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THE MYSTERY OF THE BRANE RELATION

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ABSTRACT

Using the notion of surgery on objects called Y-graphs and claspers by Goussarov and Habiro, one can define a theory of finite type invariants of closed 3-manifolds. The paper discusses upper bounds for the number of invariants, and focuses on two surprises that arise: One surprise is that the upper bounds depend on a bit more than a choice of generators for H_1. A complementary surprise a curious brane relation (in two flavors, open and closed) which shows that the upper bounds are in a certain sense independent of the choice of generators of H_1.

Keywords: Finite type invariants; Goussarov-Habiro; claspers, brane relation.

1. Introduction

1.1. *Motivation*

It is well-known that starting from a *move* (often described in terms of surgery) on a set of knotted objects (such as knots, links, braids, tangles, 3-manifolds, graphs), one can define a theory of *finite type invariants*. The question of how many invariants are there in any degree gets divided into two separate questions: one that provides upper bounds for the number of invariants, and one that provides lower bounds. Traditionally, upper bounds are obtained by providing a set of topological relations among the moves, whereas lower bounds are obtained by constructing (by quite different means) invariants.

In the paper we consider the theory of finite type invariants based on the move of surgery along objects called Y-*graphs* or *claspers* by Goussarov and Habiro (see [Gu, Ha] and also [GGP]) and study upper bounds for the number of invariants.

Following the notation of [GGP], let us briefly recall that given a Y-graph G in a manifold M, then M_G denote the result of surgery on M along G. Consider the set $\mathcal{S}(M)$ of (isomorphism classes of) 3-manifolds obtained by surgery along a disjoint union of Y-graphs in M, and the free abelian group $\mathcal{F}^{\mathrm{Y}}(M)$ on $\mathcal{S}(M)$. There is a decreasing filtration on $\mathcal{F}^{\mathrm{Y}}(M)$, where $\mathcal{F}^{\mathrm{Y}}_n(M)$ is the subgroup generated by

$$[M, G] = \sum_{G' \subset \{G_1, \dots, G_n\}} (-1)^{|G'|} M_{G'}$$

for all disjoint unions $G = G_1 \cup \ldots G_n$ of Y-graphs in M, where $|G'|$ denotes the cardinality of the set G'.

Dually, and perhaps more naturally, this filtration allows us to call a function λ on $\mathcal{S}(M)$ with values in an abelian group a *finite type invariant of type* n iff its extension on $\mathcal{F}^{\mathrm{Y}}(M)$ satisfies $\lambda(\mathcal{F}^{\mathrm{Y}}_{n+1}(M)) = 0$. Thus, the question of how many finite type invariants of type n are there translates into a question about the structure of the (graded quotient) abelian groups $\mathcal{G}^{\mathrm{Y}}_n(M) \overset{\mathrm{def}}{=} \mathcal{F}^{\mathrm{Y}}_n(M)/\mathcal{F}^{\mathrm{Y}}_{n+1}(M)$. For the case of $M = S^3$ (or any other integral homology 3-sphere), it is well-known that the topological calculus of Y-graphs or claspers developed indepedently by Goussarov and Habiro, implies the existence of upper bounds of $\mathcal{G}^{\mathrm{Y}}(M)$ in terms of an abelian group $\mathcal{A}(\phi)$ generated by (abstract) trivalent graphs, modulo the well known antisymmetry AS and IHX relations, see for instance [GGP, Sec. 4]. The case of arbitrary 3-manifolds M (needed for instance in [GL, Theorems 5–7]) seems to be missing from the literature, even though the main tools are the same as in the case of $M = S^3$. There are, however, two surprises in extending the above upper bound to all closed 3-manifolds, which are the main point of this paper: one is that the upper bound for $\mathcal{G}^{\mathrm{Y}}(M)$ is given in terms of a finitely generated (in each degree) abelian group $\mathcal{A}^{\mathrm{o}}(b)$ defined below, where b is a H_1-*spanning* link i.e., an oriented framed link in M that generates (possibly with redundances) $H_1(M, \mathbb{Z})$, see Theorem 1. In other words, the generators of $\mathcal{A}^{\mathrm{o}}(b)$ *depend* on just a bit more than a choice of generators for $H_1(M, \mathbb{Z})$, they depend on a choice of 1-cycles. The other surprise is the existence of a new relation in $\mathcal{A}^{\mathrm{o}}(b)$, the *open brane* (OBR) and the *closed brane* (BR) relation, which is also given in terms of a choice of embedded 2-cycles in M.

Of course the choice of b is not unique, and the choice of cycles in the OBR relation is not unique, however the OBR and BR relations imply that any two such choices b and b' lead to rather canonical isomorphisms between $\mathcal{A}^{\mathrm{o}}(b)$ and $\mathcal{A}^{\mathrm{o}}(b')$ as well as commutative diagrams, see Theorem 1.

If one is willing to work with rational coefficients, then the above upper bound $\mathcal{A}^{\mathrm{o}}(b)$ can be identified with an invariant $\mathcal{A}$-group $\mathcal{A}(H(M))$ that depends only on the cohomology ring $H^*(M, \mathbb{Q})$ of M, see Corollary 1.4 (although the map $\mathcal{A}(H(M)) \to \mathcal{G}^{\mathrm{Y}}(M)$ still depends on a choice of a H_1-spanning link b).

As a final comment before the details, we should mention that for finite type invariants of integral homology 3-spheres, or for $\mathbb{Q}$-valued finite type invariants of rational homology spheres the above mentioned choices of 1-cycles and 2-cycles are invisible, which partly explains why they were not discovered so far.

1.2. *Statement of the results*

Throughout, by *graph* we mean we mean one with (symmetric) univalent and trivalent vertices, together with a choice of cyclic order on each trivalent vertex. Note that graphs that contain struts, i.e., an interval with two univalent vertices and no trivalent ones, will *not* be allowed here. Univalent vertices of graphs will often be called *legs* or *leaves*. Given a set X, an X-*colored graph* is a graph G together with a function $c : \mathrm{Legs}(G) \to X$. This assignment can be extended linearly to include graphs whose univalent vertices are assigned a nonzero formal linear combination of elements of X. Below we will discuss L-colored graphs (really, $\pi_0(L)$-colored graphs), where L is some auxiliary link.

Let $\mathcal{B}(X)$ denote the abelian group spanned by X-colored graphs modulo the well-known AS, IHX and LOOP relations shown in Fig. 1. $\mathcal{B}(X)$ is graded, by declaring the degree of a graph to be the number of its trivalent vertices.

Notice that the group $\mathcal{B}(X)$ is closely related to a group that appears when one studies finite type invariants of X-component links in S^3, with some notable differences: one is that we do not allow struts, another is that we do not grade by half the number of vertices, and the third is that we allow graphs with no legs.

Notice also that the AS relation implies that $2\mathrm{LOOP} = 0 \in \mathcal{B}(X)$, which can be ignored when inverting 2.

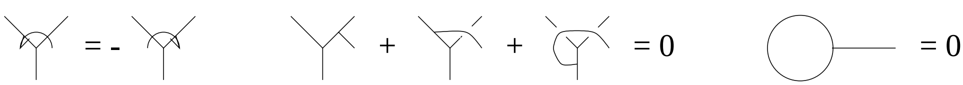


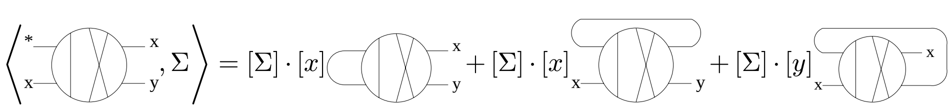
Fig. 1. The AS, IHX (all trivalent vertices oriented counterclockwise), and LOOP relations. In the LOOP relation, the appearing loop is an edge and not a leaf of the graph.

Given a H_1-spanning link b, we now define two important relations on $\mathcal{B}(b)$. Let $\cdot : H_2(M, \mathbb{Z}) \otimes H_1(M, \partial M, \mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing.

Definition 1.1. Fix a closed surface Σ in M. Let $(G, *)$ be b-colored graph, which contains a special leg colored by the special symbol $*$ (disjoint from the alphabet b). Let

$$\langle G, \Sigma \rangle := \sum_l [\Sigma] \cdot [c_l] G_l \in \mathcal{B}(b)$$

where the summation is over all legs of G except $*$ and where G_l is the result of gluing the $*$-leg of G to a c_l-colored leg of G, as shown in the following example



By convention, the summation over the empty set equals to zero. The BR (*closed brane*) relation[a] is the subgroup of $\mathcal{B}(b)$ generated by $\langle G, \Sigma \rangle = 0$ for all surfaces Σ, or really, only a generating set for $H_2(M, \mathbb{Z})$ and all graphs $(G, *)$ as above. Let $\mathcal{A}(b) = \mathcal{B}(b)/(\mathrm{BR})$.

[a]Which does not seem to be related in any meaningful way to the wonderful (mem)branes of string theory.

Definition 1.2. Fix a b-colored graph that contains a distinguished leg $*$ colored by a nullhomologous label c_0 which bounds a surface Σ_0 in M. Let

$$\langle G, \Sigma_0 \rangle := G + \sum_l [\Sigma_0] \cdot [c_l] G_l \in \mathcal{B}(b)$$

where the summation is over all legs of G except $*$ and where G_l is the result of gluing the $*$-leg of G to a c_l-colored leg of G. The OBR (*open brane*) relation is the subgroup of $\mathcal{B}(b)$ generated by $\langle G, \Sigma_0 \rangle = 0$ for all graphs G as above and all surfaces Σ_0. Note that the OBR subgroup of $\mathcal{B}(b)$ includes the BR subgroup if we assume that one of the components of b is the boundary of an embedded disk disjoint from the rest of the components of b. Let $\mathcal{A}^{\mathrm{o}}(b) = \mathcal{A}(b)/(\mathrm{OBR})$.

Theorem 1.

(i) *For every H_1-spanning link b in a manifold M, there is a group homomorphism*

$$W_{M,b} : \mathcal{A}^{\mathrm{o}}(b) \to \mathcal{G}^{\mathrm{Y}}(M)$$

which is onto, once tensored with $\mathbb{Z}[1/2]$.

(ii) *For every two H_1-spanning links b and b' in M, there are isomorphisms $W_{M,b,b'} : \mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b')$ over $\mathbb{Z}[1/2]$, such that:*

$$W_{M,b} = W_{M,b'} \circ W_{M,b,b'} \,. \tag{1}$$

1.3. *The size of $\mathcal{A}^{\mathrm{o}}(b)$*

It is natural to ask how big is the (finitely generated in each degree) abelian group $\mathcal{A}^{\mathrm{o}}(b)$ which bounds from above $\mathcal{G}^{\mathrm{Y}}(M)$.

Corollary 1.3.

(i) *If $H_1(M, \mathbb{Z})$ is torsion-free and b is a basis of H_1, then*

$$\mathcal{A}(b) \cong \mathcal{A}^{\mathrm{o}}(b).$$

(ii) *If b is $H_{1,\mathbb{Q}}$-basis and b' is H_1-spanning then*

$$\mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}^{\mathrm{o}}(b') \,.$$

(iii) *If $H_1(M, \partial M, \mathbb{Q}) = 0$, then for every H_1-spanning b we have*

$$\mathcal{B}(b) \cong \mathcal{A}(b) \,.$$

(iv) *In particular, for M a rational homology 3-sphere, we have that*

$$\mathcal{A}^{\mathrm{o}}(b) \cong_{\mathbb{Q}} \mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}(\phi) \,.$$

(v) *For M a homology-cylinder (i.e., a manifold with the same integer homology as that of $\Sigma \times I$ for a surface Σ with one boundary component) and a $H_{1,\mathbb{Q}}$-basis b, we have that*

$$\mathcal{B}(b) \cong_{\mathbb{Q}} \mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}^{\mathrm{o}}(b) \,.$$

If we are willing to work with rational coefficients, then one can define in an invariant way a group of graphs, that depends only on the cohomology ring $H^*(M, \mathbb{Q})$ as follows: $\mathcal{A}(H(M))$ is generated by graphs colored by nonzero elements of $H_1(M, \mathbb{Q})$, modulo the AS, IHX, LOOP and BR relations.

Corollary 1.4. *For every manifold M, there is a map*

$$\mathcal{A}(H(M)) \to \mathcal{G}^{\mathrm{Y}}(M),$$

onto over $\mathbb{Q}$.

For manifolds M with $b_1(M) = 0$, i.e., for rational homology 3-spheres, we show a promised *universal property* of the LMO invariant restricted to the set of rational homology spheres [LMO], or of its cousin, the Aarhus integral [A]:

Theorem 2. *The LMO invariant is the universal $\mathbb{Q}$-valued finite type invariant of rational homology spheres. In particular, for M a rational homology 3-sphere and b H_1-spanning, we have $\mathcal{A}^{\mathrm{o}}(b) \cong_{\mathbb{Q}} \mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}(\phi) \cong_{\mathbb{Q}} \mathcal{G}^{\mathrm{Y}}(M)$.*

With regards to the size of $\mathcal{A}(\phi)$, it is well-known that Lie algebras and their representation theory provides lower bounds for the abelian groups $\mathcal{A}(\phi)$. In the case of manifolds M with positive betti number, we do not know yet of lower bounds for $\mathcal{A}^{\mathrm{o}}(b)$. The little we know at present is the following:

Corollary 1.5. *Let b be a H_1-spanning link in a closed manifold M and G be a graph colored by a sublink b' of b. Assume that G has an internal edge, that is an edge between two trivalent vertices of G. If b' is not H_1-spanning (over $\mathbb{Q}$), then $G = 0 \in \mathcal{A}^{\mathrm{o}}(b)$.*
In particular, if $b_1(M) > 0$, then every graph without legs vanishes in $\mathcal{A}^{\mathrm{o}}(b)$.

Corollary 1.6. *Let b be a H_1-spanning link in a closed manifold M and G be a graph whose $r+1$ legs are colored by $x, y_1, \ldots, y_r$ so that x is primitive and linearly independent from $\{y_1, \ldots, y_r\}$. For $k = 0, \ldots, r$, let $G^{(k)}$ denote the sum of all ways of replacing k many y_i by x. Assume that G contains an internal edge. Then $G^{(k)} = 0$ in $\mathcal{A}^{\mathrm{o}}(b)$ for all k.*

We caution the reader that the above corollary by no means implies that $\mathcal{A}^{\mathrm{o}}(b)$ is zero dimensional for manifolds M with positive betti number, since for example, for manifolds with positive betti number, the coefficients of the Alexander polynomial (of the maximal torsion-free abelian cover) are finite type invariants in our sense.

2. Proofs

The proofs of the theorems and their corollaries involve algebraic alternatings of the *topological calculus of clovers*; the uninitiated reader may also look at [GGP, Sec. 3]. Clovers are mild generalizations of Y-graphs; a clover of degree 1 is by definition a Y-graph and surgery on a clover of degree n corresponds to surgery on a disjoint union of n Y-graphs. Thus, one need never talk about clovers; in that case surgery on an embedded Θ graph correponds to surgery on a disjoint union of two Y-graphs whose leaves link pairwise like a Hopf link. Sentences like the above will hopefully make the reader appreciate clovers; in addition clovers will be assist in a quick definition of the map $W_{M,b}$ of Theorem 1 as well as a motivation for the validity of Theorem 1. In view of this, we will use them freely in what follows.

Before we prove the theorems, it will be important to state some lemmas the proof of which follows by applying to the topological calculus of clovers elementary alternations, see for example [GGP, Sec. 4.1]:

Lemma 2.1. [Ha, Gu] **(Cutting a Leaf)** *Let G be a clover of degree m in a manifold M and L be a leaf of G. An arc a starting in the external vertex incident to L and ending in other point of L, splits L into two arcs L' and L''. Denote by G' and G'' the graphs obtained from G by replacing the leaf L with $L' \cup a$ and $L'' \cup a$ respectively, see Fig. 2. Then $[M, G] = [M, G'] + [M, G'']$ in $\mathcal{G}^{\mathrm{Y}}_m(M)$.*

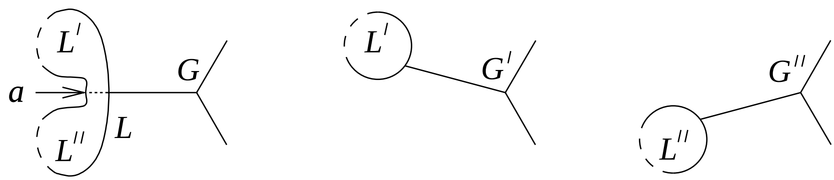
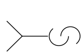
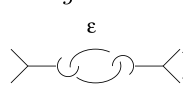
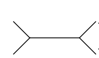


Fig. 2. Splitting a leaf.

Lemma 2.2. [Gu, Ha] **(Sliding an Edge)** *Let G be a clover of degree m in a manifold M, and let G' be obtained from G by sliding an edge of G along a tube in M. Then $[M, G] = [M, G']$ in $\mathcal{G}^{\mathrm{Y}}_m(M)$.*

The next lemma involves some mixed objects i.e., pairs (L, G) of a (framed) link L and a clover G in M. In this case, alternating means to alternate with respect sublinks of L and components of G.

Lemma 2.3. [Gu, Ha] *For all $\varepsilon = \pm 1$, consider an ε-framed unknot and a clover in M shown below. Then, we have the following identities in $\mathcal{G}^{\mathrm{Y}}(M)$:*

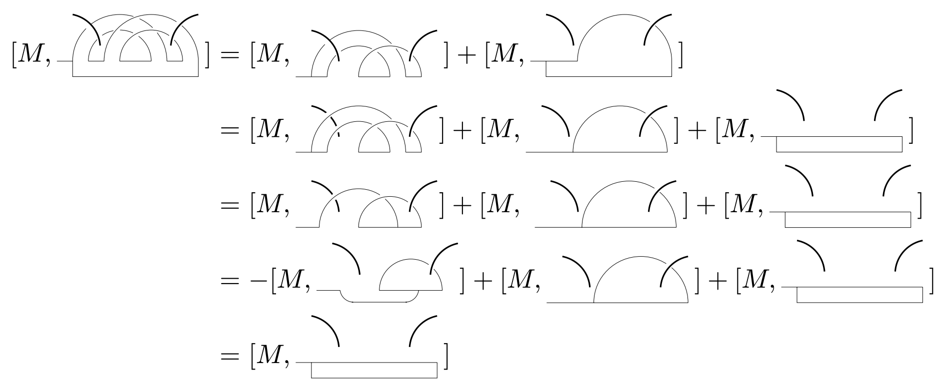
2[M,  ε] = 0 *and* 2[M, ] = -2ε[M, ] .

Lemma 2.4. *Let G be a clover with $r+1$ leaves l_i for $i = 0, \ldots, r$ in a manifold M. Assume that l_0 bounds an embedded surface Σ_0 in M. Then*

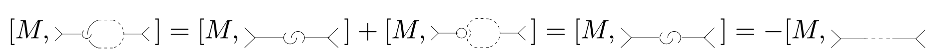
$$G + \sum_{i=1}^{r} [\Sigma_0] \cdot [l_i] G_i = 0 \in \mathcal{G}^{\mathrm{Y}}(M)$$

where G_i is the result of gluing the 0-th leg of G to its i-th leg.

Proof. Consider a graph G and a surface Σ_0 as above. Σ_0 can be thought of as an embedded disk with bands. We can assume that G is disjoint from the (interiors of the bands) of Σ_0 and thus G intersects the (interior of) Σ_0 only in the embedded disk. Cut each band along arcs (in the normal direction to the core of the band) using the Cutting and Sliding Lemmas 2.1 and 2.2 as shown



(where Σ_0 is a surface of genus 1, and the solid arcs represent arbitrary tubes in the 3-manifold). The above calculation reduces to the case of a surface Σ_0 hawith no bands, i.e., a disk. Using the Cutting and Sliding Lemmas 2.1 and 2.3 once again, we may assume that the leaf l_0 of G is zero-framed and that the disk Σ_0 intersects geometrically once a leaf of G and is otherwise disjoint from G. The following equality



(2)

which follows by Lemma 2.1, concludes our proof. □

Lemma 2.5. *Let (G, γ) be a clover in a manifold M together with a distinguished leaf γ that bounds two surfaces Σ_0 and Σ_1 in M. Then,*

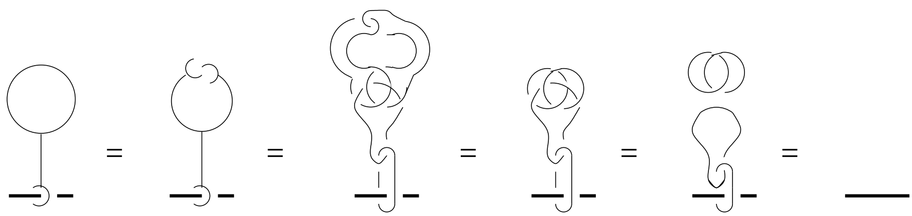
$$\langle G, \Sigma_0 - \Sigma_1 \rangle = 0 \in \mathcal{G}^{\mathrm{Y}}(M)\,.$$

Proof. It follows from two applications of the OBR relation that

$$-G = \langle G, \Sigma_0 \rangle = \langle G, \Sigma_1 \rangle \in \mathcal{G}^{\mathrm{Y}}(M)\,.$$

□

Proof of Theorem 1. First we construct the map $W_{M,b}$. Let $b = (b_1, \ldots, b_r)$ be a H_1-spanning link in M. Given a graph G with colored legs choose an arbitrary embedding of it in M. For every coloring $\sum_i a_i b_i$ of each of its univalent vertices (where a_i are integers), push $|a_i|$ disjoint copies of b_i (using the framing of b_i), orient them the same (resp. opposite) way from b_i if $a_i \geq 0$ (resp. $a_i < 0$), and finally take an arbitrary band sum of them. We can arrange the resulting knots, one for each univalent vertex of the embedding of G, to be disjoint from each other, and together with the embedding of G to form an embedded graph with leaves in M. Although the isotopy class of the embedded graph depends on the choices made, the image of $[M,G] \in \mathcal{G}^{\mathrm{Y}}_m(M)$ is well-defined. This follows from Lemma 2.1. We need to show that the relations AS, IHX, OBR, BR and LOOP are mapped to zero, which will define our map $W_{M,b}$. For the AS and IHX relations, see for example [GGP, Section 4.1]. The LOOP relation follows from Fig. 2.



The OBR and BR relations follow from Lemmas 2.4 and 2.5.

We now show that $W_{M,b}$ is onto, over $\mathbb{Z}[1/2]$. Note first that $\mathcal{G}^{\mathrm{Y}}_m(M)$ is generated by $[M,G]$ for all simple graphs of degree m, where a *simple graph* is a disjoint union of graphs of degree 1. Each of the leaves of G are isotopic to some connected sum of (possibly orientation reversed) components of b and contractible knots. Using the Cutting Lemma 2.1, we may assume that each leaf is isotopic to one of the components of b (with possibly reversed orientation) or is contractible in M. From this point on, the proof is analogous to the case of $M = S^3$. Let L be the link consisting of all contractible leaves of G. There exists a trivial, unit-framed link C in $M \smallsetminus (G \smallsetminus L)$ with the properties that

- each component of C bounds a disk that intersects L at at most two points.
- Under the diffeomorphism of M with M_C, L becomes a zero-framed unlink bounding a disjoint collection of disks D_i.

Such a link C was called *L-untying* in [GGP]. Lemma 2.1 and Eq. (2) imply that we can assume each of the disks D_i are disjoint from G and intersect C in at most two points C. Lemma 2.3 imply that $W_{M,b}$ is onto, over $\mathbb{Z}[1/2]$.

In order to show that $\mathcal{A}^{\mathrm{o}}(b)$ is independent of b, up to isomorphism, we need the following:

Lemma 2.6. *Every two H_1-spanning links b and b' in M are equivalent by a sequence of moves*:

M1: *Add one component* (*after possibly changing its orientation*) *of b to another.*
M2: *Change the framing of a component of b.*
M3: *Insert or delete a null-homologous zero-framed component of b.*

Proof. It suffices to show that under these moves b is equivalent to $b \cup b'$. Consider a component b'_i of b'. Since b is a basis of $H_1(M,\mathbb{Z})$, we can add a multiple of components of b (after perhaps changing their orientation) so that b'_i is nullhomologous, in which case we can change its framing to zero, and erase it. The lemma now follows by induction on the number of components of b'. □

Proof of Theorem 1 (Continue). If b' is obtained from b by applying one of the three moves above, we will now define $W_{M,b,b'} : \mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b')$ (abbreviated by $W_{b,b'}$ in what follows) and show that Eq. (1) holds.

For the first move, if $b = (b_1, b_2, \dots, b_r)$ and $b' = (b_1 \sharp b_2, b_2, \dots, b_r)$ (where $b_1 \sharp b_2$ is an arbitrary oriented band sum of b_1 with b_2) then $W_{b,b'}$ sends a b_1 colored vertex of an abstract graph G to a $b_1 \sharp b_2 - b_2$ colored vertex of G. It is easy to see that this defines a map $\mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b')$ whose inverse sends a $b_1 \sharp b_2$ colored vertex of G to a $b_1 + b_2$ colored vertex of G. Similarly, one can define a map $W_{b,b''}$ where $b' = (b_1 \sharp \bar{b}_2, b_2, \dots, b_r)$. Equation (1) follows from Lemma 2.1.

For the second move, let $b = (b_1, b_2, \dots, b_r)$ and $b' = (b'_1, b_2, \dots, b_r)$ where b'_1 is a knot whose framing differs from that of b_1 by $\varepsilon = \pm 1$. For graph G with n legs colored by b_1 we define

$$W_{b,b'}(G) = \sum_{I : |I| = \text{even}} \varepsilon^{|I|/2} G'_I$$

where the summation is over all functions $I : \{1, \dots, n\} \to \{0, 1\}$ such that the cardinality I of $I^{-1}(1)$ is even and G'_I is the result of gluing the b_1 colored legs l_i of G for which $I(i) = 1$ pairwise and recoloring the remaining b_1 colored legs with b'_1 colored legs. It is easy to see that $W_{b,b'}$ is well-defined (i.e., that it respects the relations in $\mathcal{A}^{\mathrm{o}}(b)$) and that its inverse is given by

$$W_{b,b'}(G') = \sum_{I : |I| = \text{even}} (-\varepsilon)^{|I|/2} G_I \, .$$

Let C denotes a $(-\varepsilon)$-framed unknot in M which bounds a disk that geometrically intersects b_1 in one point and intersects no other components of b. Then M_C is diffeomorphic to M under a diffeomorphism that sends the image of b in M_C to b' in M. Since $W_{M,b'}(G) = [M_C, G]$ and $W_{M,b}(G_I) = [M, G_I]$, Eq. (1) (or rather, its equivalent form $W_{b'} = W_b \circ W_{b',b}$) follows from the following:

Lemma 2.7. *For a graph G of degree m as above, we have in $\mathcal{G}^{\mathrm{Y}}_m(M)$:*

$$[M_C, G] = \sum_{I : |I| = \text{even}} (-\varepsilon)^{|I|/2} [M, G_I].$$

Proof. Using the Cutting Lemma 2.1 each b_1-colored leaf l_i of G can be split along an arc in two leaves; one that bounds a disk D_i intersecting C once and disjoint from b, and another that is isotopic to b_1 but disjoint from C. For $I : \{1, \dots, n\} \to \{0, 1\}$, let G'_I denote the graph $(G \smallsetminus (b_1 \text{ colored leaves of } G)) \cup \cup_{i : I(i) = 1} D_i$. Lemma 2.1 implies that $[M_C, G] = \sum_I [M_C, G'_I]$. Let G''_I denote the graph in M that corresponds to G'_I under the diffeomorphism $M = M_C$; we obviously have $[M_C, G'_I] = [M, G''_I]$.

Note that G''_I has a collection of $|I|$ leaves each of which is unknotted bounding a disk with linking number ε with every other leaf of this collection. An application of Lemma 2.4 $|I|$ times together with Lemma 2.3 implies that $[M_C, G''_I] = (-\varepsilon)^{|I|/2}[M, G_I]$ (resp. 0) for even (resp. odd) $|I|$. □

For the third move, let $b = (b_1, b_2, \dots, b_r)$ and $b' = (b_0, b_1, b_2, \dots, b_r)$ where b_0 is a null-homologous zero-framed knot, and consider the natural map $W_{b,b'} : \mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b')$. Choose a surface Σ_0 that b_0 bounds. The OBR relation in $\mathcal{A}^{\mathrm{o}}(b')$ for b_0 colored vertices defines a map $W_{b',b} : \mathcal{A}^{\mathrm{o}}(b') \to \mathcal{A}^{\mathrm{o}}(b)$; this map is independent of Σ_0 since the difference between two choices of Σ_0 equals to a choice of a closed surface and the resulting difference vanishes due to the BR relation on $\mathcal{A}^{\mathrm{o}}(b)$. It is easy to see that $W_{b',b}$ is inverse to $W_{b,b'}$. Equation (1) follows essentially by definition. This completes the proof of Theorem 1. □

Proof of Corollary 1.3. The first statement follows immediately from the fact that if b is a basis then no nontrivial linear combination is nullhomologous, thus the OBR relation is vacuous.

For the second statement, since we are using $\mathbb{Q}$ coefficients, we may assume that the link b is a basis for $H_1(M, \mathbb{Z})/(\text{torsion})$, and choose a link b^t to span the torsion part of $H_1(M, \mathbb{Z})$. Then, we have that $\mathcal{A}(b) = \mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b \cup b^t)$. There are integers n_i and surfaces Σ_i such that $n_i b^t_i = \partial\Sigma_i$ for all components of b^t. The OBR relation for b^t colored legs gives a map $\mathcal{A}^{\mathrm{o}}(b \cup b^t) \to \mathcal{A}^{\mathrm{o}}(b)$ which is independent of the choices of $\{n_i, \Sigma_i\}$ and is inverse to the map $\mathcal{A}^{\mathrm{o}}(b) \to \mathcal{A}^{\mathrm{o}}(b \cup b^t)$. Thus, $\mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}^{\mathrm{o}}(b \cup b^t)$. Since $\mathcal{A}^{\mathrm{o}}(b \cup b^t) \cong \mathcal{A}^{\mathrm{o}}(b')$ for every H_1-spanning link b', the result follows.

The third statement follows immediately from the fact that if the intersection form on M vanishes, then the BR relation is vacuous.

The forth and fifth statements are immediate consequences of those above. □

Proof of Corollary 1.4. Let b' be a H_1-spanning link and b be a $H_{1,\mathbb{Q}}$-basis. Then, we have over $\mathbb{Q}$

$$\mathcal{A}(H(M)) \cong_{\mathbb{Q}} \mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}^{\mathrm{o}}(b') \to \mathcal{G}^{\mathrm{Y}}(M)$$

which concludes the proof of the corollary. □

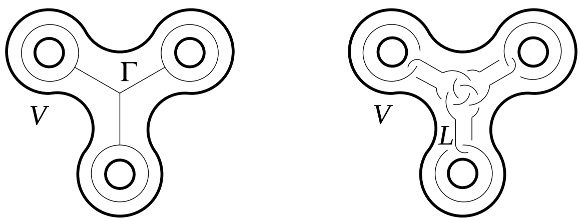
Proof of Theorem 2. The proof is a simple application of the *locality property* of the Kontsevich integral, as explained leisurely in [A, II, Sec. 4.2], and a simple counting argument.

We now give the details. We need to show that

- The part of the LMO=Aarhus integral $Z \in \mathcal{A}(\phi)$ of degree at most n is an invariant of type n.
- For a trivalent graph G of degree n in a rational homology 3-sphere M, we have that

$$Z(M_G) = G + \text{higher degree diagrams } \in \mathcal{A}(\phi).$$

For the first claim, recall that a degree 1 clover G in a manifold M is the image of an embedding $V \to M$ of a neighborhood V of the standard (framed) graph Γ of $\mathbb{R}^3$, and that surgery of M along G can be described as the result of Dehn surgery on the six component link L in V shown below



L is partitioned in three blocks L_1, L_2, L_3 of two component links each. We call each block an *arm* of G. Alternating a rational homology 3-sphere M with respect to surgery on G equals to alternating M with respect to all nine subsets of the set of arms of G.

Recall also that the Kontsevich integral of a framed link L in a 3-manifold M $Z(M, L)$ (defined by Kontsevich for links in S^3 and extended by Le-Murakami-Ohtsuki for links in arbitrary 3-manifolds [LMO, Sec. 6.2]) takes values in linear combinations of L-colored uni-trivalent graphs.

Recall also that the LMO=Aarhus integral of a rational homology 3-sphere M_L (obtained by surgery on a framed link L in a rational homology 3-sphere M) is obtained by considering the Kontsevich integral $Z(M, L)$, splitting it in a quadratic Z^q and trivalent (a better name would be “other”) part Z^t, and gluing the L-colored legs of Z^t using the inverse linking matrix of L.

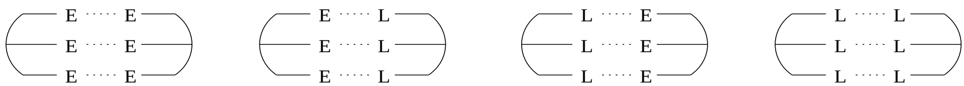
Given a clover $G = \cup_{i=1}^n G_i$ in a rational homology 3-sphere M, (where G_i are of degree 1), let L^{act} denote the link that consists of the $3n$ arms of G. When we compute $Z([M, G]) = Z([M, L^{\mathrm{act}}])$, we need to concentrate on all L^{act}-colored uni-trivalent graphs that have at least one univalent vertex on each block of G. Such graphs will have at least $3n$ univalent vertices. Since at most three univalent vertices can share a trivalent vertex, it follows that the above considered graphs will have at least n trivalent vertices; in other words it follows that $Z([M, G]) \in \mathcal{A}_{\geq n}(\phi)$.

The second claim is best shown by example. Recall that surgery on the (generic trivalent graph) Θ shown below corresponds to surgery on two clovers G_1 and G_2, each with arms $\{E_{ij}, L_{ij}\}$ for $i = 1, 2$ and $j = 1, 2, 3$. The linking matrix of the 12 component link $L^{\mathrm{act}} = E_{ij} \cup L_{ij}$ and its inverse are given by

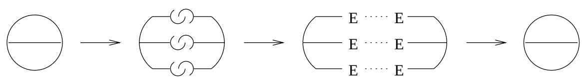
$$\begin{pmatrix} 0 & I \\ I & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -I & I \\ I & 0 \end{pmatrix}$$

where I is the identity 6×6 matrix. The relevant trivalent part $Z^t(M, L^{\mathrm{act}})$ is shown schematically in four cases here, where the graphs on the left terms of each case come from G_1 and the graphs on the right terms of each case come from G_2

and the dashed lines correspond to gluings of the univalent vertices:



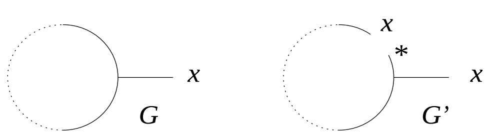
However, the last three cases all contribute zero, since LLL is a 3-component unlink whose coefficient in Z^t is a multiple of the triple Milnor invariant and thus vanishes. Thus, we are only left to glue terms in the first case, and this is summarized in the following figure



which concludes the proof. $\square$

Proof of Corollary 1.5. If G is as in the statement of the corollary, colored by a sublink b' of b which is not H_1-spanning , then we can find an $x \in b \smallsetminus b'$, and a closed surface x^* such that $[x^*][y] = \delta_{y,x}$ for all components y of b. Cut G along an edge, and color the two new leaves x and $*$ to obtain a graph $(G, *)$. By definition, we have $\langle G, x^* \rangle = G$, thus the result follows from the BR relation. $\square$

Proof of Corollary 1.6. We will first show the result for $k = 0$. Let G be as in the statement of the corollary and let G' be the graph with two more leaves than G, colored by x and $*$ respectively as shown:



The BR relation implies that

$$0 = \langle G', x^* \rangle = G + \text{LOOP} = G = G^{(0)}.$$

Now, we will show the result for all k. Let $G(n)$ be the same graph as G with $r+1$ leaves colored by $x, x+ny_1, \ldots, x+ny_r$, for $n \in \mathbb{N}$. Since x is primitive and linearly independent from $\{x+ny_1, \ldots, x+ny_r\}$, the $k = 0$ case for $G(n)$ shown above implies that $G(n) = 0$ for all n. Since $G(n) = \sum_k n^k G^{(k)}$, the result follows. $\square$

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