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THE NON-COMMUTATIVE A-POLYNOMIAL OF TWIST KNOTS

STAVROS GAROUFALIDIS

School of Mathematics,
Georgia Institute of Technology,
Atlanta, GA 30332-0160, USA
stavros@math.gatech.edu

XINYU SUN

Department of Mathematics,
Tulane University, 6823 St. Charles Ave,
New Orleans, LA 70118, USA
xsun1@tulane.edu

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ABSTRACT

The purpose of the paper is two-fold: to introduce a multivariable creative telescoping method, and to apply it in a problem of Quantum Topology: namely the computation of the non-commutative A-polynomial of twist knots.

Our multivariable creative telescoping method allows us to compute linear recursions for sums of the form $J(n) = \sum_k c(n,k)\hat{J}(k)$ given a recursion relation for $(\hat{J}(n))$ and the hypergeometric kernel $c(n,k)$. As an application of our method, we explicitly compute the non-commutative A-polynomial for twist knots with -15 and 15 crossings.

The non-commutative A-polynomial of a knot encodes the monic, linear, minimal order q-difference equation satisfied by the sequence of colored Jones polynomials of the knot. Its specialization to $q = 1$ is conjectured to be the better-known A-polynomial of a knot, which encodes important information about the geometry and topology of the knot complement. Unlike the case of the Jones polynomial, which is easily computable for knots with 50 crossings, the A-polynomial is harder to compute and already unknown for some knots with 12 crossings.

Keywords: Knots; Jones polynomial; colored Jones function; A-polynomial; C-polynomial; non-commutative A-polynomial; q-difference equations; WZ algorithm; creative telescoping; Gosper's algorithm; certificate; multi-certificate.

Mathematics Subject Classification 2010: Primary 57N10; Secondary 57M25

1. Introduction

1.1. *The goal*

The purpose of the paper is two-fold: to introduce a multivariable creative telescoping method, and to apply it in a problem of Quantum Topology: namely the computation of the non-commutative A-polynomial of twist knots.

Our multivariable creative telescoping method allows us to compute linear recursions for sums of the form $J(n) = \sum_k c(n,k)\hat{J}(k)$ given a recursion relation for $(\hat{J}(n))$ and the hypergeometric kernel $c(n,k)$. General theory implies the existence of a recursion relation for $(\hat{J}(n))$. However, in practice the computation is not manageable for twist knots, and there is no guarantee that the recursion relation will be of minimal order. Our method does not guarantee a minimal order recursion relation either, however (unlike the known methods) it is manageable and produces a minimal order recursion relation for the non-commutative A-polynomial for twist knots of $-15, \dots, 15$ twists. The non-commutative A-polynomial encodes the monic, linear, minimal order q-difference equation satisfied by the sequence of colored Jones polynomials of the knot. Our results give a new proof of the AJ-conjecture for those knots.

1.2. *The Jones polynomial of a knot*

In this section, we recall the relevant Laurent polynomial invariants of knots, such as the Jones polynomial and its colored cousins. In 1985 Jones introduced the famous *Jones polynomial* of a knot K in 3-space [17]. The Jones polynomial (an element of $\mathbb{Z}[q^{\pm 1}]$) is a powerful knot invariant which amongth other things detects cheirality, and it can be extended to a sequence $(J_K(n))$ of Laurent polynomials by taking parallels of a knot K. Technically, $J_K(n) \in \mathbb{Z}[q^{\pm 1}]$ is the quantum group invariant of the 0-framed knot using the n-dimensional representation of $\mathrm{SU}(2)$, and normalized by $J_{\mathrm{Unknot}}(n) = 1$. For a detailed definition, see [30] and also [8]. With this normalization, we have that $J_K(1) = 1$, and $J_K(2)$ is the Jones polynomial of K.

For a given knot K, the sequence of Laurent polynomials $(J_K(n))$ is not random. To be precise, $(J_K(n))$ is q-*holonomic* i.e. it satisfies a linear q-difference equation (which of course, depends on the knot) with coefficients in $\mathbb{Q}(q, q^n)$. This fact, proven in [8], is an easy consequence of two facts:

(a) $J_K(n)$ is a finite multisum of a proper q-hypergeometric term, as follows from the state-sum definition of the colored Jones function; see [8].
(b) Multisums of proper q-hypergeometric terms are q-holonomic, as follows from the WZ theory of Wilf-Zeilberger; see [31].

1.3. *The non-commutative A-polynomial of a knot and its significance*

A q-holonomic sequence is annihilated by a unique monic homogeneous linear q-difference equation of smallest degree, and the corresponding monic polynomial in two q-commuting variables E and Q is an invariant (the so-called *characteristic polynomial* of the q-holonomic sequence. We define the *non-commutative A-polynomial* $A_K(E, Q, q)$ of a knot K to be the characteristic polynomial of $(J_K(n))$.

In [10], it was conjectured by the first author (the so-called AJ-Conjecture) that the specialization $A_K(E, Q, 1)$ of $A_K(E, Q, q)$ should agree with the *A-polynomial* of a knot. The latter is an important invariant that parametrizes the $\mathrm{SL}(2, \mathbb{C})$ character variety of the knot complement, as viewed from the boundary torus. For a detailed definition of the A-polynomial, its properties and its applications to the geometry and topology of the knot complement, see [5]. Thus, $A_K(E, Q, q)$ can be thought of as a deformation (or quantization) of the character variety.

The Jones polynomial of a knot is easily computable via skein theory with knots with, say, 50 crossings; see, for example [3]. On the other hand, the A-polynomial of a knot is much harder to compute, and at present it is unknown for some knots with 12 crossings. There are two general methods to compute the A-polynomial: an exact (primarily elimination, and Puiseux expansions) developed by Boyd [4] and a numerical one developed by Culler [6].

The non-commutative A-polynomial and its possible relation with the A-polynomial of a knot is an important ingredient to the Hyperbolic Volume Conjecture and its generalization.

1.4. *Computing the non-commutative A-polynomial*

For theoretical as well as experimental reasons it would be good to have explicit formulas for the non-commutative A-polynomial. So far, an explicit formula has been given for torus knots in [13] (using properties of the Kauffman bracket skein module of the solid torus), as well as for the simplest hyperbolic 4_1 knot in [8] (using an explicit single-sum formula for the colored Jones function).

The WZ algorithm has been implemented (see [23–25]) and together with explicit state-sum formulas for the colored Jones function of an arbitrary planar projection given in [8], in principle one can obtain a linear q-difference equation for the colored Jones function of an arbitrary knot. There are two problems with this approach:

(a) The number of summation variables in the multisum formulas is generally two less than the number of crossings, and the q-multisum algorithms appear to be slow for the current machines.
(b) There is no guarantee that the various q-multisum algorithms will give a minimal order linear q-difference equation. In fact, in many cases (where symmetry is involved), it has been observed that they fail to give the minimal order q-difference equation. See [22] for well-known examples of this failure.

With respect to the first problem, we were unable to use the sofware of [2, 27] to compute a q-difference equation for our double sums.

One might wonder whether Problem (b) really occurs for the state-sums that originate in knot theory. As expected, this problem *does* occur. The knots 5_2 and 6_1 have double-sum formulas for their colored Jones function. An application of the q-multisum package of [24] was done by Takata in [28] who found out an explicit inhomogeneous q-difference equation of degree 5 and 5 respectively. On the other hand, as we shall see, there exist inhomogeneous q-difference equations of degrees 3 and 4 respectively.

In a different direction, Le used geometric methods of the Kauffman bracket skein module and was able to prove the AJ-Conjecture for most 2-bridge knots, as well as give a linear algebra algorithm that in principle computes the non-commutative A-polynomial; see [20, Theorem 1] and [20, Sec. 5.6.3]. The algorithm was implemented in Maple by the second author, but proved to be too slow to run for the 5_2 and 6_1 knots.

1.5. *A sample of our results*

The main goal of the paper is to give an explicit formula for the non-commutative A-polynomial of twist knots with p twists, where $p = -15, \dots, 15$.

Let us recall the *twist knots* K_p for integer p, shown in Fig. 1. The planar projection of K_p has $2|p| + 2$ crossings, $2|p|$ of which come from p full twists, and 2 come from the negative *clasp*.

For small p, these knots may be identified with ones from Rolfsen's table (see [3, 26]) as follows:

$$\begin{aligned} K_1 &= 3_1, \quad K_2 = 5_2, \quad K_3 = 7_2, \quad K_4 = 9_2, \\ K_{-1} &= 4_1, \quad K_{-2} = 6_1, \quad K_{-3} = 8_1, \quad K_{-4} = 10_1. \end{aligned}$$

Let E and Q denote the operators that act on a sequence $(J(n))$ of Laurent polynomials $J(n) \in \mathbb{Z}[q^{\pm 1}]$ by:

$$(EJ)(n) = J(n+1), \quad (QJ)(n) = q^n J(n). \tag{1.1}$$

Note that $EQ = qQE$. Let $(A_p^{nh}(E, Q, q), B_p(q^n, q))$ denote the *inhomogeneous* non-commutative A-polynomial of K_p. That is, $A_p^{nh}(E, Q, q)$ is monic and minimal degree (with respect to E) that satisfies the equation

$$(A_p^{nh}(E, Q, q) J_p)(n) = B_p(q^n, q), \tag{1.2}$$

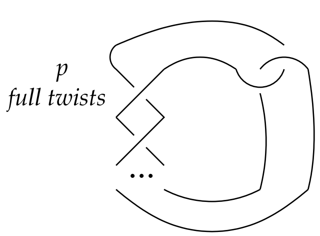

Fig. 1. The twist knot K_p, for integers p.

for all n, where $B_p(q^n,q) \in \mathbb{Q}(q^n,q)$. To convert the inhomogeneous equation above to a homogeneous one, see Sec. 3.

Theorem 1.1. (a) *For* $p = \pm 1$, *we have*:

$$\begin{aligned}
A_1^{nh}(E,Q,q) &= q^{3n+2}(q^n-1) + (q^{n+1}-1)E,\\
B_1^{nh}(q^n,q) &= (q^{2n+1}-1)q^n,\\
A_{-1}^{nh}(E,Q,q) &= q^{2n+2}(q^n-1)(q^{2n+3}-1) - (q^{n+1}+1)\\
&\quad \times (q^{4n+4} - q^{3n+3} - q^{2n+3} - q^{n+1} - q^{2n+1} + 1)(q^{n+1}-1)^2 E\\
&\quad + q^{2n+2}(q^{2n+1}-1)(q^{n+2}-1)E^2,\\
B_{-1}^{nh}(q^n,q) &= q^{n+1}(q^{2n+3}-1)(q^{n+1}+1)(q^{2n+1}-1).
\end{aligned}$$

(b) *For* $p = \pm 2$, *we have*:

$$\begin{aligned}
A_2^{nh}(E,Q,q) &= q^{7n+9}(-1+q^n)(q^{2n+4}-1)(q^{2n+5}-1) - q^{2n+5}(q^{n+1}-1)\\
&\quad \times (q^{2n+2}-1)(q^{2n+5}-1)(q^{5n+6} - 2q^{4n+5} - q^{3n+5} + q^{2n+4} - q^{2n+3}\\
&\quad - q^{2n+2} - q^{3n+2} + q^{2n+1} + q^{n+1} - 1)E + q(q^{n+2}-1)(q^{2n+1}-1)\\
&\quad \times (q^{2n+4}-1)(q^{5n+9} - q^{4n+7} - q^{3n+7} + q^{3n+6} + q^{3n+5}\\
&\quad + q^{2n+5} - q^{3n+4} + 2q^{n+2} + q^{2n+2} - 1)E^2 + (q^{n+3}-1)(q^{2n+1}-1)\\
&\quad \times (q^{2n+2}-1)E^3,\\
B_2(q^n,q) &= (q^{n+1}+1)(q^{2n+1}-1)(q^{n+2}+1)(q^{2n+5}-1)(q^{2n+3}-1)q^{2n+4},\\
A_{-2}^{nh}(E,Q,q) &= (-1+q^n)(q^{2n+5}-1)(q^{2n+6}-1)(q^{2n+7}-1)q^{4n+8} - q^{2n+5}\\
&\quad \times (q^{n+1}-1)(q^{2n+6}-1)(q^{2n+7}-1)(q^{2n+2}-1)(q^{6n+9} - q^{5n+8}\\
&\quad - q^{4n+8} + q^{4n+7} + q^{3n+7} + q^{4n+6} - q^{3n+6} - q^{2n+6} - q^{3n+5}\\
&\quad - q^{2n+5} - q^{4n+4} + q^{3n+3} + q^{2n+3} - q^{n+3} - q^{2n+2} - 2q^{n+2}\\
&\quad - q^{2n+1} + q + 1)E + q(q^{n+2}-1)(q^{2n+1}-1)(q^{2n+4}-1)(q^{2n+7}-1)\\
&\quad \times (1 - q^{2n+3} - 2q^{n+2} - q^{n+3} + q^{2n+5} + 2q^{5n+8} + 3q^{4n+8} - q^{3n+7}\\
&\quad - q^{4n+6} - 2q^{3n+6} - q^{2n+6} + q^{3n+5} - q^{2n+2} + q^{4n+5} + q^{2n+4}\\
&\quad - q^{6n+10} - q^{6n+14} - q^{5n+11} - q^{7n+15} - q^{6n+11} + q^{6n+12} + q^{8n+16}\\
&\quad + q^{5n+13} + q^{6n+13} + q^{5n+9} + q^{3n+9} + 2q^{3n+4} - 2q^{7n+14} + 3q^{4n+9}\\
&\quad + 2q^{3n+8} - 2q^{5n+10} + 2q^{5n+12})E^2 - (q^{n+3}-1)(q^{2n+6}-1)\\
&\quad \times (q^{2n+2}-1)(q^{2n+1}-1)(q^{6n+16} + q^{6n+15} - q^{5n+14} - 2q^{5n+13}\\
&\quad - q^{4n+13} - q^{4n+12} + q^{4n+10} + q^{3n+10} - q^{4n+9} - q^{3n+9} - q^{4n+8}
\end{aligned}$$

$$
\begin{aligned}
&- q^{3n+8} - q^{2n+7} + q^{3n+6} + q^{2n+6} + q^{2n+5} - q^{2n+3} - q^{n+3} + 1)E^3 \\
&+ q^{4n+12}(q^{n+4} - 1)(q^{2n+1} - 1)(q^{2n+2} - 1)(q^{2n+3} - 1)E^4, \\
B_{-2}(q^n, q) &= (q^{n+3} + 1)(q^{2n+5} - 1)(q^{2n+7} - 1)q^{2n+6}(q^{n+2} + 1) \\
&\times (q^{n+1} + 1)(q^{2n+1} - 1)(q^{2n+3} - 1).
\end{aligned}
$$

The formulas quickly become too lengthy to type. For more information, see Appendix B for $p = \pm 3$ as well as the data file [12] for $p = -15, \ldots, 15$. Theorem 1.1 gives a new proof of the AJ-Conjecture for twist knots with $-15, \ldots, 15$ twists.

1.6. *Plan of the proof*

In Sec. 2, we outline the main strategy. The idea is to use the recursion relation of the cyclotomic function $(\hat{J}_p(n))$ of the twist knot K_p (from [11]) as well as a single-sum relation between $J_p(n)$ and $\hat{J}_p(n)$, together with some new ideas of Creative Telescoping and some guessing. In Sec. 3, we review the method of Creative Telescoping and in Sec. 4, we present a multi-certificate version that takes into account the product of a hypergeometric summand with a q-holonomic one.

We conclude with three appendices: in Appendix A, we present an alternative method that uses generating functions (that was kindly communicated to us by Zeilberger). In Appendix B, we give the non-commutative polynomial of twist knots K_p for $p = -3, 3$, and in Appendix C, we give the A-polynomial of the same knots.

2. The Strategy

In this section, we will describe our strategy to obtain a formula for $A_p(E, Q, q)$.

(a) We consider the cyclotomic function $\hat{J}_K(n) \in \mathbb{Z}[q^{\pm 1}]$ introduced by Habiro in [14], who used the notation $J_K(P''_n)$.

(b) The relation between the cyclotomic and the colored Jones functions is given by:

$$J_K(n) = \sum_{k=0}^{n} c(n,k)\hat{J}_K(k), \tag{2.1}$$

where the *cyclotomic kernel* $c(n, k)$ is a proper q-hypergeometric term given for $0 \le k \le n$ by:

$$
\begin{aligned}
c(n,k) &= \frac{\{n-k\}\{n-k+1\}\cdots\{n+k-1\}\{n+k\}}{\{n\}} \\
&= (-1)^k q^{-k(k+1)/2}(q^{1-n};q)_k(q^{1+n};q)_k,
\end{aligned} \tag{2.2}
$$

where

$$\{n\} = q^{n/2} - q^{-n/2},$$

and the *quantum factorial* is defined by:

$$(x;q)_n = \begin{cases} (1-x)\cdots(1-xq^{n-1}), & \text{if } n > 0; \\ 1, & \text{if } n = 0; \\ \dfrac{1}{(1-xq^{-1})\cdots(1-xq^{n})}, & \text{if } n < 0. \end{cases}$$

Please note that we are using the unbalanced quantum factorials (common in discrete math) and not the balanced ones (common in the representation theory of quantum groups).

(c) The cyclotomic function $(\hat{J}_K(n))$ is q-holonomic, as shown in [8], and its characteristic polynomial $C_K(E,Q,q)$ is defined to be the non-commutative C-polynomial of a knot. Let us abbreviate $\hat{J}_{K_p}(n)$, $J_{K_p}(n)$, $A_{K_p}(E,Q,q)$ and $C_{K_p}(E,Q,q)$ for the twist knots K_p by $\hat{J}_p(n)$, $J_p(n)$, $A_p(E,Q,q)$ and $A_p(E,Q,q)$ respectively.

(d) In [11] we gave an explicit formula for $C_p(E,Q,q)$.

(e) Using the explicit formula for $C_p(E,Q,q)$ as well as the relation (2.1) and a version of creative telescoping (and some guessing), we deduce a linear q-difference equation for $(J_p(n))$, which specializes to the A-polynomial of K_p when $q=1$.

(f) Since the A-polynomial of K_p is irreducible (see [15]), and the non-commutative A-polynomial of K_p specializes to the A-polynomial, it follows that our q-difference equation is indeed of minimal order. This computes $A_p(E,Q,q)$.

For completeness and concreteness, we give a formula for $\hat{J}_p(n)$, using [21, Theorem 5.1] (compare also with [10, Sec. 3]):

$$\begin{aligned} \hat{J}_p(n) &= \sum_{k=0}^{n} q^{n(n+3)/2+pk(k+1)+k(k-1)/2}(-1)^{n+k+1}\frac{(q^{2k+1}-1)(q;q)_n}{(q;q)_{n+k+1}(q;q)_{n-k}} \\ &= \sum_{k=0}^{n} q^{n(n+3)/2+pk(k+1)+k(k-1)/2}(-1)^{n+k+1}\frac{(q^{2k+1}-1)(q^{n-k+1};q)_k}{(q;q)_{n+k+1}}. \end{aligned} \tag{2.3}$$

Observe that since $(q^{n-k+1};q)_k = 0$ for $k > n > 0$, we can assume that the k-summation in the above equation is for $0 \le k < +\infty$.

Equations (2.1)–(2.3) imply that $J_p(n)$ is given by a double-sum formula of a proper q-hypergeometric summand. As explained earlier, the `qMultisum.m` implementation of the WZ algorithm given in [23, 24] and used in [28] is slow to run, and gives q-difference equations of higher than actual degree. An application of our multicertificate version of Creative Telescoping is the following theorem, proved in Sec. 4.4.

Theorem 2.1. *The minimal inhomogeneous recursion for $J_p(n)$ for $-15 \le p \le 15$ is a linear q-difference equation of order*

$$\begin{cases} 2p-1, & \text{if } \ 0 < p \le 15; \\ 2|p|, & \text{if } \ -15 \le p < 0. \end{cases}$$

The inhomogeneous recursion is given explicitly by Theorem 1.1 for $p = -2, \dots, 2$, Appendix B for $p = \pm 3$ and the data file [12] for $p = -15, \dots, 15$.

3. A Brief Review of Creative Telescoping

In this section, we recall briefly some key ideas of Zeilberger on recursion relations of combinatorial sums. An excellent reference is [25]. For a longer introduction, see also [11, Sec. 3].

A *term* is $F(n,k)$ called *hypergeometric* if both $\frac{F(n+1,k)}{F(n,k)}$ and $\frac{F(n,k+1)}{F(n,k)}$ are rational functions over n and k. In other words,

$$\frac{F(n+1,k)}{F(n,k)} \in \mathbb{Q}(n,k), \quad \frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(n,k). \tag{3.1}$$

Examples of hypergeometric terms are $F(n,k) = (an+bk+c)!$ (for integers a, b, c), and ratios of products of such. The latter are actually called *proper hypergeometric*. A key problem is to construct recursion relations for sums of the form:

$$S(n) = \sum_k F(n,k), \tag{3.2}$$

where $F(n,k)$ is a proper hypergeometric term. The summation set can be the set of all integers or an interval thereof. Let us first suppose that summation is over entire set of integers. Sister Celine [7] (see also [25]) proved the following:

Theorem 3.1. *Given a proper hypergeometric term $F(n,k)$, there exist a natural number $I \in \mathbb{N}$ and a set of functions $a_i(n) \in \mathbb{Q}(n)$, $0 \le i \le I$, such that*

$$\sum_{i=0}^{I} a_i(n) F(n+i,k) = 0. \tag{3.3}$$

The important part of the above theorem is that the functions $a_i(n)$ are independent of k. Therefore if we take the sum over k on both sides, we get

$$\sum_{i=0}^{I} a_i(n) \sum_k F(n+i,k) = 0. \tag{3.4}$$

In other words, we have:

$$\sum_{i=0}^{I} a_i(n) S(n+i) = 0. \tag{3.5}$$

So, Eq. (3.3) produces a recursion relation, which is inhomogeneous if we are summing over an interval. How can we find functions $a_i(n)$ that satisfy Eq. (3.3)? The idea is simple: divide Eq. (3.3) by $F(n,k)$, and use (3.1) to convert the divided equation into a *linear equation* over the field $\mathbb{Q}(n,k)$, with unknowns $a_i(n)$ for $i = 0,\dots,I$. Clearing denominators, we get linear equation over $\mathbb{Q}(n)[k]$ with the same unknowns $a_i(n)$. Thus, the coefficients of every power of k must vanish, and this gives a linear system of equations over $\mathbb{Q}(n)$ with unknowns $a_i(n)$. If there are more unknowns than equations, one is guaranteed to find a nonzero solution. By a counting argument, one may see that if we choose I high enough (this depends on the complexity of the term $F(n,k)$), then we have more equations than unknowns.

Although it can be numerically challenging to find $a_i(n)$ that satisfy Eq. (3.3), it is routine to check the equation once $a_i(n)$ are given. Indeed, one only need to divide the equation by $F(n,k)$, and then check that a function in $\mathbb{Q}(n,k)$ is identically zero. The latter is computationally easy task in the field $\mathbb{Q}(n,k)$.

This algorithm produces a recursion relation for $S(n)$. However, it is known that the algorithm does not always yield a recursion relation of the smallest order.

Applying Gosper's algorithm, Wilf and Zeilberger invented another algorithm, the *WZ algorithm*, also called *creative telescoping*. Instead of looking for 0 on the right-hand side of Eq. (3.3), they instead looked for a function $G(n,k)$ such that

$$\sum_{i=0}^{N} a_i(n)F(n+i,k) = G(n,k+1) - G(n,k). \tag{3.6}$$

Summing over k, and using telescoping cancellation of the terms in the right-hand side, we get a recursion relation for $S(n)$. How to find the $a_i(n)$ and $G(n,k)$ that satisfy (3.6)? The idea is to look for a *rational function* $\mathrm{Cert}(n,k)$ (the so-called *certificate* of (3.6)) such that

$$G(n,k) = \mathrm{Cert}(n,k)F(n,k). \tag{3.7}$$

Dividing out (3.6) by $F(n,k)$ and proceeding as before, one reduces this to a problem of linear algebra. As before, given $a_i(n)$ and $\mathrm{Cert}(n,k)$, it is routine to check whether (3.6) holds.

Now, let us rephrase the above equations using operators. We define operators E, E_k, n and k that act on a function $F(n,k)$ by:

$$(EF)(n,k) = F(n+1,k), \quad (E_kF)(n,k) = F(n,k+1), \tag{3.8}$$

$$(nF)(n,k) = nF(n,k), \qquad (kF)(n,k) = kF(n,k). \tag{3.9}$$

The operators E and n (and also E_k, k) do not commute. Instead, we have:

$$En = (n+1)E, \quad E_kk = (k+1)E_k.$$

On the other hand, n, E commute with k, E_k. Then we can rewrite Eq. (3.6) as

$$\left(\sum_{i=0}^{I} a_i(n)E^i\right) F(n,k) = (E_k - 1)G(n,k) = (E_k - 1)\mathrm{Cert}(n,k)F(n,k). \tag{3.10}$$

Implementation of the algorithms are available in various platforms, such as, Maple and Mathematica. See, for example, [23, 33].

Let us mention now how one deals with boundary terms. In the applications below, one considers not quite the unrestricted sums of Eq. (3.2), but rather restricted ones of the form:

$$S'(n) = \sum_{k=0}^{\infty} F(n,k), \tag{3.11}$$

where $F(n,k)$ is a proper hypergeometric term. When we apply the Creating Telescoping summation to (3.6), we are left with some boundary terms $R(n) \in \mathbb{Q}(n)$. In that case, Eq. (3.5) becomes:

$$\left(\sum_{i=0}^{I} a_i(n)E^i\right) S'(n) = R(n).$$

This is an inhomogeneous equation of order I which can be converted into a homogeneous recursion of order $I+1$ by following trick: apply the operator

$$(E-1)\frac{1}{R(n)}$$

on both sides of the recursion. We get

$$\left(\frac{1}{R(n+1)}E - \frac{1}{R(n)}\right)\left(\sum_{i=0}^{I} a_i(n)E^i\right) S'(n) = 0,$$

i.e.

$$\left(\frac{a_I(n+1)}{R(n+1)}E^{I+1} + \sum_{i=1}^{I}\left(\frac{a_{i-1}(n+1)}{R(n+1)} - \frac{a_i(n)}{R(n)}\right)E^i - \frac{a_0(n)}{R(n)}\right) S'(n) = 0.$$

In Quantum Topology we are using q-factorials rather than factorials. The previous results translate without conceptual difficulty to the q-world, although the computer implementation is slower. A *term* $F(n,k)$ is called *q-hypergeometric* if

$$\frac{F(n+1,k)}{F(n,k)}, \frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(q, q^n, q^k).$$

Examples of q-hypergeometric terms are the quantum factorials of linear forms in n, k, and ratios of products of quantum factorials and q raised to quadratic functions of n and k. The latter are called *q-proper hypergeometric*.

Sister Celine's algorithm and the WZ algorithm work equally well in the q-case. In either algorithms, we can replace the operators E, n, E_k, k of (3.8) by the operators E, Q, E_k, Q_k defined by:

$$(EF)(n,k) = F(n+1,k), \quad (E_kF)(n,k) = F(n,k+1), \tag{3.12}$$

$$(QF)(n,k) = q^n F(n,k), \qquad (Q_kF)(n,k) = q^k F(n,k). \tag{3.13}$$

Observe that E, Q (and also E_K, Q_k) q-commute, i.e. we have:

$$EQ = qQE, \quad E_k Q_k = qQ_k E_k. \tag{3.14}$$

On the other hand, E, Q commute with E_k, Q_k. With these modifications, and with the replacement of the field $\mathbb{Q}(n)$ by $\mathbb{Q}(q, q^n)$, the rest of the proofs still apply naturally. The implementations of the q-case include [19, 24, 33].

4. Multi-Certificate Creative Telescoping and Theorem 2.1

4.1. *Multi-certificate creative telescoping*

In a nut-shell, the method of creative telescoping works as follows. To find the recursion such that

$$\sum_{i=0}^{m} a_i(n) \left(\sum_{k \geq 0} F(n+i, k) \right) = b(n),$$

it suffices to find a rational function $\mathrm{Cert}(n, k) \in \mathbb{Q}(q, q^n, q^k)$ such that $G(n, k) := \mathrm{Cert}(n, k) F(n, k)$ satisfies:

$$\left(\sum_{i=0}^{m} a_i(n) E^i \right) F(n, k) = \sum_{i=0}^{m} a_i(n) F(n+i, k) = G(n, k+1) - G(n, k) = (E_k - 1) G(n, k).$$

If this can be done, we sum both sides for $0 \leq k < +\infty$, and we obtain:

$$\sum_{i=0}^{m} a_i(n) \left(\sum_{k \geq 0} F(n+i, k) \right) = G(n, 0).$$

For twist knots K_p, we have from Eq. (2.1):

$$J_p(n) = \sum_{k=0}^{n} c(n, k) \hat{J}_p(k), \tag{4.1}$$

where $c(n, k)$ is proper q-hypergeometric (given by (2.2)), and $\hat{J}_p(n)$ satisfies a linear q-difference equation of degree $|p|$ from [11].

Without loss of generality, suppose that $p > 0$. Suppose the minimal order recursion of $\hat{J}_p(k)$ is:

$$\left(\sum_{i=0}^{p} r_i(k) E_k^i \right) \hat{J}_p(k) = 0, \tag{4.2}$$

with $r_p(k) = 1$ and $r_i(k) \in \mathbb{Q}(q, q^k)$ for $i = 0, \ldots, p$. The idea is to look for p certificates $\{C_0(n, k), \ldots, C_{p-1}(n, k)\}$, such that

$$\left(\sum_{i=0}^{m} a_i(n) E^i \right) c(n, k) \hat{J}_p(k) = (E_k - 1) \left(\sum_{j=0}^{p-1} C_j(n, k) E_k^j \right) c(n, k) \hat{J}_p(k). \tag{4.3}$$

4.2. *A first reduction to linear algebra*

Our goal in this section, stated in Proposition 4.1 below, is to translate the functional equation (4.3) into a system of linear equations with unknowns $a_i(n) \in \mathbb{Q}(q, q^n)$ and $C_j(n,k) \in \mathbb{Q}(q, q^n, q^k)$ for $i = 0, \ldots, m$ and $j = 0, \ldots, p-1$. Since $c(n,k)$ is proper q-hypergeometric, we have:

$$\frac{Ec(n,k)}{c(n,k)} = s(n,k) \in \mathbb{Q}(q, q^n, q^k), \quad \frac{E_k c(n,k)}{c(n,k)} = t(n,k)\mathbb{Q}(q, q^n, q^k).$$

Observe that

$$\begin{aligned} 0 &= \sum_{i=0}^{p} r_i(k)\hat{J}_p(k+i) \\ &= \sum_{i=0}^{p} r_i(k)\frac{c(n,k+p)}{c(n,k+i)}\hat{J}_p(k+i)c(n,k+i) \\ &= \sum_{i=0}^{p} r_i(k)\frac{c(n,k+p)}{c(n,k+i)}\hat{J}_p(k+i)c(n,k+i) \\ &= \left(\sum_{i=0}^{p} r_i(k)\frac{c(n,k+p)}{c(n,k+i)}E_k^i\right)\hat{J}_p(k)c(n,k). \end{aligned}$$

So if we define

$$\begin{aligned} R_i(n,k) &= r_i(k)\frac{c(n,k+p)}{c(n,k+i)} \\ &= r_i(k)\prod_{j=0}^{p-i-1} t(n,k+i+j) \in \mathbb{Q}(q, q^n, q^k), \end{aligned}$$

then we obtain that

$$\left(\sum_{i=0}^{p} R_i(n,k)E_k^i\right)\hat{J}_p(k)c(n,k) = 0. \tag{4.4}$$

Notice that since $r_p(k) = 1$, it follows that $R_p(n,k) = 1$ too.

Proposition 4.1. *Equation* (4.3) *is equivalent to the following system of linear equations*:

$$\sum_{i=0}^{m} a_i(n)\prod_{j=0}^{i-1} s(n+j,k) = -\sum_{j=0}^{p-1} C_{p-1}(n,k-j+1)R_j(n,k-j) \tag{4.5}$$

and

$$C_{j-1}(n,k+1) = C_j(n,k) + C_{p-1}(n,k+1)R_j(n,k), \quad 1 \le j \le p-1, \tag{4.6}$$

in the unknowns $a_i(n) \in \mathbb{Q}(q, q^n)$ *for* $i = 0, \ldots, m$ *and* $C_j(n,k) \in \mathbb{Q}(q, q^n, q^k)$ *for* $j = 0, \ldots, p-1$.

Proof. For convenience we define $C_{-1}(n,k) = 0$. Then using the commutation relation

$$(E_k - 1)C_j(n,k) = C_j(n,k+1)E_k - C_j(n,k)$$

and Eq. (4.4) we obtain that:

$$\begin{aligned}
&\left(\sum_{i=0}^{m} a_i(n)E^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = (E_k - 1)\left(\sum_{j=0}^{p-1} C_j(n,k)E_k^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(\sum_{j=0}^{p-1} C_j(n,k+1)E_k^{i+1} - \sum_{j=0}^{p-1} C_j(n,k)E_k^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(\sum_{j=1}^{p} C_{j-1}(n,k+1)E_k^{i} - \sum_{j=0}^{p-1} C_j(n,k)E_k^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(-C_{p-1}(n,k+1)\sum_{j=0}^{p-1} R_j(n,k)E_k^j\right.\\
&\qquad \left. + \sum_{j=1}^{p-1} C_{j-1}(n,k+1)E_k^{i} - \sum_{j=0}^{p-1} C_j(n,k)E_k^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(\sum_{j=0}^{p-1}(-C_{p-1}(n,k+1)R_j(n,k) + C_{j-1}(n,k+1)\right.\\
&\qquad \left. - C_j(n,k))E_k^j\right) c(n,k)\hat{J}_p(k).
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{i=0}^{m} a_i(n)\left(\prod_{j=0}^{i-1} s(n+j,k)\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(\sum_{i=0}^{m} a_i(n)E^i\right) c(n,k)\hat{J}_p(k)\\
&\quad = \left(\sum_{j=0}^{p-1}(-C_{p-1}(n,k+1)R_j(n,k) + C_{j-1}(n,k+1)\right.\\
&\qquad \left. - C_j(n,k))E_k^j\right) c(n,k)\hat{J}_p(k).
\end{aligned}$$

If we divide both sides by $c(n,k)$, which is hypergeometric, we obtain a new recursion on $\hat{J}_p(k)$ of order $p-1$. Since $J_p(k)$ satisfies a minimal order recursion of degree p, the last equality implies that the coefficient of each E_k^i is 0 for all i. Hence

$$\begin{cases} -C_{p-1}(n,k+1)R_j(n,k) + C_{j-1}(n,k+1) - C_j(n,k) = 0, & \text{if } 1 \leq j \leq p-1, \\ \displaystyle\sum_{i=0}^{m} a_i(n)\frac{c(n+i,k)}{c(n,k)} = -C_{p-1}(n,k+1)R_0(n,k) - C_0(n,k), & \text{if } j = 0. \end{cases}$$

The first equation implies (4.6). In particular,

$$C_0(n,k+p-1) = C_{p-1}(n,k) + \sum_{j=0}^{p-2} C_{p-1}(n,k+j+1)R_{p-j-1}(n,k+j).$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{m} a_i(n)\frac{c(n+i,k)}{c(n,k)} &= \sum_{i=0}^{m} a_i(n)\prod_{j=0}^{i-1} s(n+j,k) \\ &= -\sum_{j=0}^{p-1} C_{p-1}(n,k-j+1)R_j(n,k-j), \end{aligned}$$

which proves (4.5) and concludes the proof of the proposition. □

4.3. *A second reduction to linear algebra*

Proposition 4.1 reduces the problem of finding p certificates $C_j(n,k)$ to a problem of finding a single certificate $C_{p-1}(n,k)$. Since $C_{p-1}(n,k) \in \mathbb{Q}(q,q^n,q^k)$ is a rational function, we can write it in the form:

$$C_{p-1}(n,k) = \frac{N_p(n,k)}{D_p(n,k)}, \tag{4.7}$$

where $N_p(n,k) = \sum_{i=0}^{r_d} d_i(n)q^{ki}$ and $D_p(n,k) = \sum_{i=0}^{r_e} e_i(n)q^{ki}$, in which $d_i(n)$ and $e_i(n)$ are in $\mathbb{Q}(q,q^n)$. By making the proper choices of $D_p(n,k)$ and the values of m and r_d, we can clear denominators and convert Eq. (4.6) as a *linear equation* in $\mathbb{Q}(q,q^n)[q^k]$ with unknowns $a_i(n)$, $d_i(n)$, and $e_i(n)$.

Setting every coefficient of every power of q^k to zero, we obtain a system of linear equations in the unknowns $a_i(n)$, $d_i(n)$, and $e_i(n)$ and coefficients in the field $\mathbb{Q}(q,q^n)$. A nontrivial solution is guaranteed by Sister Celine's method for the case of $J_p(n)$. At any rate, we can solve the system of equations using software like *Maple* or *Mathematica*.

Now comes the tricky part, and an educated guess for the case of twist knots. Since

$$\frac{c(n+i,k)}{c(n,k)} = \prod_{j=0}^{i-1} s(n+i,k),$$

and $R_j(n,k)$ are all polynomials in $\mathbb{Q}(q,q^n)[q^k]$, the most natural choice of $D_p(n,k)$ is the one such that $D_p(n,k-j+1)$ divides the polynomial $R_j(n,k-j)$ for all j. Let $D_p(n,k)$, the denominator of the certificate $C_{p-1}(n,k)$, be

$$\begin{cases} q^{pk}\displaystyle\prod_{i=1-p}^{p-1}(1-q^{k-n-i}), & \text{if } \; p>0, \\ \displaystyle\prod_{i=1}^{2|p|}(1-q^{k-n+|p|-i}), & \text{if } \; p<0. \end{cases}$$

4.4. *Proof of Theorem* 2.1

Let us fix a nonzero integer p such that $-15 \leq p \leq 15$ and let $d_p = 2p-1$ (respectively, $2|p|$) when $p>0$ (respectively, $p<0$). Our algorithm produces explicit operators

$$A_p^{nh}(E,Q,q) = \sum_{j=0}^{d_p} \alpha_{p,j}(Q,q)E^j, \quad \alpha_{p,j}(Q,q) \in \mathbb{Z}[Q^{\pm 1}, q^{\pm 1}]$$

and $B_p(Q,q) \in \mathbb{Z}[Q^{\pm 1}, q^{\pm 1}]$ such that for all $n \in \mathbb{N}$ we have

$$\sum_{j=0}^{d_p} \alpha_{p,j}(q^n,q)J_p(n+j) = B_p(q^n,q). \tag{4.8}$$

Moreover,

$$A_p^{nh}(E,Q,1) = A_p(E,Q^2)F_p(Q), \quad F_p(Q) \in \mathbb{Q}(Q) \tag{4.9}$$

where $A_p(L,M)$ is the A-polynomial of the twist knot K_p, computed explicitly in [15].

It is well-known that a solution to a linear inhomogeneous q-difference equation satisfies a linear homogeneous equation of degree one more, simply by dividing the right-hand side of (4.8) by $B_p(q^n,q)$, shifting n to $n+1$ and subtracting. It follows that

$$A_p^h J_p = 0, \quad A_p^h(E,Q,q) = E\left(\frac{1}{B_p(Q,q)}A_p^{nh}(E,Q,q)\right) - \frac{1}{B_p(Q,q)}A_p^{nh}(E,Q,q).$$

Let $A_p(E,Q,q)$ denote the minimal order recursion for J_p. *A priori*, $A_p^h(E,Q,q)$ need not coincide with $A_p(E,Q,q)$, but it is always the case that $A_p(E,Q,q)$ is a right factor of $A_p^h(E,Q,q)$.

Due to the special structure of J_p, it was shown in [10] that $E-1$ divides $A_p(E,Q,1)$. In addition, (4.9) implies that

$$A_p^h(E,Q,1) = A_p(E,Q^2)(E-1)\frac{F_p(Q)}{B_p(Q,1)}, \quad \frac{F_p(Q)}{B_p(Q,1)} \in \mathbb{Q}(Q).$$

Since $A_p(E, Q^2)$ is irreducible (see [16]), it follows that either $A_p(E, Q, q) = A_p^h(E, Q, q)$ or $A_p(E, Q, q)$ has degree 1, i.e. J_p is *closed form* [20, Proposition 2.2], implies that the colored Jones function of every alternating knot (in particular, of every 2-bridge knot, and therefore, of the colored Jones function J_p of every twist knot K_p) is not closed form. Theorem 2.1 follows. □

Remark 4.2. There is an alternative way to deduce that J_p is not closed form as follows. If J_p is closed form, then dividing the inhomogeneous equation (4.8) by $J_p(n)$ and using $J_p(n+j)/J_p(n) \in \mathbb{Q}(q^n, q)$, it follows that $J_p(n) = R(q^n, q)$ for some function $R(Q, q) \in \mathbb{Q}(Q, q)$. Thus, Eq. (4.8) becomes

$$\sum_{j=0}^{d_p} \alpha_{p,j}(Q, q) R(Q, q) = B_p(Q, q).$$

There is an algorithm to find all rational function solutions to a linear difference equation; see [1]. Applying Abramov's algorithm to the above equation and our explicit formula for J_p concludes that J_p is not closed form.

5. Odds and Ends

5.1. *A generalization of Theorem* 2.1

In fact, the multi-certificate proof of Theorem 2.1 implies the following result.

Theorem 5.1. *If $c(n, k)$ is proper q-hypergeometric term and $(\hat{J}(n))$ is q-holonomic, and*

$$J(n) = \sum_{k=0}^{n} c(n, k)\hat{J}(k),$$

then $J(n)$ is q-holonomic. A linear q-difference equation for $(J(n))$ can be constructed from a linear q-difference equation for $(\hat{J}(n))$ and $c(n, k)$.

Note that we are not assuming that $(\hat{J}(n))$ is given by a multisum of a proper q-hypergeometric summand. Instead, we are using a q-difference equation for $(\hat{J}(n))$. A software package that accompanies the proof of Theorem 2.1 was developed by the second author.

Remark 5.2. Our proof of Theorem 2.1 reduces to solving a system of $2|p|$ linear equations over the field $\mathbb{Q}(q, q^n)$. When $-15 \le p \le 15$, this system can be solved explicitly by symbolic software. Le's algorithm for computing the non-commutative A-polynomial of a 2-bridge knot, also requires a system of linear equations $(2|p|)!$ over the field $\mathbb{Q}(q, q^n)$ in the case of twist knots; see [20]. However, an implementation of Le's algorithm exceeded the capacity of our symbolic software for $p = 1$ and $p = -1$.

5.2. *Is there a recursion of the non-commutative A-polynomial with respect to the number of twists?*

Recall that $A_p(L,M)$ denotes the A-polynomial of the twist knot K_p. In [15], Hoste–Shanahan use a trace identity in $\mathrm{SL}(2,\mathbb{C})$ in order to give a second order linear recursion relation for the sequence (A_p).

There is supporting evidence that $A_p^{nh}(L,M,1)$ is annihilated by the following operator

$$\begin{cases} M^2(M-1)^4(M+1)^4(L+M)^4-(M-1)^2(M+1)^2(M^4-LM^4 \\ \quad +2LM^3+L^2M^2+M^2+2LM^2+2LM+L^2-L)P+P^2, & \text{if } p>0; \\ 1-(M-1)^2(M+1)^2(M^4-LM^4+2LM^3+L^2M^2+M^2+2LM^2 \\ \quad +2LM+L^2-L)P+M^2(M-1)^4(M+1)^4(L+M)^4P^2, & \text{if } p<0, \end{cases} \tag{5.1}$$

where

$$PA_p^{nh}(L,M,1)=A_{p+1}^{nh}(L,M,1).$$

Equation (5.1) may be proven using the recursion on $\hat{J}_p(k)$ and its simplification when $q=1$; see [11, Theorem 2]. Unfortunately, there is equally strong evidence that the sequence $A_p^{nh}(E,Q,q)$ does not satisfy a linear recursion with respect to p.

Appendix A. A Generating Functions Approach

In this appendix, we present an alternative approach to get a recursion relation for $J_K(n)$ given Eq. (2.1) and a recursion relation for $\hat{J}_K(n)$. This idea was communicated to us by Zeilberger, and may be useful in its own right. We were not able to compute the non-commutative A-polynomial for twist knots this way.

To explain the idea, let us recall first that a sequence $(a(n))$ of rational numbers is holonomic if and only if the generating series

$$F(z)=\sum_{n=0}^{\infty}a(n)z^n$$

is holonomic, i.e. it is annihilated by an element of the Weyl algebra $\mathbb{Q}\langle z, d/dz\rangle$; see [32]. The q-analogue of this is the following. Consider a sequence $(a(n))$ with $a(n)\in\mathbb{Q}(q)$, and the generating series

$$F(z,q)=\sum_{n=0}^{\infty}a(n)z^n\in\mathbb{Q}(q)[[z]]. \tag{A.1}$$

There are two operators Q and Z that act on the elements $F(z,q)$ of $\mathbb{Q}(q)[[z]]$ by:

$$(QF)(z,q)=F(qz,q), \quad (ZF)(z,q)=zF(z,q).$$

It is easy to see that $QZ = qZQ$, and that $(a(n))$ is q-holonomic iff the generating series $F(z, q)$ is q-holonomic.

Now, let us consider two sequences $(J(n))$ and $(\hat{J}(n))$ of rational functions that are related by:

$$J(n) = \sum_{k=0}^{n} c(n,k)\hat{J}(k) \tag{A.2}$$

where the kernel $c(n, k)$ is given by (2.2). $c(n, k)$ can be slightly simplified into $q^{-nk}\frac{(q;q)_{n+k}}{(q;q)_{n-k-1}(1-q^n)}$. We will absorb the factor $\frac{1}{1-q^n}$ in the colored Jones function and define

$$\begin{aligned}
\gamma(n,k) &:= q^{-nk}\frac{(q;q)_{n+k}}{(q;q)_{n-k-1}} \\
&= c(n,k)(1-q^n), \\
\check{J}(n) &:= \sum_{k=0}^{n} \gamma(n,k)\hat{J}(k) \\
&= \hat{J}(n)(1-q^n).
\end{aligned}$$

Proposition A.1. *Let*

$$H(k,z) = \sum_{i=0}^{\infty} \gamma(k+i,k)z^i, \tag{A.3}$$

then we have:

$$H(k,z) = (-1)^k q^{\frac{-k(k+1)}{2}} \frac{(q;q)_{2k+1}}{z^k(\frac{q}{z};q)_k(z;q)_{k+2}}. \tag{A.4}$$

Proof. We will use the idea of the WZ-algorithm to the hypergeometric summand:

$$H_1(k,i,z) = \gamma(k+i,k)z^i. \tag{A.5}$$

We claim that:

$$\begin{aligned}
&(z - q^{k+1})H_1(k+1,i,z) + \frac{(1-q^{2k+2})(1-q^{2k+3})}{q^{k+1}(1-zq^{k+2})}H_1(k,i,z) \\
&\quad = G_1(k,i,z) - G_1(k,i-1,z),
\end{aligned} \tag{A.6}$$

where

$$G_1(k,i,z) = \frac{-z(1-q^{2k+i+1})(zq^{k+2} - 1 - zq^{3k+i+4} + q^{4k+i+5})}{q^{2k+i+1}(1-zq^{k+2})}H_1(k,i,z). \tag{A.7}$$

Equation (A.6) can be verified by dividing both sides by $H_1(k, i, z)$, and then it reduces to an identity in the field $\mathbb{Q}(z, q, q^k)$ which can be readily checked. Now summing both sides of Eq. (A.5) over k, and we get the desired result. ◻

Consider the generating function of $\check{J}(n)$:

$$F(z,q) = \sum_{n=0}^{\infty} \check{J}(n) z^n. \tag{A.8}$$

Proposition A.2. *We have*:

$$F(z,q) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{-k(k+1)}{2}} \frac{(q;q)_{2k+1}}{(\frac{q}{z};q)_k (z;q)_{k+2}} \hat{J}_p(k). \tag{A.9}$$

Proof. We will interchange the order of summation and use Proposition A.1. We get:

$$\begin{aligned}
F(z,q) &= \sum_{n=0}^{\infty} \check{J}(n) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \gamma(n,k) \right) \hat{J}_p(k) z^n \\
&= \sum_{k=0}^{\infty} \hat{J}_p(k) \left(\sum_{n=k}^{\infty} \gamma(n,k) z^n \right) \\
&= \sum_{k=0}^{\infty} \hat{J}_p(k) z^k \left(\sum_{i=0}^{\infty} \gamma(k+i,k) z^i \right) \\
&= \sum_{k=0}^{\infty} H(k,z) \hat{J}_p(k) z^k \\
&= \sum_{k=0}^{\infty} (-1)^k q^{\frac{-k(k+1)}{2}} \frac{(q;q)_{2k+1}}{(\frac{q}{z};q)_k (z;q)_{k+2}} \hat{J}(k).
\end{aligned}$$

□

One can use a q-difference equation for $\hat{J}(n)$ and Proposition (A.1) to get a q-difference equation for $F(z,q)$. This will be explored in another publication.

Appendix B. The Non-Commutative A-Polynomial for $p = \pm 3$

In this section, we give the inhomogeneous non-commutative A-polynomial for $p = \pm 3$. The reader may compare the size of the output with Theorem 1.1.

$$\begin{aligned}
A_3^{nh}(E,Q,q) = {} & q^{11n+20}(-1+q^n)(q^{2n+8}-1)(q^{2n+7}-1)(q^{2n+6}-1)(q^{2n+9}-1) \\
& - q^{4n+14}(q^{n+1}-1)(q^{2n+2}-1)(q^{2n+7}-1)(q^{2n+8}-1)(q^{2n+9}-1) \\
& \times (-1 - q^{3+2n} + q^{8+4n} + q^{1+n} + q^{4+3n} + q^{3n+3} + q^{9+4n} - q^{5+4n} \\
& - q^{7+4n} - q^{10+5n} - q^{8+6n} + q^{4n+3} + q^{1+2n} - q^{6+4n} - q^{9+5n} \\
& - q^{2n+2} - q^{10+6n} - q^{7+3n} - q^{5n+4} + q^{11+7n} - 2q^{9+6n} + q^{10+7n}
\end{aligned}$$

$$
\begin{aligned}
&- q^{5n+5} + q^{6+2n} + q^{8+5n} - q^{2+3n})E + q^{2n+9}(q^{n+2}-1)(q^{2n+1}-1)\\
&\times (q^{2n+4}-1)(q^{2n+8}-1)(q^{2n+9}-1)(-1 - 2q^{13+6n} - q + q^{3+2n}\\
&- q^{8+4n} - q^{17+8n} + 2q^{16+7n} - 2q^{4+3n} + 2q^{12+4n} - 2q^{3n+9} - q^{18+7n}\\
&- 2q^{18+8n} - 4q^{9+4n} - q^{14+7n} - 2q^{10+4n} - q^{19+8n} + 2q^{2+n} + 2q^{7+4n}\\
&+ q^{10+5n} - q^{11+3n} + q^{15+6n} + q^{8+3n} + 4q^{5n+12} + q^{17+6n} + q^{8+2n}\\
&- 2q^{5+2n} - 2q^{10+3n} + q^{7+5n} + q^{17+7n} - q^{14+4n} + 2q^{6+4n} - q^{9+5n}\\
&- q^{13+7n} + q^{2n+2} - 2q^{14+6n} + q^{10+6n} + q^{4+n} + 5q^{11+5n} + 3q^{7+3n}\\
&+ 2q^{15+7n} + 2q^{3+n} - q^{8+5n} - 2q^{2n+4} + 2q^{11+4n} + 2q^{7+2n} + 2q^{11+6n}\\
&+ q^{20+9n} - q^{15+5n} - 2q^{5+3n} + 3q^{6n+16} - 2q^{14+5n} + q^{6+3n}\\
&+ q^{12+6n})E^2 - q^3(q^{n+3}-1)(q^{2n+1}-1)(q^{2n+2}-1)(q^{2n+6}-1)\\
&\times (q^{2n+9}-1)(-1 + q^{3+2n} + q^{8+4n} - 4q^{12+4n} + 2q^{3n+9} - q^{18+7n}\\
&+ q^{9+4n} - 2q^{8n+22} - q^{10+4n} - q^{24+8n} - 2q^{7n+22} - q^{18+6n} + q^{2+n}\\
&- q^{7+4n} - 2q^{17+5n} - 3q^{11+3n} + q^{6n+21} + 2q^{15+6n} + 2q^{8+3n}\\
&- 2q^{5n+12} + 2q^{20+7n} - 3q^{17+6n} + q^{8+2n} - 2q^{5+2n} - q^{10+3n} - q^{17+7n}\\
&+ 2q^{14+4n} + q^{4n+15} + q^{9n+25} + 2q^{14+6n} + q^{4+n} + 2q^{19+7n} - 2q^{11+5n}\\
&- q^{7+3n} - 2q^{16+5n} + 2q^{6n+20} - q^{23+7n} - 2q^{6+2n} + 2q^{3+n} + q^{2n+4}\\
&+ q^{5n+13} - 5q^{11+4n} - q^{7+2n} + 2q^{6n+19} + 2q^{15+5n} - q^{5+3n} - q^{3n+12}\\
&+ q^{19+5n} - q^{6n+16} + 4q^{14+5n} - 2q^{23+8n} + q^{9n+26} - 2q^{6+3n})E^3\\
&+ q(q^{n+4}-1)(q^{2n+1}-1)(q^{2n+2}-1)(q^{2n+3}-1)(q^{2n+8}-1)\\
&\times (-1 - q + q^{12+4n} + q^{4n+17} + q^{2n+9} + q^{17+5n} + q^{11+3n} - q^{6n+21}\\
&- q^{8+3n} - q^{8+2n} + q^{18+5n} + q^{5+2n} + q^{10+3n} - q^{14+4n} - q^{13+3n}\\
&- q^{5n+21} + 2q^{4+n} + q^{25+7n} - q^{16+5n} + q^{3+n} + q^{2n+4} - q^{14+3n}\\
&- q^{13+4n} + q^{n+5} + q^{3n+12} + q^{2n+10})E^4 + (q^{n+5}-1)(q^{2n+1}-1)\\
&\times (q^{2n+2}-1)(q^{2n+3}-1)(q^{2n+4}-1)E^5,
\end{aligned}
$$

$$
\begin{aligned}
B_3(q^n,q) &= q^{3n+12}(q^{n+1}+1)(q^{n+2}+1)(q^{n+3}+1)(q^{n+4}+1)(q^{2n+1}-1)\\
&\quad \times (q^{2n+3}-1)(q^{2n+5}-1)(q^{2n+7}-1)(q^{2n+9}-1),\\
A^{nh}_{-3}(E,Q,q) &= q^{6n+18}(-1+q^n)(q^{2n+7}-1)(q^{2n+8}-1)(q^{2n+9}-1)(q^{2n+10}-1)\\
&\quad \times (q^{2n+11}-1) - q^{4n+14}(q^{n+1}-1)(q^{2n+2}-1)(q^{2n+8}-1)\\
&\quad \times (q^{2n+9}-1)(q^{2n+10}-1)(q^{2n+11}-1)(1 + q + q^{5n+12} + q^{4+2n}
\end{aligned}
$$

$$
\begin{aligned}
&- q^{2n+7} - q^{13+7n} + q^{8+4n} - q^{2n+2} + q^2 - q^{5+n} + q^{8n+14} - q^{8+3n} \\
&- q^{2n+9} - q^{4+n} - q^{2+n} - q^{7+6n} - q^{8+2n} - q^{11+5n} - q^{1+2n} + q^{4+3n} \\
&- q^{7+3n} - q^{10+5n} + q^{11+6n} - q^{11+4n} + q^{9+4n} - q^{5+4n} + q^{6+5n} \\
&- 2q^{3+n} + q^{12+6n} - q^{13+6n} + q^{3n+3} - q^{5+3n} - q^{4n+4} + q^{6+4n} \\
&+ q^{10+3n})E + q^{2n+9}(q^{n+2} - 1)(q^{2n+1} - 1)(q^{2n+4} - 1)(q^{2n+9} - 1) \\
&\times (q^{2n+10} - 1)(q^{2n+11} - 1)(1 - q^{9n+24} + 3q^{10+4n} + q + q^{16+7n} \\
&- 3q^{8+4n} - q^{4+2n} - 3q^{4+n} - 2q^{6n+19} + 4q^{5+3n} + 3q^{15+7n} - 3q^{3n+9} \\
&+ q^{4+3n} - q^{11+6n} + q^{19+8n} + q^2 - q^{18+6n} + q^{24+10n} + q^{4n+17} \\
&- 2q^{12+6n} + 2q^{14+7n} + q^{25+10n} + q^{10+6n} + 3q^{17+5n} - q^{2n+9} \\
&- 2q^{9n+23} + 3q^{6n+16} - 2q^{3+2n} - 4q^{14+5n} + q^{18+5n} - q^{2n+10} - q^{2+n} \\
&+ q^{17+6n} + 2q^{11+5n} + 2q^{8n+21} - q^{23+8n} - 2q^{8+3n} - q^{2+2n} - 2q^{17+7n} \\
&+ 3q^{10+5n} + 3q^{3n+12} + 4q^{15+6n} + 2q^{5+2n} - q^{21+9n} + 2q^{9+5n} \\
&- q^{7+5n} + 2q^{7+2n} + q^{7n+22} - 2q^{5n+12} + 2q^{8n+20} - 3q^{18+7n} \\
&- q^{8n+16} + 3q^{6+3n} + q^{5+4n} - q^{6+n} + q^{13+4n} - 2q^{17+8n} + 6q^{11+4n} \\
&- 2q^{5+n} + q^{9+6n} + 2q^{13+3n} - 6q^{5n+13} - 3q^{3+n} + 4q^{12+4n} - 2q^{19+7n} \\
&- q^{6n+20} - 3q^{7+4n} + q^{14+3n} + 2q^{6+2n} + 2q^{16+5n} - q^{18+8n} + 3q^{11+3n} \\
&- 3q^{13+6n} + 2q^{7n+21} + q^{20+7n} - 2q^{22+9n} + q^{13+7n})E^2 - q^3 \\
&\times (q^{n+3} - 1)(q^{2n+1} - 1)(q^{2n+2} - 1)(q^{2n+6} - 1)(q^{2n+10} - 1) \\
&\times (q^{2n+11} - 1)(1 + 3q^{9n+24} - 4q^{10+4n} - 2q^{16+7n} - q^{8+4n} - q^{4+2n} \\
&- q^{4+n} + 7q^{6n+19} + q^{5+3n} - q^{15+7n} - 3q^{3n+9} + q^{19+8n} + 4q^{18+6n} \\
&- q^{28+10n} + q^{36+12n} - q^{4n+17} + 2q^{14+4n} + 7q^{25+8n} - 3q^{10+3n} \\
&- 3q^{17+5n} - 4q^{6n+21} + q^{9n+23} - 8q^{6n+16} - q^{3+2n} - q^{14+5n} - q^{2+n} \\
&- q^{17+6n} + q^{8+2n} + q^{11+5n} - 3q^{8n+21} + q^{23+8n} - 3q^{23+6n} + q^{10n+32} \\
&- q^{27+8n} - q^{18+4n} - 2q^{14+6n} + 3q^{31+10n} - q^{7n+28} + q^{17+7n} \\
&- q^{34+11n} - 2q^{10+5n} + q^{3n+12} - 2q^{33+11n} - 6q^{15+6n} + q^{5+2n} \\
&- q^{9+5n} + q^{9n+32} + 2q^{26+8n} - 3q^{9+4n} - 3q^{27+9n} - 4q^{8n+22} \\
&- 3q^{8n+28} + 3q^{7+2n} - q^{11n+32} - 8q^{7n+22} - 2q^{24+6n} + 4q^{5n+12} \\
&- q^{8n+20} + 4q^{18+7n} + 2q^{9n+31} - 3q^{9n+28} - q^{8n+30} + 3q^{6+3n} \\
&- q^{22+5n} - q^{8n+29} - q^{4n+15} + 7q^{13+4n} + q^{11+4n} - q^{5+n} + 2q^{13+3n}
\end{aligned}
$$

$$
\begin{aligned}
&+5q^{5n+13}+6q^{24+8n}-2q^{3+n}+6q^{12+4n}+5q^{19+7n}+3q^{10n+30}\\
&-6q^{22+6n}+q^{7+4n}+3q^{9n+25}+q^{14+3n}+3q^{6+2n}+2q^{19+5n}\\
&-8q^{16+5n}-q^{10n+27}-6q^{15+5n}+q^{13+6n}+q^{9n+30}-3q^{4n+16}\\
&+3q^{7+3n}-6q^{7n+21}-q^{20+7n}+q^{29+10n}-q^{35+11n}-3q^{23+7n}\\
&+2q^{25+7n})E^3+q(q^{n+4}-1)(q^{2n+1}-1)(q^{2n+2}-1)(q^{2n+3}-1)\\
&\times(q^{2n+8}-1)(q^{2n+11}-1)(1+4q^{24+6n}+q^{10+4n}+q-q^{33+8n}\\
&+3q^{30+7n}-q^{4+2n}-q^{26+8n}+q^{36+10n}-2q^{4+n}+2q^{31+7n}+2q^{15+5n}\\
&-3q^{3n+12}+3q^{3n+9}-4q^{20+5n}+q^{37+10n}+4q^{23+7n}-3q^{7n+27}\\
&+q^{25+6n}+q^{32+7n}-2q^{18+5n}+q^{4n+17}+3q^{23+5n}+q^{10+3n}\\
&-2q^{26+7n}+q^{38+10n}-q^{2n+11}+3q^{22+6n}+q^{17+6n}+q^{24+5n}\\
&+3q^{24+7n}+2q^{8+2n}-q^{6+n}+q^{29+6n}-2q^{27+8n}-q^{18+4n}+2q^{15+3n}\\
&-q^{34+8n}-2q^{13+3n}-3q^{9n+34}-3q^{6n+20}-2q^{5+2n}-q^{8n+28}\\
&-q^{9n+32}+q^{16+3n}+q^{9+4n}+q^{7+2n}+q^{7n+22}-6q^{19+5n}-q^{36+9n}\\
&+2q^{2n+9}+2q^{8n+30}-3q^{9n+33}+6q^{23+6n}+3q^{16+5n}+2q^{22+5n}\\
&+2q^{8n+29}+4q^{4n+15}-3q^{13+4n}-2q^{4n+19}-q^{11+4n}-2q^{5+n}\\
&-q^{5n+13}-q^{3+n}+3q^{7n+29}-2q^{12+4n}+3q^{4n+16}+2q^{17+5n}+q^{14+3n}\\
&-q^{20+4n}-q^{6+2n}-2q^{11+3n}-3q^{6n+19}+2q^{8n+31}+q^{7+3n}+2q^{8+3n}\\
&-2q^{35+9n})E^4-(q^{n+5}-1)(q^{2n+1}-1)(q^{2n+2}-1)(q^{2n+3}-1)\\
&\times(q^{2n+4}-1)(q^{2n+10}-1)(1-q^{15+3n}+q^{18+4n}-q^{31+6n}+q^{4n+19}\\
&-q^{23+6n}+q^{16+3n}-q^{14+3n}+q^{2n+9}+q^{2n+10}+q^{34+8n}+q^{20+5n}\\
&-q^{24+6n}+q^{26+5n}-q^{30+7n}-q^{32+7n}+q^{36+8n}+q^{26+6n}-q^{5+2n}\\
&+q^{19+5n}-q^{29+6n}-q^{6n+30}-q^{21+4n}-q^{7n+33}-q^{23+5n}-q^{5+n}\\
&-q^{24+5n}+q^{35+8n}-q^{11+2n}-q^{5n+21}-2q^{31+7n}+q^{4n+16}-q^{4n+15}\\
&+q^{10+3n}-q^{14+4n})E^5+q^{6n+30}(q^{n+6}-1)(q^{2n+1}-1)(q^{2n+2}-1)\\
&\times(q^{2n+3}-1)(q^{2n+4}-1)(q^{2n+5}-1)E^6,
\end{aligned}
$$

$$
\begin{aligned}
B_{-3}(q^n,q)&=q^{3n+15}(q^{n+1}+1)(q^{n+2}+1)(q^{n+3}+1)(q^{n+4}+1)(q^{n+5}+1)\\
&\quad\times(q^{2n+1}-1)(q^{2n+3}-1)(q^{2n+5}-1)(q^{2n+7}-1)\\
&\quad\times(q^{2n+9}-1)(q^{2n+11}-1).
\end{aligned}
$$

For a computer data of the non-commutative A-polynomial of twist knots, see [12].

Appendix C. The A-Polynomial for $p = -3, \ldots, 3$

For comparison, we give a formula of the A-polynomial $A_p(L, M)$ of the twist knot K_p, taken from [15].

p	$A_p(L, M)$
1	$L + M^6$
-1	$-L + LM^2 + M^4 + 2LM^4 + L^2M^4 + LM^6 - LM^8$
2	$-L^2 + L^3 + 2L^2M^2 + LM^4 + 2L^2M^4 - LM^6 - L^2M^8 + 2LM^{10}$
	$+L^2M^{10} + 2LM^{12} + M^{14} - LM^{14}$
-2	$L^2 - L^3 - 3L^2M^2 + L^3M^2 - 2LM^4 - L^2M^4 + 3LM^6 + 3L^2M^6 + M^8$
	$+3LM^8 + 6L^2M^8 + 3L^3M^8 + L^4M^8 + 3L^2M^{10} + 3L^3M^{10} - L^2M^{12}$
	$-2L^3M^{12} + LM^{14} - 3L^2M^{14} - LM^{16} + L^2M^{16}$
3	$L^3 - 2L^4 + L^5 - 4L^3M^2 + 4L^4M^2 - 2L^2M^4 + 2L^3M^4 + 3L^4M^4$
	$+5L^2M^6 + 5L^3M^6 + LM^8 + L^2M^8 + 6L^3M^8 - LM^{10} - 4L^2M^{10}$
	$-4L^3M^{12} - L^4M^{12} + 6L^2M^{14} + L^3M^{14} + L^4M^{14} + 5L^2M^{16}$
	$+5L^3M^{16} + 3LM^{18} + 2L^2M^{18} - 2L^3M^{18} + 4LM^{20} - 4L^2M^{20}$
	$+M^{22} - 2LM^{22} + L^2M^{22}$
-3	$-L^3 + 2L^4 - L^5 + 5L^3M^2 - 6L^4M^2 + L^5M^2 + 3L^2M^4 - 6L^3M^4$
	$-10L^2M^6 - 5L^3M^6 + 4L^4M^6 - 3LM^8 - 3L^3M^8 + 5LM^{10} + 12L^2M^{10}$
	$+10L^3M^{10} + M^{12} + 4LM^{12} + 10L^2M^{12} + 20L^3M^{12} + 10L^4M^{12}$
	$+4L^5M^{12} + L^6M^{12} + 10L^3M^{14} + 12L^4M^{14} + 5L^5M^{14} - 3L^3M^{16}$
	$-3L^5M^{16} + 4L^2M^{18} - 5L^3M^{18} - 10L^4M^{18} - 6L^3M^{20} + 3L^4M^{20}$
	$+LM^{22} - 6L^2M^{22} + 5L^3M^{22} - LM^{24} + 2L^2M^{24} - L^3M^{24}$

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