-1}^{-m_{i-1}}\cdots x_1^{-m_1}X_1^{m_1}\cdots X_{i-1}^{m_{i-1}}X_i^{m_i+1}X_{i+1}^{m_{i+1}}\cdots X_r^{m_r}.$$

By moving x_i^{-1} to the front and X_i in front of $X_1^{m_1}$, and using *Lemma 1* and $x_jx_i = qx_ix_j$, we have

$$M_iH(m_1, \ldots, m_r; x_1, \ldots, x_r) =$$

$$q^{m_r+m_{r-1}+\cdots+m_{i+1}-m_1-m_2-\cdots-m_{i-1}}x_i^{-1}[x_r^{-m_r}\cdots x_1^{-m_1}X_i]X_1^{m_1}\cdots X_r^{m_r}.$$

By moving X_i next to x_i^{-1} and using *Lemma 2* this equals to

$$q^{m_r+m_{r-1}+\cdots+m_{i+1}-m_1-m_2-\cdots-m_{i-1}}x_i^{-1}$$

$$\cdot[(Q_{i2}\cdots Q_{ir})^{m_1}(Q_{i1}^{-1}Q_{i3}\cdots Q_{ir})^{m_2}$$

$$\cdot(Q_{i1}^{-1}Q_{i2}^{-1}Q_{i4}\cdots Q_{ir})^{m_3}\cdots(Q_{i1}^{-1}Q_{i2}^{-1}\cdots Q_{i,r-1}^{-1})^{m_r}X_i]$$

$$\cdot x_r^{-m_r}\cdots x_1^{-m_1}X_1^{m_1}\ldots X_r^{m_r},$$

which is equal to

$$q^{m_r+m_{r-1}+\cdots+m_{i+1}-m_1-m_2-\cdots-m_{i-1}}x_i^{-1}$$

$$\cdot(q^{-m_2-m_3-\cdots-m_r}a_{i1}x_1 + q^{m_1-m_3-\cdots-m_r}a_{i2}x_2 + \cdots$$

$$+ q^{m_1+m_2+\cdots+m_{r-1}}a_{ir}x_r)H(m_1, \ldots, m_r; x_1, \ldots, x_r).$$

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Multiplying out and rearranging, we get that the discrete function $H(m_1, \ldots, m_r; x_1, \ldots, x_r)$ is annihilated by the r operators ($i = 1, 2, \ldots, r$):

$$\mathcal{P}_i := \sum_{j=1}^{i-1} -q^{-m_j-2m_{j+1}-\cdots-2m_{i-1}-m_i} a_{ij} x_j + (M_i - a_{ii}) x_i$$

$$+ \sum_{j=i+1}^{r} -q^{m_i+2m_{i+1}+\cdots+2m_{j-1}+m_j} a_{ij} x_j.$$

Now comes a nice surprise. Let us define b_{ij} to be the coefficient of x_j in $\mathcal{P}_i$. For example, for $r = 3$ we have

$$B = \begin{pmatrix} M_1 - a_{11} & -q^{m_1+m_2} a_{12} & -q^{m_1+2m_2+m_3} a_{13} \\ -q^{-m_1-m_2} a_{21} & M_2 - a_{22} & -q^{m_2+m_3} a_{23} \\ -q^{-m_1-2m_2-m_3} a_{31} & -q^{-m_2-m_3} a_{32} & M_3 - a_{33} \end{pmatrix}.$$

Lemma 7. *B is a right-quantum matrix.*

Proof. It is easy to see that the entries in each column of B q-commute. To prove Eq. **5**, consider the following three cases for a 2-by-2 submatrix C of B: C contains two (resp. one, resp. zero) diagonal entries of B, and prove it case by case, using the fact that the operators M_i and q^{m_j} commute with the a_{ij} and satisfy the commutation relations of Eq. **6**.

Now we eliminate $x_1, x_2, \ldots, x_{r-1}$ by left-multiplying $\mathcal{P}_i$ by the minor of b_{ir} in $B = (b_{ij}) \times (-q)^{i-r}$, for each $i = 1, 2, \ldots, r$, and adding them all up. Since B is right-quantum (by *Lemma 7*), *Lemma 5* implies that the coefficients of $x_1, \ldots, x_{r-1}$ all vanish, and $\det_q(B) x_r H = 0$. After left-multiplying by x_r^{-1}, which commutes with the entries in B, we obtain that

$$\det_q(B) H(m_1, \ldots, m_r; x_1, \ldots, x_r) = 0.$$

Since the entries of B do not contain any x_i, it follows that $\det_q(B)$ annihilates every coefficient of H, in particular its constant term. Taking the constant term yields

$$\det_q(B) G(m_1, \ldots, m_r) = 0.$$

Here comes the next surprise.

Lemma 8. (i) *We have*

$$\det_q(B) = \sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \det_q(A_J) M_{\bar{J}},$$

where $\bar{J} = \{1, \ldots, r\} - J$ and $M_J = \prod_{j \in J} M_j$.

(ii) *In particular,*

$$\det_q(B)|_{M_1 = \cdots = M_r = 1} = \mathrm{Ferm}(A).$$

Proof. Let us expand $\det_q(B)$ as a sum over permutations $\pi \in S_r$. We have

$$\det_q(B) = \sum_{\pi \in S_r} (-q)^{-\mathrm{inv}(\pi)} b_{\pi_1 1} b_{\pi_2 2} \cdots b_{\pi_r r}$$

$$= \sum_{\pi \in S_r} \prod_{i=1}^{r} (-q)^{-\mathrm{inv}(\pi, i)} b_{\pi_i i},$$

where $\mathrm{inv}(\pi, i)$ is the number of $j > i$ such that $\pi_i > \pi_j$. Now, $b_{ij} = \delta_{ij} M_i - q_{ij} a_{ij}$, where q_{ij} is a monomial in the variables q^{m_k}, and $\prod_i q_{\pi_i i} = 1$. Moreover, if $\pi_i = i$, then for each j with $i < j \neq \pi_j$, the exponent of q^{m_i} in q_{ij} is 2 if $\pi_j < i$ and 0 if $\pi_j > i$.

Since $\prod_i q_{\pi_i i} = 1$, we can move the monomials q_{ij} in the left of $\prod_i (-q)^{-\mathrm{inv}(\pi,i)} b_{\pi_i i}$ and then cancel them. The monomials commute with all entries of the matrix b_{ij}, except with the diagonal ones. Commuting q^{2m_i} with $b_{ii} = \delta_{\pi_i i} M_{ii} - q_{\pi_i i} a_{\pi_i i}$ gives $b_{ii} q^{2m_i} = q^{2m_i}(\delta_{\pi_i i} q^2 M_{ii} - q_{\pi_i i} a_{\pi_i i})$. In other words, commuting replaces M_i by $q^2 M_i$. Thus, we have:

$$\det_q(B) = \sum_{\pi \in S_r} \prod_{i=1}^{r} (-q)^{-\mathrm{inv}(\pi,i)} (\delta_{\pi_i i} q^{2\mathrm{inv}(\pi,i)} M_i - a_{\pi_i i})$$

$$= \sum_{\pi \in S_r} \sum_{J \subset \{1, \ldots, r\}} \prod_{i \in J} (-q)^{-\mathrm{inv}(\pi,i)} \delta_{\pi_i i} q^{2\mathrm{inv}(\pi,i)} M_i$$

$$\cdot \prod_{i \notin J} (-q)^{-\mathrm{inv}(\pi,i)} (-a_{\pi_i i}).$$

Now, rearrange the summation. Observe that every permutation π of $\{1, \ldots, r\}$ gives rise to a permutation π' on the set $\{1, \ldots, r\} - \mathrm{Fix}(\pi)$, where $\mathrm{Fix}(\pi)$ is the fixed point set of π. Moreover, $\mathrm{inv}(\pi', i) = \mathrm{inv}(\pi, i) - |\{j \in J : j > i\}|$. Using this, part i follows. Part ii follows from part i and the definition of $\mathrm{Ferm}(A)$.

Hence,

$$\sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \det_q(A_J) M_{\bar{J}} G(m_1, \ldots, m_r) = 0.$$

Summing over $\mathbb{N}^r$, we get

$$\sum_{m_1, \ldots, m_r = 0}^{\infty} \sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \det_q(A_J) M_{\bar{J}} G(m_1, \ldots, m_r) = 0.$$

For a subset $J = \{k_1, \ldots, k_j\}$ of $\{1, \ldots, r\}$, we denote by $G_J(m_{k_1}, \ldots, m_{k_j})$ the evaluation $G(m_1, \ldots, m_r)$ at $m_i = 0$ for all $i \notin J$, and we define

$$S_J = \sum_{m_{k_1}, \ldots, m_{k_j} = 0}^{\infty} G(m_1, \ldots, m_r).$$

Using telescoping cancellation, the inclusion–exclusion principle, and *Lemma 8* (part ii), the above equation becomes

$$\sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \mathrm{Ferm}(A_J) S_J = 0.$$

Using induction (with respect to r), together with $S_\emptyset = 1$, we obtain that $\mathrm{Ferm}(A) S_{\{1, \ldots, r\}} = 1$.

Some Remarks on the Boson–Fermion Correspondence

Let us give some motivation for *Theorem 1* from the point of view of quantum topology.

For a reference on quantum space and quantum algebra, see Chapter IV of ref. 7 and ref. 8.

Recall that a vector (column or row) of r indeterminate entries $x_1, \ldots, x_r$ lies in r-dimensional quantum space $A^{r|0}$ if its entries satisfy

$$x_j x_i = q x_i x_j$$

for all $1 \leq i < j \leq r$.

Recall that a *right (left) endomorphism* of $A^{r|0}$ is a matrix $A = (a_{ij})$ of size r whose entries commute with the coordinates x_i of a vector $x = (x_1, \ldots, x_r)^T \in A^{r|0}$ and in addition, Ax (*left*, $x^T A$) lie in $A^{r|0}$. Recall also that an endomorphism of $A^{r|0}$ is one that is right and left endomorphism.

It is easy to see (e.g., in theorem IV.3.1 of ref. 7) that A is a right-quantum (i.e., a right-endomorphism) if for every 2-by-2 submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A we have

$$ca = qac, \quad db = qbd, \quad ad = da + q^{-1}cb - qbc.$$

Moreover, A is left-quantum if for every 2-by-2 submatrix of A (as above) we have

$$ba = qab, \quad dc = qcd, \quad ad = da + q^{-1}bc - qcb.$$

Finally, A is quantum if for every 2-by-2 submatrix of A (as above) we have

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc,$$
$$ad = da + q^{-1}cb - qbc. \qquad [7]$$

The set of quantum matrices A are the points of the r-dimensional quantum algebra $M_q(r)$, which is defined to be the quotient of the free algebra in noncommuting variables x_{ij} for $1 \leq i, j, \leq r$, modulo the left ideal generated by the commutation relations of Eq. **7**.

The algebra $M_q(r)$ has interesting and important structure. $M_q(r)$ is Noetherian and has no zero divisors; in addition, a basis for the underlying vector space is given by the set of *sorted monomials* $\{\Pi_{i,j} a_{ij}^{n_{ij}} | n_{ij} \geq 0\}$, where the product is taken lexicographically. An important quotient of $M_q(r)$ is the quantum group $SL_q(r) := M_q(r)/(\det_q - 1)$, which is a Hopf algebra (see theorem IV.4.1 in ref. 7) whose representation theory gives rise to the quantum group invariants of knots, such as the celebrated Jones Polynomial.

Observing that

$$\mathrm{tr}\, S^n(A) = \sum_{m_1 + \ldots m_r = n} G(m_1, \ldots, m_r)$$
$$\mathrm{tr}\Lambda^n(A) = \sum_{J \subset \{1, \ldots, r\}, |J| = n} \det_q(A_J)$$

Theorem 1 implies that

Theorem 2. *If A is in $M_q(r)$, then*

$$\frac{1}{\mathrm{Ferm}(A)} = \sum_{n=0}^{\infty} \mathrm{tr} S^n(A).$$

Since the algebra $M_q(r)$ has a vector space basis given by sorted monomials, it should be possible to give an alternative proof of the quantum MacMahon Master Theorem using *combinatorics on words*, as was done in ref. 9 for several proofs of the MacMahon Master Theorem. We hope to return to this alternative point of view in the near future.

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1. Andrews GE (1975) in *Problems and Prospects for Basic Hypergeometric Functions*, ed Aseky R (Academic, New York), pp 191–224.
2. MacMahon PA (1917) *Combinatory Analysis* (Cambridge Univ Press, Cambridge, UK), Vol 1.
3. Fadeev L, Reshetikhin N, Takhtadjian L (1990) *Lenin Math J* 1:193–225.
4. Huynh V, Lê TTQ (2006) ArXiv: math.GT/0503296.
5. Zeilberger D (1980) *SIAM J Math Anal* 11:919–934.
6. Zeilberger D (1990) *J Comput Appl Math* 32:321–368.
7. Kassel C (1995) *Quantum Groups*, Graduate Texts in Mathematics (Springer, New York), Vol 155.
8. Manin Y (1988) *Quantum Groups and Noncommutative Geometry* (University of Montreal, Montreal).
9. Foata D, Zeilberger D (1999) *Trans Am Math Soc* 351:2257–2274.