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The loop expansion of the Kontsevich integral, the null-move and S-equivalence☆

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Abstract

The Kontsevich integral of a knot is a graph-valued invariant which (when graded by the Vassiliev degree of graphs) is characterized by a universal property; namely it is a universal Vassiliev invariant of knots. We introduce a second grading of the Kontsevich integral, the Euler degree, and a geometric null-move on the set of knots. We explain the relation of the null-move to S-equivalence, and the relation to the Euler grading of the Kontsevich integral. The null-move leads in a natural way to the introduction of trivalent graphs with beads, and to a conjecture on a rational version of the Kontsevich integral, formulated by the second author and proven in Geom. Top 8 (2004) 115 (see also Kricker, preprint 2000, math/GT.0005284).

MSC: primary 57N10; secondary 57M25

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1. Introduction

1.1. What is the Kontsevich integral?

The Kontsevich integral of a knot in S^3 is a graph-valued invariant, which is characterized by a universal property, namely it is a universal $\mathbb{Q}$-valued Vassiliev invariant of knots [1]. Using the language of physics, the Kontsevich integral is the Feynmann diagram expansion (i.e., perturbative

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expansion) of the Chern–Simons path integral, expanded around a trivial flat connection, [35,38]. This explains the shape of the graphs (namely, they have univalent and trivalent vertices only), their vertex-orientation and their Vassiliev degree, namely half the number of vertices.

In physics one often takes the logarithm of a Feynmann diagram expansion, which is given by a series of connected graphs, and one studies the terms with a fixed number of loops (i.e., with a fixed first betti number).

The purpose of this paper is to study the Kontsevich integral, graded by the Euler degree (rather than the Vassiliev degree, or the number of loops), to introduce a null-move on knots, and to relate the Euler degree to the null-move.

This point of view explains in a natural way a Rationality Conjecture of the Kontsevich integral (formulated by the second author and proven by Garoufalidis [13]), the relation of the null-move to S-equivalence, the relation of the null-move to cyclic branched covers [14], and offers an opportunity to use Vassiliev invariants as obstructions to the existence of Seifert forms [17].

1.2. The Kontsevich integral, and its grading by the Euler degree

We begin by explaining the Euler expansion of the Kontsevich integral Z, defined for links in S^3 by Kontsevich [25], and extended to an invariant of links in arbitrary closed 3-manifolds by the work of Le–Murakami–Ohtsuki [28]. Our paper will focus on the above invariant $Z(M,K)$ of knots K in integral homology 3-spheres M:

$$Z:\ \mathrm{Pairs}(M,K) \to \mathscr{A}(*),$$

where $\mathscr{A}(*)$ is the completed vector space over $\mathbb{Q}$ generated by unitrivalent graphs with vertex-orientations modulo the well-known antisymmetry AS and IHX relations. The graphs in question have a Vassiliev degree (given by half the number of vertices) and the completion of $\mathscr{A}(*)$ refers to the Vassiliev degree.

We now introduce a second grading on unitrivalent graphs: the *Euler degree* $e(G)$ of a unitrivalent graph G is the number of trivalent vertices that remain when we shave-off all legs of G. It is easy to see that $e(G) = -2\chi(G)$ where χ is the Euler characteristic of G, which explains the naming of this degree. The AS and IHX relations are homogeneous with respect to the Euler degree, thus we can let Z_n denote the Euler degree n part of the Kontsevich integral.

Note that there are finitely many unitrivalent graphs with Vassiliev degree n, but infinitely many connected graphs with Euler degree n. For example, wheels with n legs have Vassiliev degree n but Euler degree 0. The Kontsevich integral of a knot, graded by the Vassiliev degree is a universal Vassiliev finite type invariant. On the other hand, Z_n are not Vassiliev invariants; they are rather power series of Vassiliev invariants.

Why consider the Euler degree? A deep and unexpected geometric reason is the content of the next section.

1.3. The $\mathbb{Z}$-null-move

A beautiful theory of Goussarov and Habiro is the study of the geometric notion of *surgery on a clasper*, see [18,19,21,12] (in the latter claspers were called clovers). Following the notation of [12], given a clasper G in a 3-manifold N, we let N_G denote the result of surgery. Clasper surgery can be described in terms of twisting genus 3 handlebodies in N, or alternatively in terms of surgery on a framed link in N. We will refer the reader to [12] for the definition and conventions of surgery on a clasper. We will *exclude* claspers with no trivalent vertices, that is, ones that generate I-moves.

In the present paper we are interested in pairs (M,K) and claspers $G \subset M \setminus K$ whose leaves are *null homologous links* in $M \setminus K$. We will call such claspers $\mathbb{Z}$-*null*, or simply *null*. The terminology is motivated by the fact that the leaves of such claspers are sent to 0 under the map: $\pi_1(M \setminus K) \to H_1(M \setminus K) \cong \mathbb{Z}$.

In order to motivate our interest in null-claspers, recall the basic and fundamental principle: surgery on a clasper preserves the homology and (when defined) the linking form. Applying that principle in the $\mathbb{Z}$-cover of a knot complement, it follows that surgery on a null-clasper preserves the Blanchfield form, as we will see below.

Surgery on null-claspers describes a move on the set of knots in integral homology 3-spheres. Given this move, one can define in the usual fashion a notion of *finite type invariant s* and a dual notion of *n-equivalence*, as was explained by Goussarov and Habiro. Explicitly, we can consider a decreasing filtration $\mathscr{F}^{\mathrm{null}}$ on the vector space generated by all pairs (M,K) as follows: $\mathscr{F}^{\mathrm{null}}_{\mathrm{n}}$ is generated by $[(M,K),G]$, where $G=\{G_1,\dots,G_n\}$ is a disjoint collection of null claspers in $M \setminus K$, and where

$$[(M,K),G] = \sum_{I \subset \{0,1\}^n} (-1)^{|I|} (M,K)_{G_I} \tag{1}$$

(where $|I|$ denotes the number of elements of I and N_{G_I} stands for the result of simultaneous surgery on claspers $G_i \subset N$ for all $i \in I$).

Definition 1.1. A function f: Pairs$(M,K) \to \mathbb{Q}$ is a *finite type invariant of null-type n* iff $f(\mathscr{F}^{\mathrm{null}}_{n+1}) = 0$.

We will discuss the general notion of n-null-equivalence in a later publication, at present we will consider the special case of $n=0$.

Definition 1.2. Two pairs (M,K) and (M',K') are *null-equivalent* iff one can be obtained from the other by a sequence of null moves.

In order to motivate the next lemma, recall a result of Matveev [32], who showed that two closed 3-manifolds M and M' are equivalent under a sequence of clasper surgeries iff they have the same homology and linking form. This answer is particularly pleasing since it is expressed in terms of abelian algebraic invariants.

Lemma 1.3. *The following are equivalent*: (M,K) *and* (M',K');

(a) *are null-equivalent*,
(b) *are* S*-equivalent and*
(c) *have isometric Blanchfield form.*

Proof. For a definition of *S-equivalence* and *Blanchfield forms*, see for example [22,30]. It is well-known that (M,K) and (M',K') are S-equivalent iff they have isometric Blanchfield forms;

for an algebraic-topological proof of that combine Levine and Kearton [31,23], or alternatively Trotter [37]. Thus (b) is equivalent to (c).

If (M',K') is obtained from (M,K) by surgery on a null-clasper G, then G lifts to the universal abelian cover $\tilde{X}$ of the knot complement $X=M\setminus K$. The lift $\tilde{G}$ of G is a union of claspers translated by the group $\mathbb{Z}$ of deck transformations. Furthermore, $\widetilde{X_G}$ is obtained from $\tilde{X}$ by surgery on $\tilde{G}$. Since clasper surgery preserves the homology and linking form, it follows that (M',K') and (M,K) have isometric Blanchfield forms, thus (a) implies (c).

Suppose now that (M',K') is S-equivalent to (M,K). Then, by Matveev's result, the integral homology 3-sphere M can be obtained from S^3 by surgery on some null-clasper G. Moreover, if K is a knot in M that bounds a Seifert surface Σ, we can always arrange (by an isotopy on G) that G is disjoint from Σ, thus G is (M,K)-null. Thus, if (M,K) and (M',K') are S-equivalent pairs, modulo null-moves we can assume that $M=M'=S^3$. In that case, Naik and Stanford show that S-equivalent pairs (S^3,K) and (S^3,K') are equivalent under a sequence of double Δ-moves, that is null-moves where all the leaves of the clasper bound disjoint disks, [33]. Thus, (b) implies (a) and the lemma follows. □

It follows that if (M,K) and (M',K') are null-equivalent, then they have the same Alexander module, in particular the same Alexander polynomial, and the same algebraic concordance invariants which were classified by Levine [31,30].

Remark 1.4. There is a well-known similarity between $\mathbb{Q}/\mathbb{Z}$-valued linking forms of rational homology spheres and $\mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$-valued Blanchfield forms. From this point of view, integral homology 3-spheres correspond to knots with trivial Alexander polynomial. This classical analogy extends to the world of finite type invariants as follows: one can consider finite type invariants of rational homology 3-spheres (using surgery on claspers) and finite type invariants of knots in integral homology 3-spheres (using surgery on null-claspers). In our paper, we will translate well-known results about finite type invariants and n-equivalence of integral homology 3-spheres to the setting of knots with trivial Alexander polynomial.

Are there any nontrivial finite type invariants of null-type? The next result answers this, and reveals an unexpected relation between the null-move and the Euler degree:

Theorem 1. *For all n, Z_n is a null-type n invariant of pairs (M,K) with values in $\mathscr{A}_n(*)$.*

In the above, $\mathscr{A}_n(*)$ stands for the subspace of $\mathscr{A}(*)$ generated by diagrams of Euler-degree n.

The above theorem has implications for the *loop expansion* of the Kontsevich integral defined as follows. The logarithm of the Kontsevich integral

$$\log Z\colon\ \text{Pairs}(M,K)\to\mathscr{A}^c(*)$$

takes values in the completed vector space (with respect to the Vassiliev degree) $\mathscr{A}^c(*)$ generated by connected vertex-oriented unitrivalent graphs, modulo the AS and IHX relations. Let us define the *loop degree* of a graph to be the number of loops, i.e., the first betti number. The AS and IHX relations are homogeneous with respect to the loop degree, thus we can define Q_n^{loop} to be the $(n+1)$-loop degree of $\log Z$. Since a connected trivalent graph with $n+1$ loops has Euler degree $2n$, and since $\log Z$ is an additive invariant of pairs under connected sum, it follows that

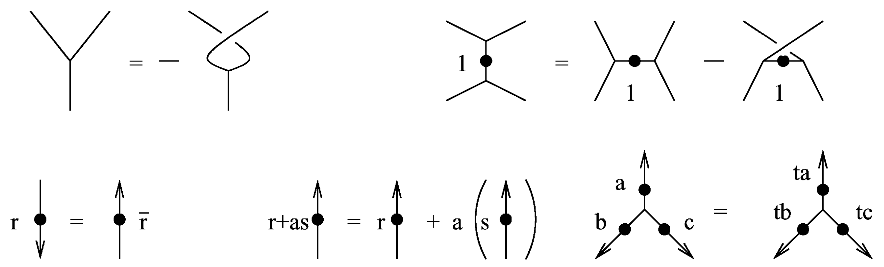


Fig. 1. The AS, IHX (for arbitrary orientations of the edges), Orientation Reversal, Linearity and Holonomy Relations.

Corollary 1.5. *For all n, Q_n^{loop} is an additive null-type $2n$ invariant of pairs (M,K) with values in the loop-degree $(n+1)$ part of $\mathscr{A}^c(*)$.*

Our next task is to study the graded quotients $\mathscr{G}_n^{\mathrm{null}} = \mathscr{F}_n^{\mathrm{null}}/\mathscr{F}_{n+1}^{\mathrm{null}}$; knowing this would essentially answer the question of how many null-finite type invariants there are.

Since null-equivalence has more than one equivalence classes, we ought to study $\mathscr{F}^{\mathrm{null}}$ and $\mathscr{G}^{\mathrm{null}}$ on each null-equivalence class. This is formalized in the following way. Given a pair (M,K), let $\mathscr{F}^{\mathrm{null}}(M,K)$ denote the subspace of $\mathscr{F}^{\mathrm{null}}$ generated by all pairs (M',K') which are null-equivalent to (M,K). It is easy to see that $\mathscr{F}^{\mathrm{null}}(M,K) = \mathscr{F}^{\mathrm{null}}(M',K')$ when (M,K) and (M',K') are null-equivalent and that we have a direct sum decomposition

$$\mathscr{F}^{\mathrm{null}} = \oplus_{0-\mathrm{null\ \ equiv.\ \ classes}(M,K)}\mathscr{F}^{\mathrm{null}}(M,K).$$

Fixing (M,K), we can define $\mathscr{G}_n^{\mathrm{null}}(M,K) = \mathscr{F}_n^{\mathrm{null}}(M,K)/\mathscr{F}_{n+1}^{\mathrm{null}}(M,K)$ accordingly. The simplest case to study are the graded quotients $\mathscr{G}^{\mathrm{null}}(S^3,\mathcal{O})$ for the unknot $\mathcal{O}$ in S^3. This is equivalent to the study of null-finite type invariant of knots with trivial Alexander polynomial.

It turns out in a perhaps unexpected way that $\mathscr{G}^{\mathrm{null}}(S^3,\mathcal{O})$ can be described in terms of trivalent graphs with *beads*. Let us define these here. Let $\Lambda = \mathbb{Z}[t,t^{-1}]$ denote the group-ring of the integers. It is a ring with involution $r \to \bar{r}$ given by $\bar{t} = t^{-1}$ and augmentation map $\varepsilon\colon \Lambda \to \mathbb{Z}$ given by $\varepsilon(t) = 1$. Consider a trivalent graph G with oriented edges. A *Λ-coloring* of G is an assignment of an element of Λ to every edge of G. We will call the assignment of an edge e, the *bead* of e. See also [16].

Definition 1.6. $\mathscr{A}(\Lambda)$ is the completed vector space over $\mathbb{Q}$ generated by pairs (G,c) (where G is a trivalent graph, with oriented edges and vertex-orientation and c is a Λ-coloring of G), modulo the relations: AS, IHX, Linearity, Orientation Reversal, Holonomy and Graph Automorphisms (Fig. 1).

$\mathscr{A}(\Lambda)$ is graded by the Euler degree of a graph, and the completion is with respect to the above grading.

Remark 1.7. We could have defined $\mathscr{G}^{\mathrm{null}}(M,K)$ and $\mathscr{A}(\Lambda)$ over $\mathbb{Z}$ rather than over $\mathbb{Q}$. We will not use special notation to indicate this; instead when we wish, we will state explicitly which coefficients we are using.

Theorem 2. *Over $\mathbb{Z}$, there is degree-preserving map*:

$$\mathscr{A}(\Lambda) \to \mathscr{G}^{\mathrm{null}}(S^3, \mathcal{O})$$

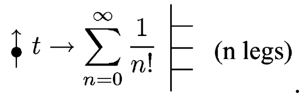
which is onto, over $\mathbb{Z}[\frac{1}{2}]$. Since $\mathscr{A}_{\mathrm{odd}}(\Lambda) = 0$, it follows that over $\mathbb{Z}[\frac{1}{2}]$, $\mathscr{F}_n^{\mathrm{null}}(S^3, \mathcal{O})$ is a 2-step filtration, i.e., satisfies $\mathscr{F}_{2n+1}^{\mathrm{null}}(S^3, \mathcal{O}) = \mathscr{F}_{2n+2}^{\mathrm{null}}(S^3, \mathcal{O})$ for all n.

1.4. Further developments

In view of Theorems 1 and 2, it is natural to ask if there is any relation among the algebras $\mathscr{A}(*)$ and $\mathscr{A}(\Lambda)$. In order to make contact with later work, we introduce here an important *hair* map

$$\mathrm{Hair} : \mathscr{A}(\Lambda) \to \mathscr{A}(*),$$

which is defined by replacing a bead t by an exponential of hair:



The Kontsevich integral (graded by the Euler degree, and evaluated on Alexander polynomial 1 knots) fails to be a universal finite type invariant (with respect to the null-move) because it takes values in the "wrong" space, namely $\mathscr{A}(*)$ rather than $\mathscr{A}(\Lambda)$. It is natural to conjecture the existence of an invariant Z^{rat} that fits in a commutative diagram

$$\begin{tikzcd} & \mathscr{A}(\Lambda) \arrow[d, "\mathrm{Hair}"] \\ (\Delta = 1) - \mathrm{knots} \arrow[ur, "Z^{\mathrm{rat}}"] \arrow[r, "Z"] & \mathscr{A}(*) \end{tikzcd}$$

This Rationality Conjecture was formulated by the second author (in a version for all knots) and was proven in [13].

In an earlier version of the paper, we formulated a conjectural relation of the 2-loop part of the Z^{rat} invariant and the Casson–Walker invariant of cyclic branched coverings of a knot. This relation has been settled by Garoufalidis [14, Corollary 1.4].

Remark 1.8. In the study of knot theory via surgery, knots with Alexander trivial polynomial are topologically (but not smoothly) slice, [9] and also [17, Appendix]. The abelian and solvable invariants of knots with trivial Alexander polynomial, such as their Alexander module, Casson–Gordon invariants and all of their recent non-abelian invariants of Cochran–Orr–Teichner [7] vanish. However, already the Euler degree 2 part of the Kontsevich integral Z_2 (which incidentally equals to the 2-loop part of Z) does not vanish on Alexander polynomial 1 knots. The symbol of Z_2 can be computed in terms of equivariant linking numbers (see Theorem 4) and this gives good realization properties for Z_2. The Z_2 invariant offers an opportunity to settle an error in one of M. Freedman's lemmas on knots with Alexander polynomial 1 [17]. For another application, see [10].

Remark 1.9. To those who prefer moves that untie knots, it might sound disappointing that the null-move fails to do so. On the other hand, the geometric null-move describes a natural equivalence relation on knots, namely isometry of Blanchfield forms. Concordance is another well-prized

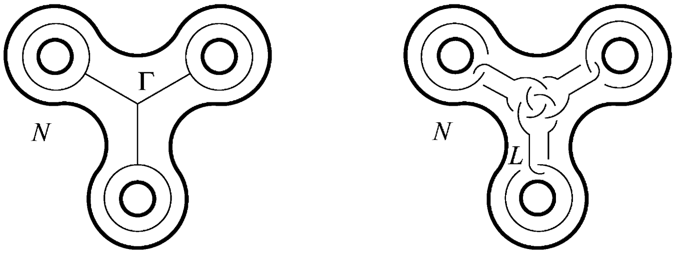


Fig. 2. Y-graph and the corresponding surgery link.

equivalence relation on knots. If one could generate concordance in terms of surgery of some type of claspers, this could open the door for constructing a plethora of concordance invariants of knots. It is known that surgery on certain claspers preserve concordance, [15,8], but it is also known that these moves do not generate concordance [15].

1.5. Plan of the paper

The paper consists of five, largely independent, sections.

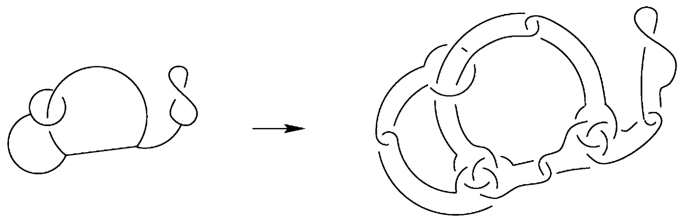
- In Section 2 we apply the topological calculus of claspers to the case of null-claspers which leads naturally to trivalent graphs with beads, and their relation to the graded quotients $\mathcal{G}(S^3, \mathcal{O})$.
- In Section 3 we give a detailed study of the behavior of the Aarhus integral under surgery on claspers. In particular, counting arguments above the critical degree imply that the Euler degree n part of the Kontsevich integral is a finite type invariant of null-type n. This provides a conceptual relation between the null-move and the Euler degree of graphs. This counting leads naturally to the study of the hairy strut part of the Kontsevich integral. Struts correspond to linking numbers and the goal in the rest of the sections is to show that hairy struts correspond to equivariant linking numbers.
- In Section 4 we review linking numbers and equivariant linking numbers, and give an axiomatic description of the latter, motivated by the calculus of claspers.
- In the final Section 5 we show that the hairy strut part of the Kontsevich integral satisfies the same axioms as the equivariant linking numbers, and as a consequence of a uniqueness result, the two are equal.

2. Surgery on claspers and the null-filtration

2.1. A brief review of surgery on claspers

In this section we recall briefly the definition of surgery on claspers, with the notation of [12]. We refer the reader to [12] for a detailed description. A framed graph G in a 3-manifold M is called a *clasper of degree* 1 (or simply, a *Y-graph*, Fig. 2), if it is the image of Γ under a smooth embedding $\phi_G: N \to M$ of a neighborhood N of Γ. The embedding ϕ_G can be recovered from G up to isotopy. Surgery on G can be described either by surgery on the corresponding framed six component link, or in terms of cutting a neighborhood of G (which is a handlebody of genus 3), twisting by a fixed diffeomorphism of its boundary and gluing back.

A clasper of higher degree is a (thickening of) an embedded trivalent graph, with some distinguished loops which we call *leaves*. The degree of a clasper is the number of trivalent vertices, excluding those of the leaves. Surgery on claspers of higher degree is defined similarly. For example, a clasper of degree 2 and its corresponding surgery link are shown below:



Surgery on a clasper of degree n can be described in terms of surgery on n claspers of degree 1. We will exclude claspers of degree 0, that is with no trivalent vertices. These correspond to I-moves in the language of Goussarov and Habiro.

2.2. A review of finite type invariants of integral homology 3-spheres

As we mentioned in the introduction (see Remark 1.4), the study of the graded quotients $\mathcal{G}(S^3, \mathcal{O})$ is entirely analogous to the study of the graded quotients $\mathcal{G}(M)(S^3)$ of finite type invariants of integral homology 3-spheres. For a detailed description of the latter invariants, we refer the reader to Garoufalidis [11]. Let us recall here some key ideas, for later use.

Let $\mathcal{F}(S^3)$ denote the vector space over $\mathbb{Q}$ generated by integral homology 3-spheres, and let $\mathcal{F}_n(S^3)$ denote the subspace of $\mathcal{F}(S^3)$ generated by $[M, G]$ for all disjoint collections of claspers $G = \{G_1, \ldots, G_n\}$ in integral homology 3-spheres M, where

$$[M, G] = \sum_{I \subset \{0,1\}^n} (-1)^{|I|} M_{G_I}.$$

Compare with Eq. (1).

Definition 2.1. A function f: integral homology 3-spheres $\to \mathbb{Q}$ is a *finite type invariant of type n* iff $f(\mathcal{F}_{n+1}(S^3)) = 0$.

Let $\mathcal{G}_n(S^3) = \mathcal{F}_n(S^3)/\mathcal{F}_{n+1}(S^3)$ denote the graded quotients. These quotients can be described in terms of a vector space $\mathcal{A}(\phi)$ defined as follows:

Definition 2.2. $\mathcal{A}(\phi)$ is the completed vector space over $\mathbb{Q}$ generated by trivalent graphs with vertex-orientations, modulo the AS and IHX relations.

In [12] we defined a degree-preserving map

$$\mathscr{A}(\phi) \to \mathscr{G}(S^3) \tag{2}$$

as follows: Given an abstract vertex-oriented trivalent graph G, choose an embedding $\phi: G \to S^3$ that preserves its vertex-orientation. Recall that a vertex-orientation is a cyclic order of the set of three flags around a trivalent vertex. An embedding ϕ gives rise at each vertex $\phi(v)$ to a frame that consists of the tangent vectors of the images under ϕ of the three flags around v. Comparing this frame with the standard orientation of S^3, we get a number $\varepsilon_{\phi,v} = \pm 1$ at each vertex v of G. We say that ϕ is orientation preserving if $\prod_v \varepsilon_{\phi,v} = 1$.

We will consider the clasper $\phi(G) \subset S^3$ of degree n and the associated element $[S^3, \phi(G)] \in \mathscr{G}_n(S^3)$. Lemma 2.5 implies that the element $[S^3, \phi(G)] \in \mathscr{G}(S^3)$ is independent of the embedding ϕ. Further, it was shown in [12, Corollary 4.6; Theorem 4.11] and that the AS and the IHX relations hold. This defines map (2) over $\mathbb{Z}$, which is obviously degree-preserving.

We now recall the following theorem of Garoufalidis [12, Theorem 4.13]:

Theorem 3. *The map* (2) *is onto, over* $\mathbb{Z}[\frac{1}{2}]$.

Let us comment on the theorem. It is not obvious that the map (2) is onto, since $\mathscr{G}(S^3)$ is generated by elements of the form $[M, G]$ for claspers G of degree n with leaves *arbitrarily* framed links, whereas the image of the map (2) involves claspers whose leaves are linked in the pattern of 0-framed Hopf links.

Let us briefly review a proof of Theorem 3, taken from [11], which adapts well to our later needs. Observe that each leaf of a clasper in S^3 is *null homologous*. Using Lemmas 2.4 and 2.5 we can write $[S^3, G]$ as a linear combination of $[S^3, G']$ such that each G' satisfies the following properties:

- Each leaf l of G' is an unknot with framing 0 or ± 1 and bounds a disk D_l.
- The disks D_l of ± 1-framed leaves l are disjoint from each other, disjoint from G', and disjoint from the disks of the 0-framed leaves.
- Each disk of a 0-framed leaf intersects precisely one other disk of a 0-framed leaf in a single clasp.

Using [12, Lemma 4.8], and working over $\mathbb{Z}[\frac{1}{2}]$, we may assume that G' as above has no ± 1-framed leaves. Then, it follows that $[S^3, G']$ lies in the image of (2). This completes the proof of Theorem 3. □

Remark 2.3. If we fix a rational homology 3-sphere M, we could define a filtration on the vector space $\mathscr{F}(M)$ generated by all homology spheres with a fixed linking form (these are exactly 0-equivalent to M under a sequence of clasper moves) and introduce a decreasing filtration on $\mathscr{F}(M)$ and graded quotients $\mathscr{G}(M)$. There is a degree-preserving map

$$\mathscr{A}(\phi) \to \mathscr{G}(M),$$

which is onto, over $\mathbb{Z}[\frac{1}{2}, 1/|H_1(M)|]$. The proof uses the same reasoning as above together with the fact that every knot (such as a leaf of a clasper) in M is null homologous in M, once multiplied by $|H_1(M)|$.

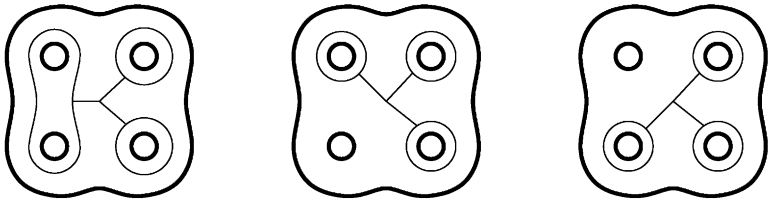


Fig. 3. Cutting a leaf. The claspers Γ, Γ' and Γ'', from left to right.

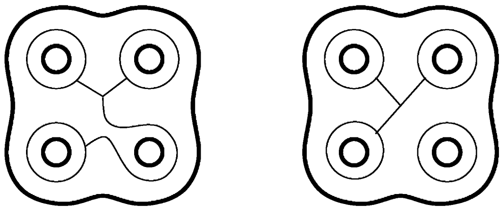


Fig. 4. Sliding an edge. The claspers Γ^s and Γ.

2.3. The "Cutting" and "Sliding" lemmas

In this section we recall the key Cutting and Sliding lemmas.

Lemma 2.4 (Goussarov [19], Habiro [21]; Cutting a leaf). *Let Γ, Γ' and Γ'' be the claspers of Fig. 3 in a handlebody V embedded in a integral homology 3-sphere M and let Γ_0 be a clasper of degree $m-1$ in the complement of V. If $G = \Gamma_0 \cup \Gamma$, $G' = \Gamma_0 \cup \Gamma'$ and $G'' = \Gamma_0 \cup \Gamma''$ then we have that*

$$[M, G] = [M, G'] + [M, G''] \bmod \mathscr{F}_{m+1}(S^3).$$

Informally, we may think that we are splitting a leaf of G into a connected sum.

Lemma 2.5 (Goussarov [19], Habiro [21]; Sliding an edge). *Let Γ^s and Γ be the claspers of Fig. 4 in a handlebody V embedded in a manifold M and let Γ_0 be a clasper of degree $m-1$ in the complement of V. If $G^s = \Gamma_0 \cup \Gamma^s$ and $G = \Gamma_0 \cup \Gamma$, then we have that*

$$[M, G^s] = [M, G] \bmod \mathscr{F}_{m+1}(S^3).$$

Informally, we may think that we are sliding an edge of G.

2.4. The graded quotients $\mathscr{G}^{\text{null}}(S^3, \mathcal{O})$

Our first goal in this section is to define the analogue of map (2) using trivalent graphs with beads. This will be achieved by introducing *finger moves* of $(S^3, \mathcal{O})$-null claspers.

Let us begin with an alternative description of the space $\mathscr{A}(\Lambda)$. Consider a unitrivalent graph G with vertex-orientation and edge orientation, and cut each edge to a pair of *flags* (i.e., half-edges). Orient the two flags $\{e_b, e_t\}$ incident to an edge e of G as follows:

Definition 2.6. The ring Λ_G is the polynomial ring over $\mathbb{Z}$ with generators $t_e^{\pm 1}$ for every flag e of G and relations: $t_{e_b} t_{e_t} = 1$ if the pair of flags (e_b, e_t) is an edge of G, and $\prod_{v \in e} t_e = 1$ for all trivalent vertices v of G.

Let us define an abelian group:

$$\mathscr{A}^C(\Lambda) = (\oplus_G \Lambda_G \cdot G)/(\mathrm{AS}, \mathrm{IHX}, \mathrm{Aut}), \tag{3}$$

where the sum is over all isomorphism classes of trivalent graphs with oriented edges and vertex-orientation.

Example 2.7. For a strut I, a vortex Y and a Theta graph Θ, the associated rings are given by

$$\Lambda_{\mathsf{I}} \cong \mathbb{Z}[t^{\pm 1}],$$

$$\Lambda_{\mathsf{Y}} \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1 t_2 t_3 - 1),$$

$$\Lambda_{\Theta} \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1 t_2 t_3 - 1, \mathrm{Sym}_2 \times \mathrm{Sym}_3),$$

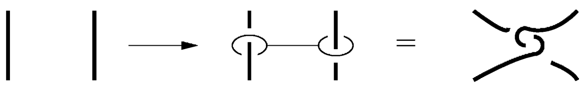
where Sym_3 acts as a permutation on the t_is and Sym_2 acts as an simultaneous involution of the powers of the t_i.

Lemma 2.8. *Consider an $(S^3, \mathcal{O})$-null clasper G of degree n, and let G^{nl} (the superscript stands for 'no leaves') denote the abstract unitrivalent graph obtained from removing the leaves of G. Then, there exists a geometric action*

$$\Lambda_{G^{nl}} \times [(S^3, \mathcal{O}), G] \to \mathscr{G}(S^3, \mathcal{O})$$

given by finger *moves.*

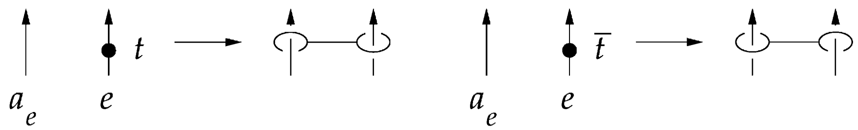
Proof. The action will be defined in terms of I-moves. Let us recall surgery on a clasper with no trivalent vertices, a so-called *I-move*:



An I-move can be thought of as a right-handed *finger move*, or a right-handed Dehn twist, and has an inverse given by a left-handed finger move (indicated by a stroke in the opposite direction in [19]). We will call right-handed I-moves positive, and left-handed ones negative.

To define the action, without loss of generality, we will assume that G is a collection of claspers of degree 1. Further, we orient the flags of G outward, towards their univalent vertices.

Consider a monomial $c = \prod_e t_e^{n(e)}$ where the product is over the set of flags of G. For each flag e of G choose a segment a_e of $\mathcal{O}$, such that segments of different flags are nonintersecting. For each flag e of G, choose a collection I_{e,a_e} of $n(e)$ I-claspers such that when $n(e)=1$, we have:



Let $I_c = \cup_e I_{e,a_e}$ be the collection of I-claspers for all flags e of G, and let $I_c \cdot G$ denote the image of G after clasper surgery on I_c. This defines an element $[(S^3, \mathcal{O}), I_c \cdot G] \in \mathcal{G}(S^3, \mathcal{O})$. We claim that this element depends on c alone, and not on the intermediate choices of arcs and the I-claspers. Indeed, Lemma 2.5 implies that if an edge of $I_c \cdot G$ passes through two oppositely oriented arcs of $\mathcal{O}$, resulting in a clasper G', then $[(S^3, \mathcal{O}), I_c \cdot G] = [(S^3, \mathcal{O}), G'] \in \mathcal{G}(S^3, \mathcal{O})$. This implies easily our previous claim.

Thus, we can define the action of c on G by $c \cdot [(S^3, \mathcal{O}), G] = [(S^3, \mathcal{O}), I_c \cdot G]$. It is easy to see that the relations of the ring $\Lambda_{G^{nl}}$ hold. $\square$

We can now define map

$$\mathcal{A}^C(\Lambda) \to \mathcal{G}^{\mathrm{null}}(S^3, \mathcal{O}) \tag{4}$$

over $\mathbb{Z}$ as follows (the superscript stands for "clasper"): consider a trivalent graph G with a vertex-orientation and an orientation of the edges, and let $\phi: G \to B^3 \subset (S^3, \mathcal{O})$ be an orientation preserving embedding in a small ball B^3. Consider the element $[S^3, \phi(G)] \in \mathcal{G}(S^3, \mathcal{O})$, as in map (2). Using Lemma 2.8 and the fact that G is trivalent, we get a map

$$\Lambda_G \to \mathcal{G}(S^3, \mathcal{O})$$

induced by the action $\Lambda_G \times [S^3, \phi(G)] \to \mathcal{G}(S^3, \mathcal{O})$. Just as in the map (2), the AS, IHX and Aut relations are preserved. This defines the map (4).

We now give a version of the ring Λ_G that uses half the number of generators:

Definition 2.9. If G is a vertex-oriented graph with oriented edges, let us define the ring Λ_G^V over $\mathbb{Z}$ with generators $t_e^{\pm 1}$ for every edge e of G and relations: $\prod_{v \in e} t_e = 1$ for all trivalent vertices v of G.

The next lemma identifies Λ_G with Λ_G^V:

Lemma 2.10. *If G has oriented edges, then we have a canonical isomorphism*:

$$\Lambda_G \cong \Lambda_G^V.$$

Proof. If e is an oriented edge, let (e_b, e_t) denote the pair of flags such that e has the same orientation as e_b and opposite from e_t. This gives rise to a map $t_{e_b} \in \Lambda_G \to t_e \in \Lambda_G^V$ and $t_{e_t} \in \Lambda_G \to t_e^{-1} \in \Lambda_G^V$, which is an isomorphism. $\square$

Lemma 2.11. *We have a canonical isomorphism*

$$\mathcal{A}(\Lambda) \cong \mathcal{A}^C(\Lambda)$$

Proof. It follows immediately using Lemma 2.10 above. □

Combining the above lemma with the map (4), we get the desired map

$$\mathscr{A}(\Lambda) \to \mathscr{G}^{\mathrm{null}}(S^3, \mathcal{O})$$

of Theorem 2.

Proof (Of Theorem 2). We now claim that the proof of Theorem 3 works without change, and proves Theorem 2. Indeed, Lemmas 2.4 and 2.5 work for $(S^3, \mathcal{O})$-null claspers as stated. Furthermore, if $G \subset S^3 \setminus \mathcal{O}$ is an $(S^3, \mathcal{O})$-null clasper, then each leaf of G lies in the commutator group $[\pi, \pi] = 1$, where $\pi = \pi_1(S^3 \setminus \mathcal{O}) \cong \mathbb{Z}$. Since $1 = [\pi, \pi] = [[\pi, \pi], [\pi, \pi]]$, this implies that each leaf of G bounds a surface in $S^3 \setminus \mathcal{O}$ whose bands are null homologous links in $S^3 \setminus \mathcal{O}$. The surfaces for different leaves may intersect each other. We can apply now the proof of Theorem 3 exactly as stated, to conclude the proof of Theorem 2. □

We end this section with an alternative description of the ring Λ_G, and thus of the vector space of graphs $\mathscr{A}(\Lambda)$, that was introduced by the second author [36] and studied by P. Vogel in unpublished work. We thank P. Vogel for explaining us his unpublished work.

Lemma 2.12. *For a trivalent graph G with oriented edges we have a canonical isomorphism*

$$\Lambda_G^V \cong \mathbb{Z}[H^1(G, \mathbb{Z})].$$

Proof. Using Lemma 2.10, we will rather describe a canonical isomorphism $\Lambda_G^V \cong \mathbb{Z}[H^1(G, \mathbb{Z})]$. Recall the exact sequence

$$0 \to C^0(G, \mathbb{Z}) \to C^1(G, \mathbb{Z}) \to H^1(G, \mathbb{Z}) \to 0,$$

where $C^0(G, \mathbb{Z})$ and $C^1(G, \mathbb{Z})$ are the abelian groups of $\mathbb{Z}$-valued functions on the vertices and oriented edges of G. Let $c \in C^1(G, \mathbb{Z})$. This gives rise to the element $\prod_e t_{e_b}^{c(e)} \in \Lambda_G$ where the product is taken over all edges of G. It is easy to see that this element depends on the image of c in $H^1(G, \mathbb{Z})$ and gives rise to a map $\mathbb{Z}[H^1(G, \mathbb{Z})] \to \Lambda_G$. It is easy to see that this map is an isomorphism. □

Let us define a vector space

$$\mathscr{A}^{RV}(\Lambda) = (\oplus_G \mathbb{Z}[H_1(G, \mathbb{Z})] \cdot G)/(\mathrm{AS}, \mathrm{IHX}, \mathrm{Aut}),$$

where the sum is over trivalent graphs with oriented edges.

Lemmas 2.11 and 2.12 imply that

Corollary 2.13. *There are canonical isomorphisms*:

$$\mathscr{A}(\Lambda) \cong \mathscr{A}^C(\Lambda) \cong \mathscr{A}^{RV}(\Lambda).$$

The map

$$\mathscr{A}^{RV}(\Lambda) \to \mathscr{G}(S^3, \mathcal{O})$$

(which is the composite of (4) with the identification of the above corollary), can be defined more explicitly without appealing to finger moves. Indeed, observe that given an embedding $\varphi\colon G \to S^3 \setminus \mathcal{O}$, then taking linking number in S^3 of $\mathcal{O}$ with 1-cycles in $\varphi(G)$, we get a canonical element $\varphi^*(\mathcal{O}) \in H^1(G, \mathbb{Z})$.

Lemma 2.14. *If $\varphi, \psi\colon G \to S^3 \setminus \mathcal{O}$ are two embeddings of a trivalent graph of degree $2n$ such that $\varphi^*(\mathcal{O}) = \psi^*(\mathcal{O}) \in H^1(G, \mathbb{Z})$, then $[(S^3, \mathcal{O}), \varphi(G)] = [(S^3, \mathcal{O}), \psi(G)] \in \mathcal{G}^{\text{null}}_{2n}(S^3, \mathcal{O})$.*

Proof. Our Sliding Lemma implies that an edge of G can slide past two arcs of K with opposite orientations. It can also slide past another edge. It is easy to see that this implies our result. □

Using the above lemma, and given a vertex-oriented trivalent graph G and an element $c \in H^1(G, \mathbb{Z})$, let $\varphi\colon G \to S^3 \setminus \mathcal{O}$ be any embedding with associated element c. This defines a map $\mathbb{Z}[H^1(G, \mathbb{Z})] \cdot G \to \mathcal{G}(S^3, \mathcal{O})$. The AS, IHX and Aut relations are satisfied and this gives rise to a well-defined map

$$\mathcal{A}^{RV}(\Lambda) \to \mathcal{G}(S^3, \mathcal{O}).$$

Remark 2.15. Given a cocommutative Hopf algebra H and an arbitrary graph G with oriented edges, Vogel associates an abelian group Λ^V_G. In case $H = \mathbb{Z}[t, t^{-1}]$ with comultiplication $\Delta(t) = t \otimes t$ and antipode $s(t) = t^{-1}$ the abelian group Λ^V_G is the one given by Definition 2.9.

Remark 2.16. All the results of this section work without change if we replace the pair $(S^3, \mathcal{O})$ with a pair (M, K) of a knot K in a integral homology 3-sphere M with trivial Alexander polynomial.

3. The behavior of the Aarhus integral under surgery on claspers

In this section, we specialize the general principle of defining/calculating the Aarhus invariant to the case of links obtained by surgery on claspers. The reader is referred to [4] for a detailed discussion of the Aarhus integral. In the present paper we will only be interested in pairs (M, K) of knots K in integral homology 3-spheres M, and $(N, \emptyset)$ of integral homology 3-spheres N. In case $M = S^3$, we will be using the term "Kontsevich integral" rather than LMO invariant or Aarhus integral. Hopefully this will not cause any confusion or historical misunderstanding.

3.1. A brief review of the Aarhus integral

Given a nondegenerate framed link L in a S^3 (i.e., whose linking matrix is invertible over $\mathbb{Q}$) the Aarhus invariant $Z(S^3_L, \emptyset)$ of S^3_L is obtained from the Kontsevich integral $Z(S^3, L)$ in the following way:

- Consider $Z(S^3, L)$, an element of the completed (with respect to the Vassiliev degree) vector space $\mathcal{A}(\circlearrowleft_L)$ of chord diagrams on L-colored disjoint circles.
- After some *suitable basing* of L (defined below) we can lift $Z(S^3, L)$ to an element of the completed algebra $\mathcal{A}(\uparrow_L)$ of unitrivalent graphs on L-colored vertical segments.

- Symmetrize the legs on each L-colored segment to convert $Z(S^3, L)$ to an element of the completed algebra $\mathscr{A}(*_L)$ of unitrivalent graphs with symmetric L-colored legs.
- Separate the *strut* part $Z^q(S^3, L)$ from the other part $Z^t(S^3, L)$:

  $$Z(S^3, L) = Z^q(S^3, L)\, Z^t(S^3, L),$$

  where Z^t contains no diagrams that contain an L-labeled strut component. It turns out that the strut-part is related to the linking matrix of L as follows:

  $$Z^q(S^3, L) = \exp\left(\frac{1}{2}\sum_{x,y\in L} l_{xy} \begin{array}{c} x \\ | \\ y \end{array}\right).$$

- Glue the L-colored legs of the graphs of $Z^t(S^3, L)$ pairwise in all possible ways and multiply the created edges by the entries of the negative inverse linking matrix of L.
- Finally renormalize by a factor that depends on the signature of the linking matrix of L. The end result, $Z(S^3_L, \emptyset)$ depends only on the 3-manifold S^3_L and not on the surgery presentation L of it or the basing of L.

What is a "suitable basing of L?" Ideally, we wish we could choose a base point on each component of L to convert L into a union of L-labeled intervals. Unfortunately, it is more complicated: a suitable basing of L is a string-link representative of L, equipped with *relative scaling* (i.e., a parenthetization) between the strings. Such objects were called *q-tangles* (see [27]) by Le-Murakami, *non-associative tangles* by Bar-Natan (see [3]), and were described in the equivalent language of *dotted Morse links* in [4, Part II, Section 3].

Note that certain parts (such as the L-labeled struts) of the Kontsevich integral of a based link L are independent of the basing.

Remark 3.1. The above discussion of the Aarhus integral also works when we start from pairs (M, L) of nondegenerate framed links L in rational homology 3-sphere M.

The discussion also works in a relative case when we start from pairs $(M, L \cup L')$ (with L nondegenerate in a rational homology 3-sphere M) and do surgery on L alone.

In what follows, we will assume silently that the links in question are suitably based, using parenthesized string-link representatives.

3.2. The Aarhus integral and surgery on claspers

We will be interested in links that describe surgery on claspers. Consider a clasper G of degree 1 in S^3 (a so-called Y-graph). Surgery on the clasper G can be described by surgery on a six component link $E \cup L$ associated to G, where E (resp. L) is the three component link that consists of the edges (resp. leaves) of G. The linking matrix of $E \cup L$ and its negative inverse are given as follows:

$$\begin{pmatrix} 0 & I \\ I & \mathrm{lk}(L_i, L_j) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{lk}(L_i, L_j) & -I \\ -I & 0 \end{pmatrix}. \tag{5}$$

The six component link $E \cup L$ is partitioned in three blocks of two component links $A_i = \{E_i, L_i\}$ each for $i = 1, 2, 3$, the *arms* of G. A key feature of surgery on G is the fact that surgery on any proper subset of the set of arms does not alter M. In other words, alternating M with respect to surgery on G equals to alternating M with respect to surgery on all nine subsets of the set of arms $A = \{A_1, A_2, A_3\}$. That is,

$$Z([S^3, G]) = Z([S^3, A]). \tag{6}$$

Due to the locality property of the Kontsevich integral (explained in [26] and in a leisure way in [4, II, Section 4.2]), the nontrivial contributions to the right-hand side of Eq. (6) come from the part of $Z^t(S^3, A_1 \cup A_2 \cup A_3)$ that consists of graphs with legs *touch* (i.e., are colored by) all three arms of G.

The above discussion generalizes to the case of an arbitrary disjoint union of claspers.

3.3. The Aarhus integral is a universal invariant of integral homology 3-spheres

In this section, we will explain briefly why the Aarhus integral (evaluated at integral homology 3-spheres and graded by the Euler degree) is a universal finite type invariant. We will use essentially the same ideas to prove Theorem 1 in the following section. The ideas are well-known and involve elementary counting arguments, see [4,26] and also [11].

We begin by showing the following well-known proposition, which we could not find in the literature.

Proposition 3.2. *The Euler degree n part of $Z(\cdot, \emptyset)$ is a type n invariant of integral homology 3-spheres with values in $\mathscr{A}_n(\phi)$.*

Proof. Since $\mathscr{A}_{\mathrm{odd}}(\phi) = 0$, it suffices to consider the case of even n.

Recalling Definition 2.1, suppose that $G = \{G_1, \ldots, G_m\}$ (for $m \geqslant 2n+1$) is a collection of claspers in S^3 each of degree 1, and let A denote the set of arms of G. Eq. (6) and its following discussion implies that

$$Z([S^3, G]) = Z([S^3, A])$$

and that the nonzero contribution to the right-hand side come from diagrams in $Z^t(S^3, A)$ that touch all arms. Thus, contributing diagrams have at least $3(2n+1) = 6n+3$ A-colored legs, to be glued pairwise. Since pairwise gluing needs an even number of univalent vertices, it follows that we need at least $6n+4$ A-colored legs.

Note that $Z^t(S^3, A)$ contains no struts. Thus, at most three A-colored legs meet at a vertex, and after gluing the A-colored legs we obtain trivalent graphs with at least $(6n+4)/3 = 2n+4/3$ trivalent vertices, in other words of Euler degree at least $2n+2$. Thus, $Z_{2n}([S^3, G]) = 0 \in \mathscr{A}_{2n}(\phi)$, which implies that Z_{2n} is a invariant of integral homology 3-spheres of type $2n$ with values in $\mathscr{A}_{2n}(\phi)$. $\square$

Sometimes the above vanishing statement is called *counting above the critical degree*. Our next statement can be considered as *counting on the critical degree*. We need a preliminary definition.

Definition 3.3. Consider a clasper G in S^3 of degree $2n$, and let $G^{break} = \{G_1, \ldots, G_{2n}\}$ denote the collection of degree 1 claspers G_i which are obtained by inserting a Hopf link in the edges of G.

Then the *complete contraction* $\langle G\rangle \in \mathscr{A}(\phi)$ of G is defined to be the sum over all ways of gluing pairwise the legs of G^{break}, where we multiply the resulting elements of $\mathscr{A}(\phi)$ by the product of the linking numbers of the contracted leaves.

Proposition 3.4. *If G is a clasper of degree $2n$ in S^3, then*

$$Z_{2n}([S^3, G^{break}]) = \langle G\rangle \in \mathscr{A}_{2n}(\phi).$$

Proof. It suffices to consider a collection $G = \{G_1, \ldots, G_{2n}\}$ of claspers in S^3 each of degree 1. Let A denote the set of arms of G. The counting argument of the above proposition shows that the contributions to $Z_{2n}([S^3, G]) = Z_{2n}([S^3, A])$ come from complete contractions of a disjoint union $D = Y_1 \cup \ldots \cup Y_{2n}$ of $2n$ *vortices*. A vortex is the diagram Y, the next simplest unitrivalent graph after the strut. Furthermore, the $6n$ legs of D should touch all $6n$ arms of G. In other words, there is a 1-1 correspondence between the legs of such D and the arms of G.

Consider a leg l of D that touches an arm $A_l = \{E_l, L_l\}$ of G. If l touches L_l, then due to the restriction of the negative inverse linking matrix of G (see Section 3.2), it needs to be contracted to another leg of D that touches E_l. But this is impossible, since the legs of D are in 1-1 correspondence with the arms of G.

Thus, each leg of D touches precisely one edge of G. In particular, each component Y_i of D is colored by three edges of G.

Note that the set of edges of G is an algebraically split link. Given a vortex colored by three edges of G, the coefficient of it in the Kontsevich integral equals to the *triple Milnor invariant* (as is easy to show, see for example Ref. [20]) and vanishes unless all three edges are part of a degree 1 clasper G_i. When the triple Milnor invariant does not vanish, it equals to 1 using the orientation of the clasper G_i.

Thus, the diagrams D that contribute are a disjoint union of $2n$ vortices $Y = \{Y_1, \ldots, Y_{2n}\}$ and these vortices are in 1-1 correspondence with the set of claspers $\{G_1, \ldots, G_{2n}\}$, in such a way that the legs of each vortex Y_i are colored by the edges of a unique clasper G_j.

After we glue the legs of such Y using the negative inverse linking matrix of G, the result follows. □

Remark 3.5. Let us mention that the discussion of Propositions 3.2 and 3.4 uses the unnormalized Aarhus integral; however since we are counting above the critical degree, we need only use the degree 0 part of the normalization which equals to 1; in other words we can forget about the normalization.

The above proposition is useful in realization properties of the Z_{2n} invariant, but also in proving the following *Universal Property*:

Proposition 3.6. *For all n, the composite map of Eq.* (2) *and Proposition* 3.4

$$\mathscr{A}_{2n}(\phi) \to \mathscr{G}_{2n}(S^3) \xrightarrow{Z_{2n}} \mathscr{A}_{2n}(\phi)$$

is the identity. Since the map on the left is onto, it follows that the map (2) *is an isomorphism, over* $\mathbb{Q}$.

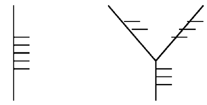


Fig. 5. Hairy struts and hairy vortices.

We may call the map $G \to Z_{2n}([S^3, G])$ (where G is of degree $2n$) the *symbol* of Z_{2n}.

Remark 3.7. In the above Propositions 3.2–3.6, S^3 can be replaced by any integral homology 3-sphere (or even a rotational homology 3-spheres) M.

3.4. Proof of Theorem 1

Our goal in this section is to modify (when needed) the discussion of Sections 3.1–3.3 and deduce in a natural way the proof of Theorem 1. It is clear that struts and vortices and their coefficients in the Kontsevich integral play an important role.

As we will see presently, the same holds here, when we replace struts and vortices by their *hairy* analogues (Fig. 5):

We begin by considering an $(S^3, \mathcal{O})$-null clasper G. We know already that we can compute $Z([S^3, G])$ from the Kontsevich integral $Z(S^3, G \cup \mathcal{O})$ of the corresponding link by integrating along G-colored struts. Choose string-link representatives of $G \cup \mathcal{O}$ with relative scaling. We will denote edges and leaves of G by E and L, respectively.

The diagrams that appear in $Z(S^3, G \cup \mathcal{O})$ are *hairy*, in the sense that upon removal of all $\mathcal{O}$-colored legs, they are simply unitrivalent graphs whose legs are colored by G.

Let us separate the hairy strut part $Z^{hq}(S^3, G \cup \mathcal{O})$ from the other part $Z^{ht}(S^3, G \cup \mathcal{O})$:

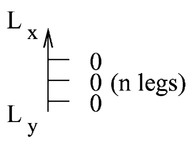
$$Z(S^3, G \cup \mathcal{O}) = Z^{hq}(S^3, G \cup \mathcal{O})\, Z^{ht}(S^3, G \cup \mathcal{O})$$

using the disjoint union multiplication, where Z^{ht} contains no diagrams that contain an L-labeled hairy strut component. Let us write the hairy strut part as follows:

$$Z^{hq}(S^3, G \cup \mathcal{O}) = \exp\left(\frac{1}{2} \sum_{x,y \in L \cup E} f(L_x, L_y, \mathcal{O})\right),$$

where $E \cup L$ is the corresponding link of leaves and edges of G and

$$f(L_x, L_y, \mathcal{O}) = \sum_{n=0}^{\infty} \mu_{L_x, L_y, O \ldots n \text{ times} \ldots 0}$$



In the above equation, $\mu_{L_x, L_y, O \ldots n \text{ times} \ldots 0}$ is the coefficient in $f(L_x, L_y, \mathcal{O})$ of the hairy diagram appearing on the right. We will call $f(\cdot, \cdot, \mathcal{O})$ the *hairy linking matrix*.

Lemma 3.8. *The hairy linking matrix of G and its negative inverse are given by*

$$\begin{pmatrix} 0 & I \\ I & f(L_i,L_j,\mathcal{O}) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} f(L_i,L_j,\mathcal{O}) & -I \\ -I & 0 \end{pmatrix}.$$

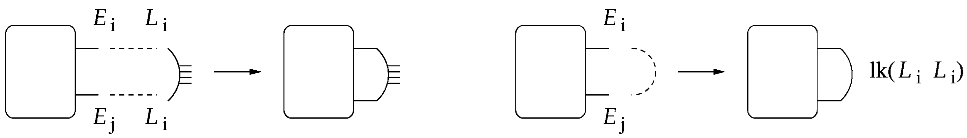
Proof. Since $\{E_i,E_j,\mathcal{O}\}$ is an unlink and $\{E_i,L_{j\neq i},\mathcal{O}\}$ is the disjoint union of an unknot E_i and the link $\{L_{j\neq i},\mathcal{O}\}$, it suffices to consider the case of a hairy strut $\big|^{E_i}_{L_i}$. In this case, E_i is a meridian of L_i. The formula for the Kontsevich integral of the Long Hopf Link of [5] (applied to E_i), together with the fact that L_i has linking number zero with $\mathcal{O}$, imply that the hairy part $\big|^{E_i}_{L_i}$ vanishes. □

Lemma 3.9. *$Z([(S^3,\mathcal{O}),G])\in\mathcal{A}(*_{\mathcal{O}})$ can be computed from $Z^{ht}(S^3,G\cup\mathcal{O})\in\mathcal{A}(*_{G\cup\mathcal{O}})$ by gluing pairwise the legs of the G-colored graphs of $Z^{ht}(S^3,G\cup\mathcal{O})$ using the negative inverse hairy linking matrix.*

Proof. We have that

$$Z^{t}(S^3,G\cup\mathcal{O})=\exp\left(\frac{1}{2}\sum_{x,y\in L\cup E}\Big|^{x}_{y} f(x,y,\mathcal{O})-\mathrm{lk}(x,y)\right)Z^{ht}(S^3,G\cup\mathcal{O}).$$

In other words, the diagrams that contribute in $Z([(S^3,\mathcal{O}),G])$ are those whose components either lie in $Z^{ht}(S^3,G\cup\mathcal{O})$ or are hairy struts of the shape $\big|^{L_i}_{L_j}$. Because of the restriction of the linking matrix (5), the pair of legs $\{L_i,L_j\}$ of a hairy strut of the above shape must be glued to a pair of legs labeled by $\{E_i,E_j\}$ as follows:



The result of this gluing is equivalent to gluing pairs of (E_i,E_j) colored legs using the negative inverse hairy linking matrix. This concludes the proof of the lemma. □

Given this lemma, the proof of Proposition 3.2 works without change and proves Theorem 1.

Furthermore, observe that the coefficients of hairy vortices with legs colored by the edges of G and with nonzero number of hair vanish. This is true since the Borromean rings (i.e., the edges of G) form an unlink in the complement of $\mathcal{O}$. Thus, the proof of Proposition 3.4 implies the following:

Theorem 4. *If G is a $(S^3,\mathcal{O})$-null clasper of degree $2n$, then*

$$Z_{2n}([(S^3,\mathcal{O}),G])=\langle G\rangle\in\mathcal{A}_{2n}(*)$$

where complete contractions are using the matrix of Lemma 3.9 instead and put the entries of it as beads on the edges that are created by the contraction of the legs.

Let us remark that the analogue of Proposition 3.6 does not hold since Z takes values in $\mathcal{A}(*)$ rather than in $\mathcal{A}(\Lambda)$. The invariant Z^{rat} discussed in Section 1.4 takes values in $\mathcal{A}(\Lambda)$, satisfies the analogue of proposition 3.4 and is thus a universal $\mathbb{Q}$-valued invariant of Alexander polynomial 1 knots, with respect to the null-move.

4. Abelian invariants: equivariant linking numbers

In Theorem 4 we calculated the symbol $Z_{2n}[(S^3,\mathcal{O}),G] \in \mathscr{A}_{2n}(\Lambda)$ in terms of a complete contraction of G that uses the hairy linking matrix. Struts correspond to linking numbers and we will show that hairy struts correspond to equivariant linking numbers. The goal of this section is to discuss the latter.

4.1. A review of linking numbers

Recall that all links are oriented. We begin by recalling the definition of the *linking number* $\mathrm{lk}(L)$ of a link $L=(L_1,L_2)$ of two ordered components in S^3. Since $H_1(S^3,\mathbb{Z})=0$, there is an oriented surface Σ_1 that L_1 bounds. We then define $\mathrm{lk}(L)=[\Sigma_1]\cdot[L_2]\in\mathbb{Z}$ where $\cdot$ is the intersection pairing. Since $H_2(S^3)=0$, the result is independent of the choice of surface Σ_1.

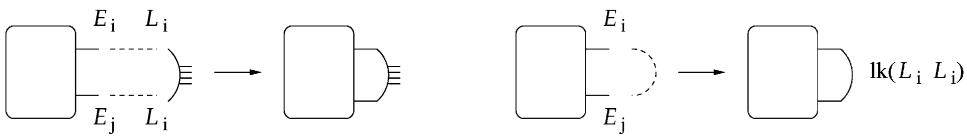
The following lemma summarizes the well-known properties of the linking numbers of two component links in S^3:

Lemma 4.1 (Symmetry). $\mathrm{lk}(L)$ *does not depend on the ordering of the components of L.*

(Cutting). *If a component L_i of L is a connected sum of L_i' and L_i'' (as in Fig.* 3), *then*

$$\mathrm{lk}(L)=\mathrm{lk}(L')+\mathrm{lk}(L'').$$

Initial condition:



Uniqueness: *If a function of two-component links satisfies the Symmetry, Cutting, and Initial Conditions, then it equals to* lk.

Proof. It is easy to see that linking numbers satisfy the above axioms.

For Uniqueness, assume that α is another such function, and let $\beta=\alpha-\mathrm{lk}$; consider a link $L=(L_1,L_2)$, and a surface Σ_1 that bounds L_1. Using the Cutting Property, as in the proof of Theorem 3, it follows that $\beta(L)$ is a linear combination of $\beta(\text{Hopf Link})$ and $\beta(\text{Unlink})$, and hence zero. $\square$

Remark 4.2. The above definition of linking number can be extended to the case of an two component links L in a rational homology 3-sphere M. In that case, the linking number takes values in $1/|H_1(M,\mathbb{Z})|\mathbb{Z}$, and Proposition 4.1 continues to hold.

Remark 4.3. The above definition of linking number can be extended to the case of a two-component link L in a possibly open 3-manifold N that satisfies $H_1(N,\mathbb{Z})=H_2(N,\mathbb{Z})=0$.

4.2. Equivariant linking numbers

In this section we review well-known results about equivariant linking numbers. These results are useful to the *surgery view* of knots, (see [29,24,34] and also [13]) and will help us identify the hairy struts with equivariant linking numbers.

We begin with the following simple situation: consider a null homologous link $L=(L_1,L_2)$ of two ordered components in $X=S^3\setminus\mathcal{O}$.

Definition 4.4. We will call such links L $(S^3,\mathcal{O})$-*null*.

Consider the universal abelian cover $\tilde{X}\to X$ corresponding to the natural map $\pi_1(X)\to H_1(X,\mathbb{Z})\cong\mathbb{Z}$. Since L is null homologous, it lifts to a link $\tilde{L}$, which is invariant under the action by the group of deck transformations $\mathbb{Z}$. Note that $\tilde{X}$ is an open 3-manifold diffeomorphic to $D^2\times\mathbb{R}$, in particular $H_1(\tilde{X},\mathbb{Z})=H_2(\tilde{X},\mathbb{Z})=0$. Using Remark 4.3, we can define linking numbers between the components of $\tilde{L}$.

Consider an arc-basing γ of L, that is an embedded arc in X that begins in one component of L and ends at the other, and is otherwise disjoint from L. Then, we can consider a lift of $L\cup\gamma$ to $\tilde{X}$ which is an arc-based two component link $(\tilde{L}_1,\tilde{L}_2)$ in $\tilde{X}$.

Definition 4.5. If (L,γ) is an $(S^3,\mathcal{O})$-null arc-based link, we define the *equivariant linking number* by

$$\widetilde{\mathrm{lk}}^{\gamma}(L)=\sum_{n\in\mathbb{Z}}\mathrm{lk}(\tilde{L}_1,t^n\tilde{L}_2)t^n\in\Lambda.$$

The above sum is finite. Moreover, since linking numbers (in $\tilde{X}$) are invariant under the action of deck transformations, it follows that the above sum is independent of the choice of lift of $L\cup\gamma$ to $\tilde{X}$.

The following lemma summarizes the properties of the equivariant linking number of $(S^3,\mathcal{O})$-null arc-based links. $\varepsilon\colon\Lambda\to\mathbb{Z}$ stands for the map $t\to 1$.

Lemma 4.6 (Symmetry). *If σ is a permutation of the two components of L then, we have*

$$\widetilde{\mathrm{lk}}^{\gamma}(L_\sigma)(t)=\widetilde{\mathrm{lk}}^{\gamma}(L)(t^{-1}).$$

Specialization:

$$\varepsilon\,\widetilde{\mathrm{lk}}^{\gamma}(L)=\mathrm{lk}(L).$$

Cutting: *Suppose that a component L_i of an $(S^3,\mathcal{O})$-null link (L,γ) is a connected sum of L_i' and L_i'' (as in Fig.* 3), *and (L',γ) and (L'',γ) are $(S^3,\mathcal{O})$-null. Then,*

$$\widetilde{\mathrm{lk}}^{\gamma}(L)=\widetilde{\mathrm{lk}}^{\gamma}(L')+\widetilde{\mathrm{lk}}^{\gamma}(L'').$$

Λ-Sliding: *If (L^s,γ) denote the result of sliding the arc-basing of the first component of (L,γ) along an oriented arc-based curve S, then*

$$\widetilde{\mathrm{lk}}^{\gamma}(L^s)=t^l\,\widetilde{\mathrm{lk}}^{\gamma}(L),$$

where $l=\mathrm{lk}(S,\mathcal{O})$.

Initial condition: *If L lies in a ball disjoint from $\mathcal{O}$, then*

$$\widetilde{\mathrm{lk}}^{\gamma}(L) = \mathrm{lk}(L).$$

Uniqueness: *If a function of $(S^3, \mathcal{O})$-null arc-based links satisfies the Symmetry, Λ-Sliding, Cutting and Initial Conditions, then it equals to the equivariant linking number.*

Proof. The Symmetry, Cutting and Λ-Sliding Properties follows immediately from Lemma 4.1.

The Specialization Property follows from the well-known fact that given a covering space $\pi: \tilde{X} \to X$ and a cycle c in X that lifts to $\tilde{c}$ in $\tilde{X}$, and a cycle c' in $\tilde{X}$, then the intersection of $\tilde{c}$ with c' equals to the intersection of c with the push-forward of c' in X.

The Uniqueness statement follows from the proof of Theorem 2 given in Section 2.4. □

Remark 4.7. Definition 4.5 and Lemma 4.6 can be extended without change to the case of a null homologous arc-based two component link (L, γ) in $M \setminus K$ where (M, K) is a knot K in a integral homology 3-sphere M with trivial Alexander polynomial. Note that $H_2(\widetilde{M \setminus K}, \mathbb{Z}) = 0$ and the condition on the Alexander polynomial ensures that $H_1(\widetilde{M \setminus K}, \mathbb{Z}) = 0$.

A further generalization of Definition 4.5 and Lemma 4.6 is possible to the case of a null homologous arc-based two component link (L, γ) in $M \setminus K$ where (M, K) is a knot K in a integral homology 3-sphere M. In that case, $\widetilde{\mathrm{lk}}^{\gamma}(L)$ lies in a *localization* Λ_{loc} of Λ defined by

$$\Lambda_{\mathrm{loc}} = \{ p/q \mid p, q \in \Lambda, \, q(1) = \pm 1 \}.$$

This specializes to the case of L being a zero-framed parallel of a null homologous knot K' in $S^3 \setminus K$ and γ a short arc between the two components of L. In this case, $\widetilde{\mathrm{lk}}^{\gamma} t(K', K')$ coincides with the self-linking η-function $\eta(K, K')$ of Kojima–Nakanishi [24].

5. Hairy struts are equivariant linking numbers

The purpose of this section is to show that the hairy struts coincide with equivariant linking numbers.

We begin with a definition. A *special link* $L \subset S^3$ is the union of $\mathcal{O}$ and an $(S^3, \mathcal{O})$-null link. Note that a special link L has a special component, namely $\mathcal{O}$.

In what follows, γ will refer to a *disk-basing* (i.e., a string-link representative) of L. Consider a disk-based special link (L, γ), together with a choice of relative scale, and its Kontsevich integral. There is an algebra isomorphism $\sigma: \mathcal{A}(\uparrow_L) \to \mathcal{A}(*_L)$ with inverse the symmetrization map $\chi: \mathcal{A}(*_L) \to \mathcal{A}(\uparrow_L)$ which is the average of all ways of placing symmetric L-colored legs on L-intervals. In what follows, we will denote by $Z(S^3, L)$ the image of the Kontsevich integral in either algebra, hopefully without causing confusion.

Consider a homotopy quotient $\mathcal{A}^h(*_L)$ of the algebra $\mathcal{A}(*_L)$ where we quotient out by all non-forests (i.e., graphs with at least one connected component which is not a forest) and by all *flavored* forests that contain a tree with at least two legs flavored by the same component of L, see also [2,20]. A *flavoring* of a graph is a decoration of its univalent vertices by a set of colors, which in our case is L. We will be interested in the Lie subalgebra $\mathcal{A}^{c,h}(*_L)$ of $\mathcal{A}^h(*_L)$ spanned by all

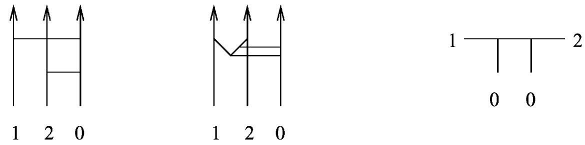


Fig. 6. On the left, $t_{20}t_{10}$, where we multiply from left to right and from the bottom to the top. In the middle, $T_2(12)$. In the right, an alternative view of $T_2(12)$ with symmetrized legs.

connected diagrams. It is spanned by hairy trees colored by $L \setminus \mathcal{O}$ such that each label of $L \setminus \mathcal{O}$ appears at most once. Note that if T and T' are A-colored and A'-colored hairy trees then

$$[T, T'] = \begin{cases} 0 & \text{if } |A \cup A'| \neq 1, \\ T \cdot_i T' & \text{if } A \cup A' = \{i\}, \end{cases}$$

where $T \cdot_i T'$ is the result of *grafting* the trees T and T' along their common leg i.

Let us define $Z^h(S^3, L)$ to be the image of $Z(S^3, L)$ in $\mathscr{A}^h(*_L)$. The group-like property of the Kontsevich integral implies that $\log Z^h(S^3, L)$ lies in $\mathscr{A}^{c,h}(*_L)$. In the case L has three components, $\mathscr{A}^{c,h}(*_L)$ has a basis consisting of the trees $T_n(12)$, for $n \geqslant 0$, together with the degree 1 struts t_{10} and t_{20}, and that

$$\mathrm{ad}^n_{t_{20}}\, t_{12} = (-1)^n T_n(12) \quad \text{and} \quad \mathrm{ad}^n_{t_{10}}\, t_{12} = T_n(12). \tag{7}$$

Here the product (and the commutator) are taken with respect to the natural multiplication on $\mathscr{A}(\uparrow_L)$ (Fig. 6).

It follows by the definition of the hairy linking matrix that for every special link L, we have

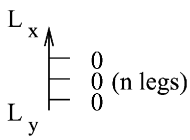
$$\log Z^h(S^3, L) = \sum_{L'} f^{\gamma}(L') \in \mathscr{A}^{c,h}(*_L), \tag{8}$$

where we explicitly denote the dependence on the disk basing γ, and where the sum is over all special sublinks L' of L that contain $\mathcal{O}$ and two more components of $L \setminus \mathcal{O}$ (with repetition).

At this point, let us convert hairy struts into a power series as follows:

Definition 5.1. If L is a special link of three components with disk-basing γ and choice of relative scaling, and

$$f(L) = \sum_{n=0}^{\infty} \mu_{L_x, L_y, O \ldots n\,\text{times} \ldots 0}$$

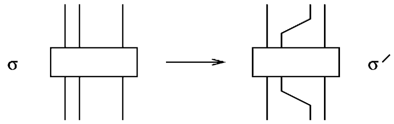


then

$$\phi^{\gamma}(L) = \sum_{n=0}^{\infty} \mu_{L_x, L_y, O \ldots n\,\text{times} \ldots 0}\, x^n \in \mathbb{Q}[[x]].$$

Lemma 5.2. *If (L, γ) is a special link of three components equipped with relative scaling, then $\phi^{\gamma}(L)$ is independent of the relative scaling of L.*

Proof. It will be more convenient to present the proof in the algebra $\mathscr{A}(\uparrow_L)$. Let σ' denote a change of scaling of a string-link σ, as shown in the following figure in case σ has three stands:



The locality of the Kontsevich integral implies that $Z(\sigma') = \Phi Z(\sigma)\Phi^{-1}$ for an associator Φ. Write $\Phi = \mathrm{e}^{\phi}$ for an element $\phi \in [\mathscr{L},\mathscr{L}]$, where $\mathscr{L}$ is the free-Lie algebra $\mathscr{L}$ of two generators a, b. The following identity:

$$\mathrm{e}^a \mathrm{e}^b \mathrm{e}^{-a} = \mathrm{e}^{\exp(\mathrm{ad}_a)b} \tag{9}$$

(valid in a free Lie algebra of two generators) implies that $\log Z(\sigma') = \mathrm{e}^{\mathrm{ad}_\phi} \log Z(\sigma) \in \mathscr{A}(\uparrow_{123})$. If we project the above equality to the quotient $\mathscr{A}^h(\uparrow_{123})$ where connected diagrams with two legs either both on the first strand or both on the second strand vanish, then it follows that we can replace $\phi \in [\mathscr{L},\mathscr{L}]$ by its image $\bar{\phi} \in [\mathscr{L},\mathscr{L}]/[\mathscr{L},[\mathscr{L},\mathscr{L}]]$. It is easy to see that $\bar{\phi} = \frac{1}{24}[a,b]$ for *any* associator Φ.

Now we can finish the proof of the lemma as follows. Consider a sting-link σ with relative scaling obtained from a disk-basing of a special link L of three components and let L' be the one obtained by a change of relative scaling of L. Projecting to $\mathscr{A}^h(\uparrow_L)$, and using the fact that $\log Z^h(S^3, L)$ lies in the center of the Lie algebra $\mathscr{A}^{c,h}(\uparrow_L)$, it follows that $\log Z^h(S^3, L') = \log Z^h(S^3, L)$. □

Lemma 5.3. *ϕ^γ satisfies the Cutting Property of Lemma* 4.6 *with* $t = \mathrm{e}^x$.

Proof. Consider the special link $(L_{1'1''20},\gamma)$, (this is an abbreviation for $((L_{1'}, L_{1''}, L_2, \mathcal{O}),\gamma))$ whose connected sum of the first two components gives (L,γ). How does the Kontsevich integral of $L_{1'1''20}$ determine that of L_{120}? The answer, though a bit complicated, is known by Bar-Natan [4, Part II, Proposition 5.4]. Following that notation, we have

$$Z(S^3, L_{120}) = \langle \exp(\Lambda_1^{1'1''}), Z(S^3, L_{1'1''20})\rangle_{1',1''} \in \mathscr{A}(*_{L_{120}}),$$

where $\langle A, B\rangle_{1',1''}$ is the operation that glues all $\{1', 1''\}$-colored legs of A to those of B (assuming that the number of legs of color $1'$ and of color $1''$ in A and B match; otherwise it is defined to be zero), and

$$\Lambda_1^{1'1''} = |_1^{1'} + |_1^{1''} + \Lambda_1^{1'1''} \quad \text{(other)},$$

where $\Lambda_1^{1'1''}$ (other) is an (infinite) linear combination of rooted trees with at least one trivalent vertex whose leaves are colored by (L_1', L_1'') and whose root is colored by L_1. We will call such trees $(1', 1''; 1)$-trees. The reader may consult [4, Part II, Proposition 5.4] for the first few terms of $\Lambda_1^{1'1''}$, which are given by *any* Baker–Cambell–Hausdorff formula, translated in terms of rooted trees.

Upon projecting the answer to the quotient $\mathscr{A}^h(*_{L_{120}})$, the above formula simplifies. Indeed, if we glue some disjoint union of trees of type $(1', 1''; 1)$ that contain at least one trivalent vertex to some hairy $L_{1'1''2}$-colored trees, the resulting connected graph will either have nontrivial homology, or at least two labels of L_2. Such graphs vanish in $\mathscr{A}^h(*_{L_{120}})$. Thus, when projecting the above formula

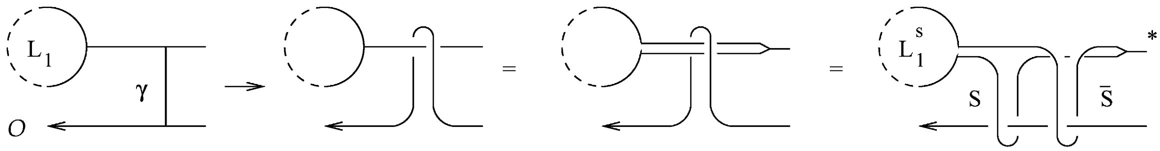


Fig. 7. A finger move along an arc γ.

to $\mathscr{A}^h(*_{L_{120}})$, we can assume that $\Lambda_1^{1'1''} = |_1^{1'} + |_1^{1''}$. Using the fact of how the Kontsevich integral of a link determines that of its sublinks and the above, it follows that

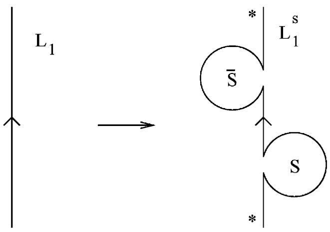
$$\begin{aligned}\log Z^h(S^3, L_{120}) &= \log \langle \exp(|_1^{1'} + |_1^{1''}), Z^h(S^3, L_{1'1''20}) \rangle_{1',1''} \\ &= \log Z^h(S^3, L_{1'20}) + \log Z^h(L_{1''20}),\end{aligned}$$

which, together with Eq. (8) concludes the proof. □

Lemma 5.4. *ϕ^γ satisfies the Λ-Sliding Property of Lemma* 4.6 *with* $t = \mathrm{e}^x$.

Proof. Consider the link (L_{S120}, γ). Recall that a slide move is given by

In an artistic way, the next figure shows the result of a slide move (compare also with Fig. 7) that replaces L_1 by $L_1^s := S \sharp L_1 \sharp \bar{S}$, where $\bar{S}$ is the orientation reversed knot.

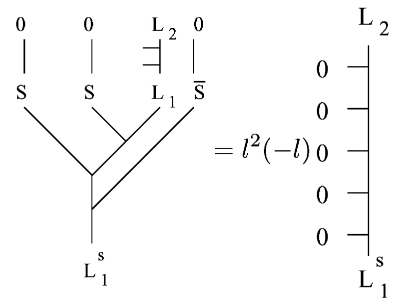


As in the previous lemma, we have that

$$Z(S^3, L_{1^s 20}) = \langle \exp(\Lambda_{1^s}^{S1}), \Delta_{S\bar{S}} Z(S^3, L_{S120}) \exp(\Lambda_{1^s}^{1\bar{S}}) \rangle_{1,S,\bar{S}} \in \mathscr{A}(*_{L_{1^s 20}}),$$

where $\Delta_{S\bar{S}}$ is the operation that replaces an S-colored leg to an $S + \bar{S}$-colored one.

Upon projecting to $\mathscr{A}^h(*_{L_{1^s 20}})$, the above formula simplifies. Indeed, the $\{S, L_1, L_2\}$-colored hairy trees are (after removal of the hair) of two shapes Y and I. When we glue, a $\{S, L_1, L_2\}$-colored hairy Y vanishes in $\mathscr{A}^h(*_{L_{1^s 20}})$. The remaining $\{S, L_1, L_2\}$-colored hairy trees are of the shapes $\mathsf{I}^S_{L_1}, \mathsf{I}^S_{L_2}, \mathsf{I}^{L_1}_{L_2}, \mathsf{I}^{\bar{S}}_{L_1}, \mathsf{I}^{\bar{S}}_{L_2}$, as well as the hairless $\mathsf{I}^S_{\mathcal{O}}$ and $\mathsf{I}^{\bar{S}}_{\mathcal{O}}$. When glued to $(S, 1; 1^s)$ and $(\bar{S}, 1; 1^s)$ rooted trees, the ones of shapes $\mathsf{I}^S_{L_1}, \mathsf{I}^{\bar{S}}_{L_1}$ vanish in $\mathscr{A}^h(*_{L_1^s, L_2, \mathcal{O}})$, thus we remain it remains to consider only trees of shape $\mathsf{I}^S_{L_2}, \mathsf{I}^{L_1}_{L_2}, \mathsf{I}^{\bar{S}}_{L_2}$ and the hairless $\mathsf{I}^S_{\mathcal{O}}$ and $\mathsf{I}^{\bar{S}}_{\mathcal{O}}$. When we glue these to $(S, 1; 1^s)$ and $(\bar{S}, 1; 1^s)$ rooted trees, the only nonzero contribution comes from gluings like



The above figure shows that each of the above gluings can be thought of as starting from a hairy graph of shape $\mathsf{I}^{L_1}_{L_2}$, and adding to it some additional hair, first on the left and then on the right; each time multiplying the result by $l^n(-l)^m$ (where $l = \mathrm{lk}_M(S, \mathcal{O})$) and n, m is the number of left and right added hair. On the other hand, adding hair is the same as commuting with t_{10} (as follows by Eq. (7)). Translating from gluings back to Lie algebras, it follows that

$$\begin{aligned} Z^h(S^3, L_{1^s 20}) &= (\mathrm{e}^{\mathsf{I}^K_S} Z^h(S^3, L_{120}) \mathrm{e}^{\mathsf{I}^K_{\bar{S}}})/(S, \bar{S} \to 1)/(1 \to 1^s) \\ &= (\mathrm{e}^{l t_{10}} Z^h(S^3, L_{120}) \mathrm{e}^{-l t_{10}})/(1 \to 1^s), \end{aligned}$$

where $E/(1 \to 1^s)$ means to replace the label 1 by 1^s in the expression E. Eq. (9) implies that

$$\log Z^h(S^3, L_{1^s 20}) = \mathrm{e}^{l\, \mathrm{ad}_{t_{10}}} f^\gamma(L_{12})(\mathrm{ad}_{t_{10}}) t_{12}/(1 \to 1^s)$$

which concludes the proof. $\square$

The following theorem is the main result of this section.

Theorem 5. *For an $(S^3, \mathcal{O})$-null arc-based link (L, γ) of three components we have that*

$$\phi^\gamma(L)(x) = \widetilde{\mathrm{lk}}^\gamma(L \setminus \mathcal{O})(\mathrm{e}^x) \in \mathbb{Q}[\mathrm{e}^{\pm x}].$$

In other words, the coefficients of hairy struts are equivariant linking numbers, in particular they are Laurent polynomials.

Proof. This follows from Lemma 4.6 once we show that ϕ^γ satisfies the Symmetry, Specialization, Λ-Sliding, Cutting and Initial Condition stated in that lemma.

The symmetry follows by Eq. (7). Specialization follows from the fact that $\phi^\gamma(L)(0)$ coincides with the coefficient of a strut in the Kontsevich integral, which equals to $\mathrm{lk}(L_1, L_2)$. The Λ-Sliding property follows from Lemma 5.4, the Cutting property follows from Lemma 5.3. The Initial Condition property follows from the fact that the Kontsevich integral is multiplicative for disjoint union of links, thus the only diagrams T_n that contribute to the Kontsevich integral (and also in ϕ^γ) in this case is T_0, which contributes the linking number of L_1 and L_2. $\square$

Remark 5.5. Theorem 5 remains true for (M, K)-null two component links L where K is a knot in an integral homology 3-sphere.

Since the coefficient μ_n of x^n in the power series $\phi^\gamma(L)$ are given by a combination of Milnor's invariants (as follows by the work of Habegger–Masbaum [20] for $M = S^3$ and forthcoming work of the first author for a general integral homology 3-sphere), we obtain that

Corollary 5.6. $\widetilde{\mathrm{lk}}^\gamma$ *is a concordance invariant of* L *and a link homotopy invariant of* $L \setminus K$.

This generalizes a result of Cochran [6], who showed that a power series expansion of the Kojima self-linking function was a generating function for a special class of repeated Milnor invariants of type μ_{11KKKK} of a two component algebraically split link (L_1, K).

5.1. Addendum

In an earlier version of the paper, we also identified hairy struts with equivariant triple Milnor linking numbers, for special links of four components. Using this, we gave an algorithm for computing the 2-loop part (or, equivalently, the Euler degree 2 part) of the Kontsevich integral of a knot with trivial Alexander polynomial. This involves a more delicate counting below the critical degree, in the language of Section 3.2. For an easier digestion of the present paper, we prefer to come back to this matter in a future publication.

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