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SOME IHX-TYPE RELATIONS ON TRIVALENT GRAPHS AND SYMPLECTIC REPRESENTATION THEORY

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Abstract. We consider two types of graded algebras (with graded actions by the symplectic Lie algebra) that arise in the study of the mapping class group, and describe their symplectic invariants in terms of algebras on trivalent graphs.

1. Introduction

Let $\mathfrak{sp}_g$ be the Lie algebra of symplectic matrices of degree $2g$ over the rational numbers. In recent studies related to the structure of the *surface mapping class group*, several authors [KM, Mo, Ha, HL] have encountered a certain distinguished quotient $\mathcal{B}$ of the exterior algebra ΛU, where U is an irreducible $\mathfrak{sp}_g$-module isomorphic to $\Lambda^3 H/H$, H is the fundamental $\mathfrak{sp}_g$-module and Λ^k is the k^{th} exterior functor. The second exterior component $\Lambda^2 U$ is decomposed as an $\mathfrak{sp}_g$-module in the following way:

$$\Lambda^2 U \cong [1^6]_{\mathfrak{sp}} \oplus [1^4]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}} \oplus [2^2 1^2]_{\mathfrak{sp}} \oplus [2^2]_{\mathfrak{sp}} \quad (g \geq 6), \tag{1}$$

where $[\lambda]_{\mathfrak{sp}}$ denotes an irreducible $\mathfrak{sp}_g$-module corresponding to a partition λ, and the algebra $\mathcal{B}$ mentioned above is defined by

$$\mathcal{B} := \Lambda U / ([2^2]_{\mathfrak{sp}}). \tag{2}$$

Here, $([2^2]_{\mathfrak{sp}})$ is the ideal of ΛU generated by the compoment $[2^2]_{\mathfrak{sp}} \subset \Lambda^2 U$ according to the decomposition (1).

In fact, the algebra $\mathcal{B}$ appears in the Hodge theoretic study of the mapping class group M_g (of a closed genus g surface) by R. Hain [Ha], who established a theory of mixed Hodge structure for the Torelli group $T_g := \ker(M_g \to \mathrm{Sp}_g(\mathbb{Z}))$. Introducing the unipotent kernel $\mathfrak{u}_g$ of the 'relative Malcev completion' of the map $M_g \to \mathrm{Sp}_g(\mathbb{Q})$, Hain showed that the universal envelope $\mathbb{U}\mathrm{Gr}^W \mathfrak{u}_g$ of the weight graded Lie algebra $\mathrm{Gr}^W \mathfrak{u}_g$ is quadratic dual to the lowest weight subalgebra of the continuous cohomology $H^*_{cts}(\mathfrak{u}_g)$. Then, using the fact $\mathrm{Gr}^W \mathfrak{u}_g$ is generated by the weight -1 component isomorphic to U, he gave a natural isomorphism

$$\mathcal{B} \cong \bigoplus_{k \geqslant 0} W_k H^k_{cts}(\mathfrak{u}_g).$$

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See [HL, Prop. 9.9]. From this, the $\mathfrak{sp}_g$-invariants of $\mathcal{B}$ are naturally mapped into the cohomology algebra $H^*(M_g, \mathbb{Q})$.

On the other hand, S.Morita [Mo] introduced a Weyl-type [W] interpretation of $(\Lambda U)^{\mathfrak{sp}}$, the space of $\mathfrak{sp}_g$-invariants of ΛU, by the algebra of trivalent graphs $\mathcal{C}(\phi)$ (see 2.1 below). Then, using the generalized Morita-Mumford-Miller classes (with twisted coefficients) from [K], N.Kawazumi and S.Morita [KM, Mo] related explicitly the trivalent graphs 'lying in' $(\Lambda U)^{\mathfrak{sp}}$ with the Morita-Mumford-Miller classes $e_n \in H^{2n}(M_g, \mathbb{Q})$, and showed that $(\Lambda U)^{\mathfrak{sp}}$ surjects onto the subalgebra $\mathbb{Q}[e_1, e_2, \dots]$ in $H^*(M_g, \mathbb{Q})$.

Motivated by the above mentioned works [Ha, HL, K, KM, Mo] (as well as by the stable independence of the classes e_i's due to Miller, Morita), one would like to understand stably the structure of the $\mathfrak{sp}_g$-invariant algebra $\mathcal{B}^{\mathfrak{sp}} = (\Lambda U/([2^2]_{\mathfrak{sp}}))^{\mathfrak{sp}}$ purely combinatorially in terms of trivalent graphs and their relations.

Let $\mathcal{C}(\phi)$ denote the commutative graded algebra generated by the trivalent graphs that (possibly) contain multiple edges and 1-loops (where a 1-loop is an edge which begins and ends on the same vertex). By definition, the multiplication in $\mathcal{C}(\phi)$ is given by disjoint union of graphs, and the degree of a trivalent graph is half the number of vertices. Let also loop denote the ideal generated by graphs containing a 1-loop.

Define IH_0 to be the ideal of $\mathcal{C}(\phi)$ 'identifying' $I = H$ (with 4 distinct edges connected to a central edge) locally in trivalent graphs.

Theorem 1. *There exists a stable isomorphism of graded algebras*

$$\mathcal{C}(\phi)/(IH_0, \mathrm{loop}) \xrightarrow{\sim} (\Lambda U/([2^2]_{\mathfrak{sp}}))^{\mathfrak{sp}}$$

which multiplies degrees by 2*. Here, "stable" means that for each degree m, the map is an isomorphism for $g \geq 3m$.*

It is easy to see that $\mathcal{C}(\phi)/(IH_0, \mathrm{loop})$ is a polynomial algebra (freely) generated by the IH_0-classes of connected trivalent graphs without 1-loops, i.e., having one generator in every positive degree (cf. 3.3 (c) below). See also [KM, p. 639] for a relevant discussion involving Kontsevich's primitive factors "$H_0(\mathrm{Out}F_n, \mathbb{Q})$" [Ko3].

In recent studies related to finite type 3-manifold invariants [GO, LMO, L], perturbative Chern-Simons theory [Wi, Ka, Ko4, RW] and graph cohomology [Ko1, Ko2], the algebra $\mathcal{A}(\phi) = \tilde{\mathcal{C}}(\phi)/(AS, IHX)$ plays a crucial role, where $\tilde{\mathcal{C}}(\phi)$ is the algebra generated by vertex-oriented trivalent graphs and AS, IHX are the relations shown in Figure 1.

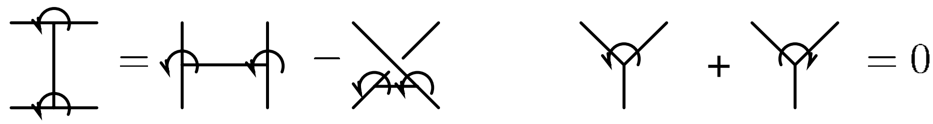


Figure 1. The *AS* and *IHX* relation on vertex-oriented trivalent graphs.

In fact, letting $\mathcal{M}$ be the vector space spanned by the isomorphism classes of (oriented) integral homology 3-spheres, and $\widehat{\mathcal{A}}(\phi)$ the completion of $\mathcal{A}(\phi)$ (with respect to the graph degrees), Le-Murakami-Ohtsuki [LMO] constructed a map $\Omega : \mathcal{M} \to \widehat{\mathcal{A}}(\phi)$ which turns out [L, GO] to be the 'universal' finite type invariant of integral homology 3-spheres, in the sense that Ω induces an isomorphism between the graded space of $\mathcal{M}$ (with respect to the Ohtsuki filtration) and $\mathcal{A}(\phi)$. The proof of Theorem 1 provides an alternative characterization of $\mathcal{A}(\phi)$ as follows. Noting the $\mathfrak{sp}$-decomposition of $\Lambda^2(\operatorname{Sym}^3 H)$ (where Sym^n denotes the n-th symmetric tensor functor)

$$\Lambda^2(\operatorname{Sym}^3 H) = [5,1]_{\mathfrak{sp}} \oplus [4]_{\mathfrak{sp}} \oplus [3^2]_{\mathfrak{sp}} \oplus [2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}} \quad (g \geq 2). \tag{3}$$

Theorem 2. *Stably, we have an isomorphism of graded algebras (which multiplies degrees by* 2*)*

$$\mathcal{A}(\phi) \xrightarrow{\sim} (\Lambda(\operatorname{Sym}^3 H)/([4]_{\mathfrak{sp}}))^{\mathfrak{sp}}.$$

In the above theorem "stably" means that for each degree m, the map is an isomorphism for $g \geq m$.

Since $[4]_{\mathfrak{sp}} \cong \operatorname{Sym}^4 H$, this result is closely related to the study of the Lie algebra $\operatorname{Ham} = \bigoplus_{m \geq 2} \operatorname{Sym}^m H$ of formal Hamiltonian vector fields with Poisson brackets [Ko2, Ko3]. In fact, the kernel Ham^1 of $\operatorname{Ham} \to \operatorname{Sym}^2 H \cong \mathfrak{sp}_g$ is generated by $\operatorname{Sym}^3 H$, thus inducing a natural homomorphism $\Lambda(\operatorname{Sym}^3 H)/([4]_{\mathfrak{sp}}) \to H^*(\operatorname{Ham}^1, \mathbb{Q})$ (hence also $\mathcal{A}(\phi) \to H^*(\operatorname{Ham}, \mathbb{Q})$). In view of comparison of two situations of our above theorems, it seems an interesting (future) subject to determine the kernel/cokernels of the latter homomorphisms.

The idea of the present paper arose during conversations of the first named author with S. Morita in July 1996 and during a visit of the second named author to Harvard in the fall of 1996. We wish to thank T. Gocho and P. Vogel for useful conversations and especially N.Kawazumi and S. Morita for enlightening communications and crucial remarks.

2. A reduction of Theorem 1

2.1. Trivalent graphs and $\mathfrak{sp}$-invariant tensors. In the course of proving Theorems 1 and 2, we will deduce several other results of independent interest, which we first formulate.

As was mentioned in the introduction, S. Morita [Mo] (using the device of H. Weyl [W]) introduced maps

$$\mathcal{C}(\phi) \to (\Lambda\Lambda^3 H)^{\mathfrak{sp}} \quad \text{and} \quad \mathcal{C}(\phi)/(\text{loop}) \to (\Lambda U)^{\mathfrak{sp}}, \tag{4}$$

and showed that they are (stable) isomorphisms of graded algebras (which multiply degrees by 2). "Stable" here means that in each degree m, the above maps are isomorphisms for $g \geq 3m$. These maps essentially place a copy of the symplectic form on each edge of a trivalent graph, but for the calculations needed below, we prefer to give coordinatewise definitions of them.

Let Γ be a *trivalent graph* with *vertex* set $\operatorname{Vert}(\Gamma)$ and *edge* set $\operatorname{Edge}(\Gamma)$ and let $\operatorname{Flag}(\Gamma)$ be the set of *flags*, where a flag is by definition a pair consisting of a vertex and an incident half-edge. Since each vertex has three adjacent flags, the cardinality $|\operatorname{Flag}(\Gamma)|$ is equal to $3|\operatorname{Vert}(\Gamma)| = 2|\operatorname{Edge}(\Gamma)|$. We call $m = 1/6|\operatorname{Flag}(\Gamma)|$ the degree of the trivalent graph Γ. A *total ordering* τ of Γ consists, by definition, of the following data:

- a linear ordering of vertices: $\operatorname{Vert}(\Gamma) = \{v_1, \ldots, v_{2m}\}$;
- a linear ordering of $\operatorname{Flag}(v) = \{f_1(v), f_2(v), f_3(v)\}$ for each $v \in \operatorname{Vert}(\Gamma)$;
- an ordering of $\operatorname{Flag}(e) = \{f_+(e), f_-(e)\}$ for each $e \in \operatorname{Edge}(\Gamma)$.

Such a τ is called $\wedge$-admissible if it satisfies the condition:

$$\operatorname{sgn}\begin{pmatrix} f_1(v_1) & f_2(v_1) & f_3(v_1) & f_1(v_2) & \ldots f_3(v_{2m}) \\ f_+(e_1) & f_-(e_1) & f_+(e_2) & f_-(e_2) & \ldots f_-(e_{3m}) \end{pmatrix} = 1$$

for every linear ordering of edges: $\operatorname{Edge}(\Gamma) = \{e_1, \ldots, e_{3m}\}$. Note that the sign is not changed under edge permutations.

Let H be an $\mathfrak{sp}_g$-module equipped with a standard symplectic basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ with $\langle x_i, y_j\rangle = \delta_{ij} = -\langle y_i, x_j\rangle$, $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = 0$ over $\mathbb{Q}$. Given a totally ordered trivalent graph (Γ, τ) of degree m, we define $f_{3i+j} := f_j(v_i)$ $(0 \leq i < 2m,\ j = 1, 2, 3)$ and $OR := \{f_+(e)\}_{e \in \operatorname{Edge}(\Gamma)}$, and shall define an $\mathfrak{sp}$-invariant $\alpha_{(\Gamma,\tau)} \in H^{\otimes 6m}$ as follows. First, regard $H^{\otimes 6m}$ as the linear combinations of the words in $\{x_i, y_i\}_{i=1}^g$ of length $6m$, and write $x_{-i} = y_i$, $x_0 = 0$. Let I be the set of indices $\mathbf{i} = (i_1, \ldots, i_{6m}) \in \{-g, -g+1, \ldots, g-1, g\}^{6m}$ such that $i_k + i_l = 0$ if and only if f_k and f_l share an edge. Then, we define

$$\alpha_{(\Gamma,\tau)} = \sum_{\mathbf{i} \in I} \operatorname{sgn}(\mathbf{i}) x_{\mathbf{i}} \quad \in (H^{\otimes 6m})^{\mathfrak{sp}},$$

where $\operatorname{sgn}(\mathbf{i}) = \prod_{f_k \in OR} \operatorname{sgn}(i_k)$ and $x_{\mathbf{i}} = x_{i_1} \cdots x_{i_{6m}}$. The image of $\alpha_{(\Gamma,\tau)}$ via the standard projection $H^{\otimes 6m} \to \Lambda^{2m}\Lambda^3 H$ is *independent* of τ (as long as τ is chosen to be $\wedge$-admissible), and will be denoted by $\alpha_\Gamma \in (\Lambda^{2m}\Lambda^3 H)^{\mathfrak{sp}}$. Since the kernel of $\Lambda^3 H \to U$ equals to $H \wedge \sum_i (x_i \wedge y_i)$, it is easy to see that the image of α_Γ in $\Lambda^{2m} U$ (denoted by the same symbol) vanishes exactly when the graph Γ has a 1-loop. Extending the map $\Gamma \mapsto \alpha_\Gamma$ linearly, we obtain the (stable) isomorphisms in (4).

Let us now introduce our basic $\mathfrak{sp}$-homomorphisms: $f_I,\ f_H,\ f_X : H^{\otimes 4} \to \Lambda^2\Lambda^3 H$ by setting the images of $t = t_1 \otimes t_2 \otimes t_3 \otimes t_4 \in H^{\otimes 4}$ as follows:

$$\begin{aligned}
f_I(t) &= \sum_{i=1}^{g} (t_1 \wedge t_2 \wedge x_i) \wedge (t_3 \wedge t_4 \wedge y_i) - (t_1 \wedge t_2 \wedge y_i) \wedge (t_3 \wedge t_4 \wedge x_i),\\
f_H(t) &= \sum_{i=1}^{g} (t_1 \wedge t_3 \wedge x_i) \wedge (t_4 \wedge t_2 \wedge y_i) - (t_1 \wedge t_3 \wedge y_i) \wedge (t_4 \wedge t_2 \wedge x_i),\\
f_X(t) &= \sum_{i=1}^{g} (t_1 \wedge t_4 \wedge x_i) \wedge (t_2 \wedge t_3 \wedge y_i) - (t_1 \wedge t_4 \wedge y_i) \wedge (t_2 \wedge t_3 \wedge x_i).
\end{aligned}$$

(We also use the same symbols to denote the compositions of f_I, f_H, f_X with the projection $\Lambda^2\Lambda^3 H \to \Lambda^2 U$ respectively.)

We now define, for scalars a, b, c, the $\mathfrak{sp}_g$-homomorphism:

$$f_{a,b,c} = a f_I + b f_H + c f_X,$$

and we denote the composite of $f_{a,b,c}$ with the projection $\Lambda^2\Lambda^3 H \to \Lambda^2 U$ by $\bar{f}_{a,b,c}$. Furthermore, given an embedding of graphs $I \hookrightarrow \Gamma$, one may associate three graphs $\Gamma = \Gamma_I, \Gamma_H, \Gamma_X$ by replacing adjacent relations of 4 edges into I by those into I, H, X as indicated in Figure 2 respectively. Using this notation, for any triple of scalars (a, b, c), we define $I_{a,b,c} \subset \mathcal{C}(\phi)$ to be the ideal generated by the $a\Gamma_I + b\Gamma_H + c\Gamma_X$ for all pairs $I \hookrightarrow \Gamma$.

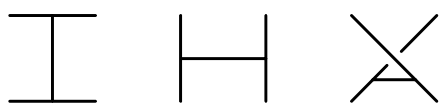


Figure 2. The graphs I, H, X.

In the next section, we will show the following.

Proposition 2.1. *The stable isomorphism of equation* (4) *induces stable isomorphisms of graded algebras*

$$\begin{aligned}
\mathcal{C}(\phi)/I_{a,b,c} &\cong (\Lambda\Lambda^3 H/(Im\, f_{a,b,c}))^{\mathfrak{sp}}\\
\mathcal{C}(\phi)/(I_{a,b,c} \cup \mathrm{loop}) &\cong (\Lambda U/(Im\, \bar{f}_{a,b,c}))^{\mathfrak{sp}}
\end{aligned}$$

which multiply degrees by 2*.*

Letting IHX (resp. IH) denote the ideal $I_{1,1,1}$ (resp. $I_{1,-1,0}$), and IH_0 denote the ideal generated by the restricted form of the IH-relation where I, H are connected by 4 distinct edges, we obtain the following:

Corollary 2.2. *For every pair* $(\mathbb{I}, \mathbb{J})$ *of the following table:*

$\mathbb{I}$	$\mathbb{J} \subset \Lambda^2 U$
IHX	$[1^4]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$
IH	$[2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$
IH_0	$[2^2]_{\mathfrak{sp}}$

we have a stable isomorphism:

$$\mathcal{C}(\phi)/(\mathbb{I} \cup \text{loop}) \cong (\Lambda U/(\mathbb{J}))^{\mathfrak{sp}}.$$

We also have:

Proposition 2.3. *The following holds*:

(a) $(\Lambda U/([1^4]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}))^{\mathfrak{sp}}$ *vanishes in positive degrees up to degree* 12.
(b) $(\Lambda U/([2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}))^{\mathfrak{sp}}$ *vanishes in positive degrees.*
(c) $(\Lambda U/([2^2]_{\mathfrak{sp}}))^{\mathfrak{sp}}$ *is a free polynomial algebra having one generator in every even positive degree.*

Theorem 1 follows immediately from Corollary 2.2 and Proposition 2.3 above.

3. Proofs for the results of Section 2

3.1. Proof of Proposition 2.1. Let J be the ideal of $\Lambda\Lambda^3 H$ generated by the image of $f_{a,b,c}$ and J_{2m} its homogeneous part of degree $2m$. It suffices to show that $J_{2m}^{\mathfrak{sp}}$ is generated by the $a\alpha_{\Gamma_I} + b\alpha_{\Gamma_H} + c\alpha_{\Gamma_X}$ for all pairs $I \hookrightarrow \Gamma$ with $\deg(\Gamma) = m$. Since $\Lambda\Lambda^3 H$ is skew-commutative, $J_{2m} = (\operatorname{Im} f_{a,b,c}) \wedge (\Lambda^{2m-2}\Lambda^3 H)$. We thus have a surjection $F_{a,b,c} : H^{\otimes 4} \otimes H^{\otimes 6m-6} \to J_{2m}$ factoring through $f_{a,b,c} \otimes id^{\otimes 6m-6}$. If we define $F_I, F_H, F_X : H^{\otimes 4} \otimes H^{\otimes 6m-6} \to \Lambda^{2m}\Lambda^3 H$ by replacing $f_{a,b,c}$ by f_I, f_H, f_X in the above, then obviously we have $F_{a,b,c} = aF_I + bF_H + cF_X$. Let $C : H^{\otimes 2} \to \mathbb{Q}$ denote the canonical contraction which maps $\sum x_i \otimes y_i - y_i \otimes x_i$ to 1, and define $C \otimes F_* : H^{\otimes 6m} \to \Lambda^{2m}\Lambda^3 H$ ($* = \{a,b,c\}, I, H, X$) in such a way that the domain components of C (resp. f_* part of F_*) share the third and sixth (resp. first, second, fourth and fifth) positions of $H^{\otimes 6m}$. Still $C \otimes F_{a,b,c}$ gives a surjection onto J_{2m}, and the semisimplicity of $\mathfrak{sp}$-representations implies that $J_{2m}^{\mathfrak{sp}}$ is generated by the images of $\mathfrak{sp}$-invariants $\alpha_{(\Gamma,\tau)}$ of $H^{\otimes 6m}$ via $C \otimes F_{a,b,c}$, where (Γ, τ) runs over trivalent graphs of degree m with $\wedge$-admissible total orderings such that $f_3 = f_+(e')$, $f_6 = f_-(e')$ for some $e' \in \operatorname{Edge}(\Gamma)$. Given such a (Γ, τ), construct three trivalent graphs $\Gamma = \Gamma_I, \Gamma_H, \Gamma_X$ by replacing I by H and X for the latter two, and give their total orderings τ_I, τ_H, τ_X so that $\tau_I = \tau$ and τ_H, τ_X differ from τ only locally in the parts 'H, X' as indicated in Figure 3.

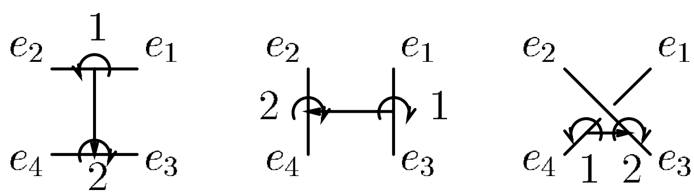
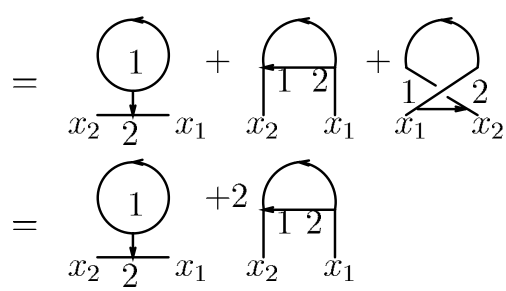


Figure 3. Three graphs I, H, X together with their total ordering, where shown are particular orderings of the trivalent vertices together with flag orientations for the respective vertices and internal edges. The external edges $e_1, ..., e_4$ are assumed (flag-)oriented in the same way on the three graphs.

Then, it is easy to see that $C \otimes F_*(\alpha_{(\Gamma,\tau)}) = \alpha_{\Gamma_*}$ for $* = I, H, X$ and hence that $C \otimes F_{a,b,c}(\alpha_{(\Gamma,\tau)}) = a\alpha_{\Gamma_I} + b\alpha_{\Gamma_H} + c\alpha_{\Gamma_X}$. Conversely, given three trivalent graphs $\Gamma_I, \Gamma_H, \Gamma_X$ which coincide except in their distinguished parts I, H, X respectively, we may associate $\wedge$-admissible total orderings τ_I, τ_H, τ_X on the three graphs $\Gamma_I, \Gamma_H, \Gamma_X$ such that they coincide with each other outside the 'IHX-parts' and such that their orderings inside the 'IHX-parts' are as indicated in Figure 3 with the middle edges shared by f_3, f_6 on the three graphs. It is not difficult then to see that $C \otimes F_{a,b,c}(\alpha_{(\Gamma_I,\tau_I)}) = a\alpha_{\Gamma_I} + b\alpha_{\Gamma_H} + c\alpha_{\Gamma_X}$. This completes the proof of Proposition 2.1. $\square$

3.2. Proof of Corollary 2.2. (i) The case $\mathbb{I} = IHX$: It is easy to see that f_{IHX} (i.e., $f_{1,1,1}$) is alternating, hence factors through $\Lambda^4 H \cong [1^4]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$. We can choose highest weight vectors for the $\mathfrak{sp}$-decomposition of $\Lambda^4 H$ as follows: $x_1 \wedge x_2 \wedge x_3 \wedge x_4$ for $[1^4]_{\mathfrak{sp}}$, $\sum_{i=1}^{g} x_i \wedge y_i \wedge x_1 \wedge x_2$ for $[1^2]_{\mathfrak{sp}}$ and $\sum_{i,j=1}^{n} x_i \wedge y_i \wedge x_j \wedge y_j$ for $[0]_{\mathfrak{sp}}$. A computation of their images in $\Lambda^2 U$ under f_{IHX} shows that $\Lambda^4 H$ is embedded into $\Lambda^2 U$. For example, with the temporary abbreviation of α_Γ by Γ (due to typesetting reasons), we have

$$f_{IHX}\Big(\sum_{i=1}^{n} x_i \otimes y_i \otimes x_1 \otimes x_2\Big) =$$



in $\Lambda^2\Lambda^3 H$ whose first (resp. second) term vanishes (resp. remains) when projected to $\Lambda^2 U$.

(ii) The case of $\mathbb{I} = IH$: Let $f_{IH} = f_I - f_H = f_{1,-1,0}$. Since $f_{IH}(t_1 \otimes t_2 \otimes t_3 \otimes t_4)$ is invariant under the variable changes of $t_1 \leftrightarrow t_4$, $t_2 \leftrightarrow t_3$ and

$t_1, t_4 \leftrightarrow t_2, t_3$ respectively, we find that f_{IH} factors through $\mathrm{Sym}^2(\mathrm{Sym}^2 H) \cong [4]_{\mathfrak{sp}} \oplus [2^2]_{\mathfrak{sp}} \oplus [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$. Since the target space does not have $[4]_{\mathfrak{sp}}$, f_{IH} is zero on this component. The highest weight vectors of the other components of $\mathrm{Sym}^2(\mathrm{Sym}^2 H)$ can be taken as $(x_1x_2)(x_1x_2) - (x_1^2)(x_2^2)$ for $[2^2]_{\mathfrak{sp}}$, $\sum_{j=1}^{g}(x_1x_j)(x_2y_j)-(x_1y_j)(x_2x_j)$ for $[1^2]_{\mathfrak{sp}}$ and $\sum_{k,l=1}^{g}(x_kx_l)(y_ky_l)-(x_ky_l)(x_ly_k)$ for $[0]_{\mathfrak{sp}}$. Mapping these vectors by f_{IH}, we find that $[2^2]_{\mathfrak{sp}}$, $[1^2]_{\mathfrak{sp}}$ and $[0]_{\mathfrak{sp}}$ remain nontrivally in $\Lambda^2 U$.

(iii) The case $\mathbb{I} = IH_0$: We have to re-examine the proof of Proposition 2.1 carefully. Observe first that the IH-relation can be classified into the three types IH_0, IH_1, IH_2 according to the number of connected edges to 'I-graph' being 4, 3 or 2.

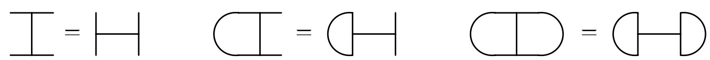


Figure 4. The three possible forms of the IH relation in trivalent graphs (with no orderings). On the left, the four external edges of I are assumed distinct.

Replace $f_{1,-1,0} = f_I - f_H$ in the proof of Theorem 1 by its composite f' with the projection to the $[2^2]_{\mathfrak{sp}}$-part, and set $J = \bigoplus J_m$ to be the ideal generated by the $\mathrm{Im}(f')$. Form a surjection $C \otimes F' : H^{\otimes 6m} \to J_{2m}$ factoring through $C \otimes f' \otimes id^{\otimes 6m-6}$. Then, we have to show that the collection of the images of $C \otimes F'(\alpha_{(\Gamma,\tau)})$ in $\Lambda^{2m} U$ (where (Γ, τ) runs over all trivalent graphs of degree m with $\wedge$-admissible total orderings) coincides exactly with the collection of the $\alpha_{\Gamma_I} - \alpha_{\Gamma_H}$ where (Γ_I, Γ_H) runs over all IH_0-related pairs of trivalent graphs. But the kernel of $\mathrm{Im}(f_{IH}) \to [2^2]_{\mathfrak{sp}}$ is isomorphic to $\Lambda^2 H = [1^2]_{\mathfrak{sp}} \oplus [0]_{\mathfrak{sp}}$ whose image in $\Lambda^2 U$ is generated by the elements of the form

$$\sum_{i,j} \mathrm{sgn}(ij)(x_k \wedge x_i \wedge x_j) \wedge (x_l \wedge x_{-i} \wedge x_{-j}) \quad (k \neq l).$$

From this it follows that, among the collection of the $C \otimes F_{IH}(\alpha_{(\Gamma,\tau)})$, those that remain under $C \otimes F'$ are exactly the $\alpha_{\Gamma_I} - \alpha_{\Gamma_H}$ coming from IH_0-related pairs, as desired.

3.3. Proof of Proposition 2.3. (b): Using the IH relation, we easily see that every (trivalent) graph can be deformed so as to have a 1-loop, hence that $\mathcal{C}(\phi)/(IH \cup \mathrm{loop})$ vanishes in positive degrees.

(c): Since IH_0 defines an equivalence relation on trivalent graphs, it follows that $\mathcal{C}(\phi)/(IH_0 \cup \mathrm{loop})$ is a polynomial algebra on the graded set of equivalence classes of connected trivalent graphs without 1-loops modulo the IH_0 relation. By induction, it is easy to see that every connected graph of degree n without 1-loops is equivalent (modulo the IH_0 relation) to the graph E_n shown in Figure 5. Thus $\mathcal{C}(\phi)/(IH_0 \cup \mathrm{loop})$ is a polynomial algebra on the classes of E_n $(n \geq 1)$.

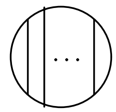


Figure 5. The graph E_n of degree n, with n vertical chords on the circle.

For (a), we collect some elementary relations in $\mathcal{C}(\phi)/(IHX \cup \text{loop})$. We understand all graphs shown below as parts of trivalent graphs. A cycle of length n in a graph will be called an n-wheel, if it has n distinct trivalent vertices.

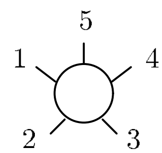


Figure 6. A wheel with five ordered legs.

Given an n-wheel w_I (where I is the set of its n external legs) and a permutation σ of the symmetric group Sym_I, we define another n-wheel $w_{\sigma(I)}$ to be the wheel whose external legs are permuted by σ.

Lemma 3.1. *For a permutation $\sigma \in Sym_I$, and an n-wheel w_I, we have:*

$$w_{\sigma I} = \text{sgn}(\sigma) w_I \quad \textit{modulo wheels with less than } n \textit{ legs.} \tag{5}$$

Proof. Apply the IHX relation, regarding I as an arc of a wheel w_I (i.e., an edge between to adjacent trivalent vertices). The wheel appears in two terms (with a different ordering of its legs) and an $n-1$ wheel appears in the third term. □

Corollary 3.2. *A $2n$-wheel w_I can be written as a linear combination of $(2n-1)$-wheels. Indeed, observe that the sign of the permutation $\sigma = (1, 2, \ldots, 2n)$ is -1, and that the graphs $w_{\sigma I}$ and w_I are isomorphic. Furthermore, a 3-wheel w_J can be written as a linear combination of 2-wheels. Indeed, observe that the graphs w_J and $w_{(12)J}$ are isomorphic. Thus we conclude that n-wheels vanish in $\mathcal{C}(\phi)/(IHX \cup \text{loop})$ for $n = 1, 2, 3, 4$.*

Lemma 3.3. *If a graph contains two pentagon cycles C_5^1, C_5^2 which intersect at two consecutive edges, then it vanishes in $\mathcal{C}(\phi)/(IHX \cup \text{loop})$.*

Proof. Apply the IHX relation, regarding I as one of the two consecutive common edges. We get that a sum of three terms vanish. One of them is the graph in consideration, and the two others contain squares, and thus by the above corollary vanish. □

Corollary 3.4. *If a graph contains a pentagon cycle, then it can be written as a linear combination of graphs that have at least* 14 *vertices.*

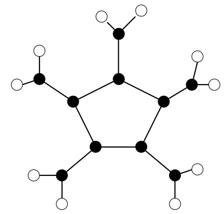


Figure 7. A pentagon with 10 vertices with 10 external legs.

Proof. By Corollary 3.2, we have only to consider a graph without (≤ 4)-wheels but with a pentagon. Figure 7 shows that such a graph has at least 10 (black) vertices near the pentagon. Furthermore, there should appear 10 (white) vertices adjacent to 5 of them. Using Lemma 3.3, we may assume that the white vertices are all distinct from the black ones, but they need not be distinct from each other. Instead, at most 3 of the white vertices can coincide with each other (because this is part of a trivalent graph), thus there are at least 4 distinct white vertices. Thus Γ has at least 14 vertices. $\square$

The *girth* $g(\Gamma)$ of a (connected) graph Γ is, by definition, the minimum length of a cycle in Γ ([B]; recall that a cycle is a closed path of distinct edges). We now have the following Lemma (for a proof, see [B, Theorem 1.2, p.105]).

Lemma 3.5. *A trivalent graph with girth at least g, contains at least $2^{(g+3)/2}-2$ (resp. $3 \cdot 2^{g/2} - 2$) vertices if g is odd (resp. g is even).*

Now, we give the proof of (a) of Proposition 2.3. Consider a connected trivalent graph Γ. If Γ has at most 8 vertices, by Lemma 3.5, it contains an n-gon for some $n = 1, 2, 3, 4$, and thus by corollary 3.2 vanishes. If Γ has girth exactly 5, then by corollary 3.4, it has at least 14 vertices. If Γ has girth at least 6, then by Lemma 3.5, it has at least 22 vertices. To sum up, if Γ has at most 12 vertices, it follows that $\Gamma = 0 \in \mathcal{C}(\phi)/(IHX \cup \text{loop})$. Thus, $\mathcal{C}(\phi)/(IHX \cup \text{loop})$ vanishes in degrees $n = 1, 2, 3, 4, 5, 6$, and Corollary 2.2 concludes our proof.

Question 3.6. Is the algebra $\mathcal{C}(\phi)/(IHX \cup \text{loop})$ trivial in positive degrees?

4. Proof of Theorem 2

Let Γ be a *vertex oriented trivalent graph* i.e., a trivalent graph such that for each vertex $v \in \text{Edge}(\Gamma)$, $\text{Flag}(v)$ is given a cyclic ordering. A *total ordering* τ of Γ is defined in the same way as in section 2.1 with an extra condition that the linear ordering of $\mathrm{Flag}(v)$ has the same sign as the given cyclic ordering on it. We say such a τ to be Sym-admissible if it satisfies the condition:

$$\mathrm{sgn}\begin{pmatrix} f_1(v_1) & f_2(v_1) & f_3(v_1) & f_1(v_2) & ...f_3(v_{2m}) \\ f_+(e_1) & f_-(e_1) & f_+(e_2) & f_-(e_2) & ...f_-(e_{3m}) \end{pmatrix} = 1$$

for every linear ordering of edges: $\mathrm{Edge}(\Gamma) = \{e_1, \ldots, e_{3m}\}$.

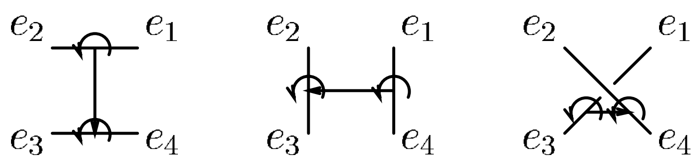


Figure 8. Three graphs I, H, X together with their total ordering, where shown are particular orderings of the trivalent vertices together with flag orientations for the respective vertices and internal edges. The external edges $e_1, ..., e_4$ are assumed (flag-)oriented in the same way on the three graphs.

Given a totally ordered, vertex oriented trivalent graph (Γ, τ) of degree m, we can define an $\mathfrak{sp}$-invariant $\alpha_{(\Gamma,\tau)} \in (H^{\otimes 6m})^{\mathfrak{sp}}$ in the same way as in section 2.1. Moreover we see that its projection image in $\Lambda^{2m}\,\mathrm{Sym}^3 H$ is independent of τ as long as τ is chosen to be Sym-admissible. We denote this well-defined image by α_Γ. The mapping $\alpha : \tilde{\mathcal{C}}(\phi) \to (\Lambda\,\mathrm{Sym}^3 H)^{\mathfrak{sp}}$ $(\Gamma \mapsto \alpha_\Gamma)$ turns out to be factoring through the AS-relation, inducing the stable isomorphism $\tilde{\mathcal{C}}(\phi)/(AS) \cong (\Lambda\,\mathrm{Sym}^3 H)^{\mathfrak{sp}}$.

Now we shall introduce our "oriented version" of the basic $\mathfrak{sp}$-homomorphisms: $f_I, f_H, f_X : H^{\otimes 4} \to \Lambda^2\,\mathrm{Sym}^3 H$ by setting the images of $t = t_1 \otimes t_2 \otimes t_3 \otimes t_4 \in H^{\otimes 4}$ as follows:

$$\begin{aligned} f_I(t) &= \sum_{i=1}^{g} (t_1 t_2 x_i) \wedge (t_3 t_4 y_i) - (t_1 t_2 y_i) \wedge (t_3 t_4 x_i), \\ f_H(t) &= \sum_{i=1}^{g} (t_4 t_1 x_i) \wedge (t_2 t_3 y_i) - (t_4 t_1 y_i) \wedge (t_2 t_3 x_i), \\ f_X(t) &= \sum_{i=1}^{g} (t_1 t_3 x_i) \wedge (t_4 t_2 y_i) - (t_1 t_3 y_i) \wedge (t_4 t_2 x_i). \end{aligned}$$

It is easy to see that the map $f_{IHX} := f_I + f_H + f_X$ factors through $\mathrm{Sym}^4 H \cong [4]_{\mathfrak{sp}}$, which is mapped onto the $[4]_{\mathfrak{sp}}$-component of $\Lambda^2\,\mathrm{Sym}^3 H$. The proof of Theorem 2 goes exactly in the same way as Theorem 1, where the only one point to be noted is that, for any embedding $I \hookrightarrow \Gamma$, the naturally induced total ordering τ_H on Γ_H is *not* Sym-admissible, thus introducing the IHX relation of Figure 1.

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