Garoufalidis *Res Math Sci* (2018) 5:11
https://doi.org/10.1007/s40687-018-0127-3

Research in the Mathematical Sciences

RESEARCH

Quantum knot invariants

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Abstract

This is a survey talk on one of the best known quantum knot invariants, the colored Jones polynomial of a knot, and its relation to the algebraic/geometric topology and hyperbolic geometry of the knot complement. We review several aspects of the colored Jones polynomial, emphasizing modularity, stability and effective computations. The talk was given in the Mathematische Arbeitstagung June 24–July 1, 2011.

Keywords: Jones polynomial, Knots, Quantum topology, Volume conjecture, Nahm sums, Stability, Modularity, Modular forms, Mock-modular forms, q-Holonomic sequence, q-Series

Mathematics Subject Classification: Primary 57N10, Secondary 57M25

Contents

1 The Jones polynomial of a knot

Quantum knot invariants are powerful numerical invariants defined by quantum field theory with deep connections to the geometry and topology in dimension three [59]. This is a survey talk on the various limits of the colored Jones polynomial [41], one of the best known quantum knot invariants. This is a 25-year-old subject that contains theorems and conjectures in disconnected areas of mathematics. We chose to present some old and recent conjectures on the subject, emphasizing two recent aspects of the colored Jones polynomial, Modularity and Stability and their illustration by effective computations. Zagier's influence on this subject is profound, and several results in this talk are joint work with him. Of course, the author is responsible for any mistakes in the presentation. We thank Don Zagier for enlightening conversations, for his hospitality and for his generous sharing of his ideas with us.

The Jones polynomial $J_L(q) \in \mathbb{Z}[q^{\pm 1/2}]$ of an oriented link L in 3-space is uniquely determined by the linear relations [41]

$$qJ_{\times}(q) - q^{-1}J_{\times}(q) = (q^{1/2} - q^{-1/2})J_{)(}(q) \qquad J_{\bigcirc}(q) = q^{1/2} + q^{-1/2}\,.$$

The Jones polynomial has a unique extension to a polynomial invariant $J_{L,c}(q)$ of links L together with a coloring c of their components that are colored by positive natural numbers that satisfy the following rules

$$\begin{aligned} J_{L\cup K, c\cup\{N+1\}}(q) &= J_{L\cup K^{(2)}, c\cup\{N,2\}}(q) - J_{L\cup K, c\cup\{N-1\}}(q), \qquad N \geq 2, \\ J_{L\cup K, c\cup\{1\}}(q) &= J_{L,c}(q), \\ J_{L,\{2,\dots,2\}}(q) &= J_L(q), \end{aligned}$$

where $(L \cup K, c \cup \{N\})$ denotes a link with a distinguished component K colored by N and $K^{(2)}$ denotes the 2-parallel of K with zero framing. Here, a natural number N attached to a component of a link indicates the N-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. For a detailed discussion on the polynomial invariants of links that come from quantum groups, see [40,56,57].

The above relations make clear that the colored Jones polynomial of a knot encodes the Jones polynomials of the knot and its 0-framed parallels.

2 Three limits of the colored Jones polynomial

In this section, we will list three conjectures, the MMR Conjecture (proven), the Slope Conjecture (mostly proven) and the AJ Conjecture (less proven). These conjectures relate the colored Jones polynomial of a knot with the Alexander polynomial, with the set of slopes of incompressible surfaces and with the $\mathrm{PSL}(2, \mathbb{C})$ character variety of the knot complement.

2.1 The colored Jones polynomial and the Alexander polynomial

We begin by discussing a relation of the colored Jones polynomial of a knot with the homology of the universal abelian cover of its complement. The homology $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}$ of the complement $M = S^3 \setminus K$ of a knot K in 3-space is independent of the knot K. This allows us to consider the universal abelian cover $\widetilde{M}$ of M with deck transformation group $\mathbb{Z}$, and with homology $H_1(\widetilde{M}, \mathbb{Z})$ a $\mathbb{Z}[t^{\pm 1}]$ module. As it is well known, this module is essentially torsion and its order is given by the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ of K [53]. The Alexander polynomial does not distinguish knots from their mirrors and satisfies $\Delta_K(1) = 1$.

There are infinitely many pairs of knots (for instance $(10_{22}, 10_{35})$ in the Rolfsen table [3,53]) with equal Jones polynomial but different Alexander polynomial. On the other hand, the colored Jones polynomial determines the Alexander polynomial. This so-called

Melvin–Morton–Rozansky Conjecture was proven in [4] and states that

$$\hat{J}_{K,n}(e^{\hbar}) = \sum_{i \geq j \geq 0} a_{K,ij} \hbar^i n^j \in \mathbb{Q}[[n,\hbar]] \tag{1}$$

and

$$\sum_{i=0}^{\infty} a_{K,ij} \hbar^i = \frac{1}{\Delta_K(e^{\hbar})} \in \mathbb{Q}[[\hbar]].$$

Here $\hat{J}_{K,n}(q) = J_{K,n}(q)/J_{\text{Unknot},n}(q) \in \mathbb{Z}[q^{\pm 1}]$ is a normalized form of the colored Jones polynomial. The above conjecture is a statement about formal power series. A stronger analytic version is known [26, Thm.1.3]; namely, for every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$ such that for all $\alpha \in U_K$ we have

$$\lim_n J_{K,n}(e^{\alpha/n}) = \frac{1}{\Delta_K(e^{\alpha})},$$

where the left-hand side is a sequence of analytic functions of $\alpha \in U_K$ converging uniformly on each compact subset of U_K to the function of the right-hand side. More is known about the summation of the series (1) along a fixed diagonal $i = j + k$ for fixed k, both on the level of formal power series and on the analytic counterpart. For further details, the reader may consult [26] and references therein.

2.2 The colored Jones polynomial and slopes of incompressible surfaces

In this section, we discuss a conjecture relating the degree of the colored Jones polynomial of a knot K with the set bs_K of boundary slopes of incompressible surfaces in the knot complement $M = S^3 \setminus K$. Although there are infinitely many incompressible surfaces in M, it is known that $\text{bs}_K \subset \mathbb{Q} \cup \{1/0\}$ is a finite set [38]. Incompressible surfaces play an important role in geometric topology in dimension three, often accompanied by the theory of normal surfaces [37]. From our point of view, incompressible surfaces are a tropical limit of the colored Jones polynomial, corresponding to an expansion around $q = 0$ [20].

The Jones polynomial of a knot is a Laurent polynomial in one variable q with integer coefficients. Ignoring most information, one can consider the degree $\delta_K(n)$ of $\hat{J}_{K,n+1}(q)$ with respect to q. Since $(\hat{J}_{K,n}(q))$ is a q-holonomic sequence [25], it follows that δ_K is a quadratic quasi-polynomial [18]. In other words, we have

$$\delta_K(n) = c_K(n)n^2 + b_K(n)n + a_K(n),$$

where $a_K, b_K, c_K : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. In [19], the author formulated the Slope Conjecture.

Conjecture 2.1 *For all knots K, we have*

$$4c_K(\mathbb{N}) \subset \text{bs}_K.$$

The motivating example for the Slope Conjecture was the case of the $(-2, 3, 7)$ pretzel knot, where we have [19, Ex.1.4]

$$\delta_{(-2,3,7)}(n) = \left\lceil \frac{37}{8}n^2 + \frac{17}{2}n \right\rceil = \frac{37}{8}n^2 + \frac{17}{2}n + a(n),$$

where $a(n)$ is a periodic sequence of period 4 given by $0, -1/8, -1/2, -1/8$ if $n \equiv 0, 1, 2, 3 \bmod 4$, respectively. In addition, we have

$$\mathrm{bs}_{(-2,3,7)} = \{0, 16, 37/2, 20\}.$$

In all known examples, $c_K(\mathbb{N})$ consists of a single element, the so-called Jones slope. How the colored Jones polynomial selects some of the finitely many boundary slopes is a challenging and interesting question. The Slope Conjecture is known for all torus knots, all alternating knots and all knots with at most 8 crossings [19] as well as for all adequate knots [15] and all 2-fusion knots [10].

2.3 The colored Jones polynomial and the PSL(2, $\mathbb{C}$) character variety

In this section, we discuss a conjecture relating the colored Jones polynomial of a knot K with the moduli space of $\mathrm{SL}(2,\mathbb{C})$-representations of M, restricted to the boundary of M. Ignoring 0-dimensional components, the latter is a one-dimensional plane curve. To formulate the conjecture, we need to recall that the colored Jones polynomial $\hat{J}_{K,n}(q)$ is q-holonomic [25], i.e., it satisfies a non-trivial linear recursion relation

$$\sum_{j=0}^{d} a_j(q, q^n)\hat{J}_{K,n+j}(q) = 0 \tag{2}$$

for all n where $a_j(u,v) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ and $a_d \neq 0$. q-holonomic sequences were introduced by Zeilberger [66], and a fundamental theorem (multisums of q-proper hypergeometric terms are q-holonomic) was proven in [61] and implemented in [52]. Using two operators M and L which act on a sequence $f(n)$ by

$$(Mf)(n) = q^n f(n), \qquad (Lf)(n) = f(n+1),$$

we can write the recursion (2) in operator form

$$P \cdot \hat{J}_K = 0 \qquad \text{where} \qquad P = \sum_{j=0}^{d} a_j(q, M)L^j.$$

It is easy to see that $LM = qML$ and M, L generate the q-Weyl algebra. Although a q-holonomic sequence is annihilated by many operators P (for instance, if it is annihilated by P, then it is also annihilated by any left multiple QP of P), one can choose a canonical recursion, denoted by $A_K(M,L,q) \in \mathbb{Z}[q,M]\langle L\rangle/(LM-qML)$ and called non-commutative A-polynomial of K, which is a knot invariant [16]. The reason for this terminology is the potential relation with the A-polynomial $A_K(M,L)$ of K [6]. The latter is defined as follows.

Let $X_M = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))/\mathbb{C}$ denote the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on M. We have an identification

$$X_{\partial M} \simeq (\mathbb{C}^*)^2/(\mathbb{Z}/2\mathbb{Z}), \qquad \rho \mapsto (M, L),$$

where $\{M, 1/M\}$ (resp., $\{L, 1/L\}$) are the eigenvalues of $\rho(\mu)$ (resp., $\rho(\lambda)$) where (μ,λ) is a meridian-longitude pair on ∂M. X_M and $X_{\partial M}$ are affine varieties and the restriction map $X_M \longrightarrow X_{\partial M}$ is algebraic. The Zariski closure of its image lifted to $(\mathbb{C}^*)^2$, and after removing any 0-dimensional components is a one-dimensional plane curve with defining polynomial $A_K(M,L)$ [6]. This polynomial plays an important role in the hyperbolic geometry of the knot complement. We are now ready to formulate the AJ Conjecture [16]; see also [21]. Let us say that two polynomials $P(M,L) =_M Q(M,L)$ are essentially equal if their irreducible factors with positive L-degree are equal.

Conjecture 2.2 *For all knots K, we have $A_K(M^2, L, 1) =_M A_K(M, L)$.*

The AJ Conjecture was checked for the 3_1 and the 4_1 knots in [16]. It is known for most 2-bridge knots [45], for torus knots and for the pretzel knots of Sect. 4; see [46,55].

From the point of view of physics, the AJ Conjecture is a consequence of the fact that quantization and the corresponding quantum field theory exists [14,31].

3 The volume and modularity conjectures

3.1 The volume conjecture

The Kashaev invariant of a knot is a sequence of complex numbers defined by [42,47]

$$\langle K\rangle_N = \hat{J}_{K,N}(e(1/N)),$$

where $e(\alpha) = e^{2\pi i\alpha}$. The Volume Conjecture concerns the exponential growth rate of the Kashaev invariant and states that

$$\lim_N \frac{1}{N}\log|\langle K\rangle_N| = \frac{\mathrm{vol}(K)}{2\pi},$$

where Vol(K) is the volume of the hyperbolic pieces of the knot complement $S^3 \setminus K$ [54]. Among hyperbolic knots, the Volume Conjecture is known only for the 4_1 knot. Detailed computations are available in [49]. Refinements of the Volume Conjecture to all orders in N and generalizations were proposed by several authors [13,17,26,28]. Although proofs are lacking, there appears to be a lot of structure in the asymptotics of the Kashaev invariant. In the next section, we will discuss a modularity conjecture of Zagier and some numerical verification.

3.2 The modularity conjecture

Zagier considered the Galois invariant spreading of the Kashaev invariant on the set of complex roots of unity given by

$$\phi_K : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{C}, \qquad \phi_K\left(\frac{a}{c}\right) = \hat{J}_{K,c}\left(e\left(\frac{a}{c}\right)\right),$$

where $(a, c) = 1$ and $c > 0$. The above formula works even when a and c are not coprime due to a symmetry of the colored Jones polynomial [36]. ϕ_K determines $\langle K\rangle$ and conversely is determined by $\langle K\rangle$ via Galois invariance.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\alpha = a/c$ and $\hbar = 2\pi i/(X + d/c)$ where $X \longrightarrow +\infty$ with bounded denominators. Let $\phi = \phi_K$ denote the extended Kashaev invariant of a hyperbolic knot K and let $F \subset \mathbb{C}$ denote the invariant trace field of $M = S^3 \setminus K$ [48]. Let $C(M) \in \mathbb{C}/(4\pi^2\mathbb{Z})$ denote the complex Chern–Simons invariant of M [35,50]. The next conjecture was formulated by Zagier.

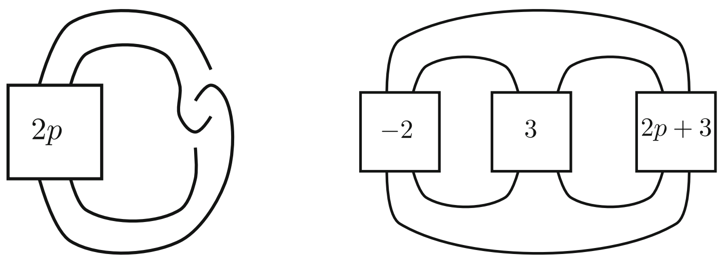
Conjecture 3.1 *[65] With the above conventions, there exist $\Delta(\alpha) \in \mathbb{C}$ with $\Delta(\alpha)^{2c} \in F(\epsilon(\alpha))$ and $A_j(\alpha) \in F(e(\alpha))$ such that*

$$\frac{\phi(\gamma X)}{\phi(X)} \sim \left(\frac{2\pi}{\hbar}\right)^{3/2} e^{C(M)/\hbar}\Delta(\alpha)\sum_{j=0}^{\infty} A_j(\alpha)\hbar^j. \tag{3}$$

When $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $X = N - 1$, and with the properly chosen orientation of M, the leading asymptotics of (3) together with the fact that $\Im(C(M)) = \mathrm{vol}(M)$ gives the volume conjecture.

4 Computation of the non-commutative *A*-polynomial

As we will discuss below, the key to an effective computation the Kashaev invariant is a recursion for the colored Jones polynomial. Proving or guessing such a recursion is at least as hard as computing the A-polynomial of the knot. The A-polynomial is already unknown for several knots with 9 crossings. For an updated table of computations see [8]. The A-polynomial is known for the 1-parameter families of twist knots K_p [39] and pretzel knots $KP_p = (-2, 3, 3 + 2p)$ [29] depicted on the left and the right part of the following figure



where an integer m inside a box indicates the number of $|m|$ half-twists, right-handed (if $m > 0$) or left-handed (if $m < 0$), according to the following figure

The non-commutative A-polynomial of the twist knots K_p was computed with a certificate by X. Sun and the author in [30] for $p = -14, \dots, 15$. The data are available from

http://www.math.gatech.edu/~stavros/publications/twist.knot.data

The non-commutative A-polynomial of the pretzel knots $KP_p = (-2, 3, 3 + 3p)$ was guessed by C. Koutschan and the author in [23] for $p = -5, \dots, 5$. The guessing method used an a priori knowledge of the monomials of the recursion, together with computation of the colored Jones polynomial using the fusion formula, and exact but modular arithmetic and rational reconstruction. The data are available from

http://www.math.gatech.edu/~stavros/publications/pretzel.data

For instance, the recursion relation for the colored Jones polynomial $f(n)$ of the $5_2 = (-2, 3, -1)$ pretzel knot is given by

$$\begin{aligned}
& b(q^n, q) - q^{9+7n}(-1+q^n)(-1+q^{2+n})(1+q^{2+n})(-1+q^{5+2n})f(n) \\
&\quad + q^{5+2n}(-1+q^{1+n})^2(1+q^{1+n})(-1+q^{5+2n})(-1+q^{1+n}+q^{1+2n}-q^{2+2n} \\
&\quad - q^{3+2n}+q^{4+2n}-q^{2+3n}-q^{5+3n}-2q^{5+4n}+q^{6+5n})f(1+n) \\
&\quad - q(-1+q^{2+n})^2(1+q^{2+n})(-1+q^{1+2n})(-1+2q^{2+n}+q^{2+2n}+q^{5+2n} \\
&\quad - q^{4+3n}+q^{5+3n}+q^{6+3n}-q^{7+3n}-q^{7+4n}+q^{9+5n}) \\
&\quad \times f(2+n) - (-1+q^{1+n})(1+q^{1+n})(-1+q^{3+n})(-1+q^{1+2n})f(3+n) = 0,
\end{aligned}$$

where

$$b(q^n, q) = q^{4+2n}(1+q^{1+n})(1+q^{2+n})(-1+q^{1+2n})(-1+q^{3+2n})(-1+q^{5+2n}).$$

The recursion relation for the colored Jones polynomial $f(n)$ of the $(-2, 3, 7)$ pretzel knot is given by

$$\begin{aligned}
&b(q^n, q) - q^{224+55n}(-1 + q^n)(-1 + q^{4+n})(-1 + q^{5+n})f(n)\\
&\quad + q^{218+45n}(-1 + q^{1+n})^3(-1 + q^{4+n})(-1 + q^{5+n})f(1 + n)\\
&\quad + q^{204+36n}(-1 + q^{2+n})^2(1 + q^{2+n} + q^{3+n})(-1 + q^{5+n})f(2 + n)\\
&\quad + (-1 + q)q^{180+27n}(1 + q)(-1 + q^{1+n})(-1 + q^{3+n})^2(-1 + q^{5+n})f(3 + n)\\
&\quad - q^{149+18n}(-1 + q^{1+n})(-1 + q^{4+n})^2(1 + q + q^{4+n})f(4 + n)\\
&\quad - q^{104+8n}(-1 + q^{1+n})(-1 + q^{2+n})\\
&\quad (-1 + q^{5+n})^3f(5 + n) + q^{59}(-1 + q^{1+n})(-1 + q^{2+n})(-1 + q^{6+n})f(6 + n) = 0,
\end{aligned}$$

where

$$\begin{aligned}
b(q^n, q) = q^{84+5n}(&1 - q^{1+n} - q^{2+n} + q^{3+2n} - q^{16+3n} + q^{17+4n} + q^{18+4n}\\
&- q^{19+5n} - q^{26+5n} + q^{27+6n} + q^{28+6n} + q^{31+6n} - q^{29+7n} - q^{32+7n}\\
&- q^{33+7n} - q^{36+7n} + q^{34+8n} + q^{37+8n} + q^{38+8n} - q^{39+9n} + q^{45+9n} - q^{46+10n}\\
&- q^{47+10n} + q^{49+10n} + q^{48+11n} - q^{50+11n} - q^{51+11n} - q^{54+11n}\\
&+ q^{52+12n} + q^{55+12n} + q^{56+12n} - q^{57+13n} - q^{62+13n} + q^{63+14n} + q^{64+14n}\\
&- q^{66+14n} + q^{67+14n} - q^{65+15n} + q^{67+15n} - q^{69+15n} + q^{71+15n} - q^{69+16n}\\
&+ q^{70+16n} - q^{72+16n} - q^{75+17n} - q^{78+17n} + q^{76+18n} + q^{79+18n} - q^{83+19n} + q^{85+19n}\\
&+ q^{84+20n} - q^{86+20n} + q^{88+20n} - q^{89+21n} + q^{91+21n} - q^{96+22n} - q^{93+23n}\\
&+ 2q^{98+24n} - q^{99+25n} - q^{108+26n} - q^{107+27n} + q^{109+27n} + q^{108+28n} - q^{110+28n}\\
&+ q^{112+28n} - q^{113+29n} + q^{115+29n} + q^{112+30n} + q^{115+30n} - q^{117+31n} - q^{120+31n}\\
&- q^{117+32n} + q^{118+32n} - q^{120+32n} - q^{119+33n} + q^{121+33n} - q^{123+33n}\\
&+ q^{125+33n} + q^{123+34n} + q^{124+34n} - q^{126+34n} + q^{127+34n} - q^{123+35n} - q^{128+35n}\\
&+ q^{124+36n} + q^{127+36n} + q^{128+36n} + q^{126+37n} - q^{128+37n} - q^{129+37n} - q^{132+37n}\\
&- q^{130+38n} - q^{131+38n} + q^{133+38n} - q^{129+39n} + q^{135+39n} + q^{130+40n} + q^{133+40n}\\
&+ q^{134+40n} - q^{131+41n} - q^{134+41n} - q^{135+41n} - q^{138+41n} + q^{135+42n}\\
&+ q^{136+42n} + q^{139+42n} - q^{133+43n} - q^{140+43n} + q^{137+44n} + q^{138+44n} - q^{142+45n}\\
&+ q^{135+46n} - q^{139+47n} - q^{140+47n} + q^{144+48n}).
\end{aligned}$$

The pretzel knots KP_p are interesting from many points of view. For every integer p, the knots in the pair $(KP_p, -KP_{-p})$ (where $-K$ denotes the mirror of K)

- are geometrically similar, in particular they are scissors congruent, have equal volume, equal invariant trace fields and their Chern–Simons invariant differs by a sixth root of unity,
- their A-polynomials are equal up to a $\mathrm{GL}(2, \mathbb{Z})$ transformation [29, Thm.1.4].

Yet, the colored Jones polynomials of $(KP_p, -KP_{-p})$ are different, and so are the Kashaev invariants and their asymptotics and even the term $\Delta(0)$ in the modularity conjecture 3.1. An explanation of this difference is given in [11].

Zagier posed a question to compare the modularity conjecture for geometrically similar pairs of knots, which was a motivation for many of the computations in Sect. 5.2.

5 Numerical asymptotics and the modularity conjecture

5.1 Numerical computation of the Kashaev invariant

To numerically verify Conjecture 3.1, we need to compute the Kashaev invariant to several hundred digits when $N = 2000$ for instance. In this section, we discuss how to achieve this.

There are multidimensional R-matrix state sum formulas for the colored Jones polynomial $J_{K,N}(q)$ where the number of summation points is given by a polynomial in N of degree the number of crossings of K minus 1 [25]. Unfortunately, this is not practical method even for the 4_1 knot.

An alternative way is to use fusion [7,32,43] which allows one to compute the colored Jones polynomial more efficiently at the cost that the summand is a rational function of q. For instance, the colored Jones polynomial of a 2-fusion knot can be computed in $O(N^3)$ steps using [23, Thm.1.1]. This method works better, but it still has limitations.

A preferred method is to guess a non-trivial recursion relation for the colored Jones polynomial (see Sect. 4) and instead of using it to compute the colored Jones polynomial, differentiate sufficiently many times and numerically compute the Kashaev invariant. In the efforts to compute the Kashaev invariant of the $(-2, 3, 7)$ pretzel knot, Zagier and the author obtained the following lemma, of theoretical and practical use.

Lemma 5.1 *The Kashaev invariant $\langle K\rangle_N$ can be numerically computed in $O(N)$ steps.*

A computer implementation of Lemma 5.1 is available.

5.2 Numerical verification of the modularity conjecture

Given a sequence of complex number (a_n) with an expected asymptotic expansion

$$a_n \sim \lambda^n n^\alpha (\log n)^\beta \sum_{j=0}^{\infty} \frac{c_j}{n^j}$$

how can one numerically compute λ, α, β and several coefficients c_j? This is a well-known numerical analysis problem [5]. An acceleration method was proposed in [62, p. 954], which is also equivalent to the Richardson transform. For a detailed discussion of the acceleration method see [22, Sect. 5.2]. In favorable circumstances, the coefficients c_j are algebraic numbers, and a numerical approximation may lead to a guess for their exact value.

A concrete application of the acceleration method was given in the appendix of [32] where one deals with several λ of the same magnitude as well as $\beta \neq 0$.

Numerical computations of the modularity conjecture for the 4_1 knot were obtained by Zagier around roots of unity of order at most 5, and extended to several other knots in [33,34]. As a sample computation, we present here the numerical data for 4_1 at $\alpha = 0$, computed independently by Zagier and by the author. The values of A_k in the table below are known for $k = 0, \dots, 150$.

$$\phi_{4_1}(X) = 3^{-1/4} X^{3/2} \exp(CX) \left(\sum_{k=0}^{\infty} \frac{A_k}{k! 12^k} h^k \right),$$

$$h = A/X \qquad A = \frac{\pi}{3^{3/2}} \qquad C = \frac{1}{\pi} \mathrm{Li}_2(\exp(2\pi i/3)).$$

k	A_k
0	1
1	11
2	697
3	724351/5
4	278392949/5
5	244284791741/7
6	1140363907117019/35
7	212114205337147471/5
8	367362844229968131557/5
9	449211928735297790783839 21/275
10	3174342130562495575602143407/7
11	699550295824437662808791404905733/455
12	1422238863146986316573269595491315893 1/2275
13	5255000379400316520126835457783180207189/175
14	4205484148170089347679282114854031908714273/25
15	1616975399001217896007199158921134595564839756 0689/14875
16	119390469635156067915857712883546381438702433035719259/14875
17	1116398659629170045249141261665722279335124967712466031771/16625
18	577848332864910742917664402961320978851712483384455237961760783/914375
19	3198465527483558758007094480403141583163892079086635997387742717 83/48125
20	523192890653480894959218049320922357395367170475082317392862964453830 3/67375
21	158555526852538710030232989409745755243229196117995383665148878914255633279/158125
22	266138687713772241962265446428426077612411819429022932150811274993281815769285 1/186875
23	1799843320784069980857785293171845353938670480452547724408088829842398128243496119/8125
24	1068857072910520399648906526268097479733304116402314182132962280539663178994210946666679/284375
25	11038592414711792337563151440072563159210647563259742536085842325190593198913696564958 19559/15925
26	848180221913649277212833106432963449310433483042794323456448440417431293021130955718815160 4709/6125

In addition, we present the numerical data for the 5_2 knot at $\alpha = 1/3$, computed in [34].

$$\phi_{5_2}(X/(3X+1))/\phi_{5_2}(X) \sim e^{C/h}(2\pi/h)^{3/2}\Delta(1/3)\left(\sum_{k=0}^{\infty} A_k(1/3)h^k\right),$$

$$h = (2\pi i)/(X+1/3),$$

$$F = \mathbb{Q}(\alpha) \qquad \alpha^3 - \alpha^2 + 1 = 0 \qquad \alpha = 0.877\cdots - 0.744\ldots i,$$

$$C = R(1-\alpha^2) + 2R(1-\alpha) - \pi i \log(\alpha) + \pi^2,$$

$$R(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x) - \frac{\pi^2}{6},$$

$$[1-\alpha^2] + 2[1-\alpha] \in \mathcal{B}(F),$$

$$-23 = \pi_1^2\pi_2 \qquad \pi_1 = 3\alpha - 2 \qquad \pi_2 = 3\alpha + 1,$$

$$\pi_7 = (\alpha^2-1)\zeta_6 - \alpha + 1 \qquad \pi_{43} = 2\alpha^2 - \alpha - \zeta_6,$$

$$\Delta(1/3) = e(-2/9)\pi_7\frac{3\sqrt{-3}}{\sqrt{\pi_1}},$$

$$A_0(1/3) = \pi_7\pi_{43},$$

$$A_1(1/3) = \frac{-952 + 321\alpha - 873\alpha^2 + (1348 + 557\alpha + 26\alpha^2)\zeta_6}{\alpha^5\pi_1^3}.$$

One may use the recursion relations [24] for the twisted colored Jones polynomial to expand the above computations around complex roots of unity [9].

6 Stability

6.1 Stability of a sequence of polynomials

The Slope Conjecture deals with the highest (or the lowest, if you take the mirror image) q-exponent of the colored Jones polynomial. In this section, we discuss what happens when we shift the colored Jones polynomial and place its lowest q-exponent to 0. Stability concerns the coefficients of the resulting sequence of polynomials in q. A weaker form of stability (0-stability, defined below) for the colored Jones polynomial of an alternating knot was conjectured by Dasbach and Lin and proven independently by Armond [2].

Stability was observed in some examples of alternating knots by Zagier, and conjectured by the author to hold for all knots, assuming that we restrict the sequence of colored Jones polynomials to suitable arithmetic progressions, dictated by the quasi-polynomial nature of its q-degree [18,19]. Zagier asked about modular and asymptotic properties of the limiting q-series.

A proof of stability in full for all alternating links is given in [27]. Besides stability, this approach gives a generalized Nahm sum formula for the corresponding series, which in particular implies convergence in the open unit disk in the q-plane. The generalized Nahm sum formula comes with a computer implementation (using as input a planar diagram of a link) and allows the computation of several terms of the q-series as well as its asymptotics when q approaches radially from within the unit circle a complex root of unity. The Nahm sum formula is reminiscent to the cohomological Hall algebra of motivic Donaldson–Thomas invariants of Kontsevich-Soibelman [44] and may be related to recent work of Witten [60] and Dimofte-Gaiotto-Gukov [12].

Let

$$\mathbb{Z}((q)) = \left\{ \sum_{n\in\mathbb{Z}} a_n q^n \mid a_n = 0,\ n \ll 0 \right\}$$

denote the ring of power series in q with integer coefficients and bounded below minimum q-degree.

Definition 6.1 Fix a sequence $(f_n(q))$ of polynomials $f_n(q) \in \mathbb{Z}[q]$. We say that $(f_n(q))$ is 0-*stable* if the following limit exists

$$\lim_n f_n(q) = \Phi_0(q) \in \mathbb{Z}[[q]],$$

i.e., for every natural number $m \in \mathbb{Z}$, there exists a natural number $n(m)$ such that the coefficient of q^m in $f_n(q)$ is constant for all $n > n(m)$.

We say that $(f_n(q))$ is *stable* if there exist elements $\Phi_k(q) \in \mathbb{Z}((q))$ for $k = 0, 1, 2, \dots$ such that for every $k \in \mathbb{N}$ we have

$$\lim_n q^{-nk} \left(f_n(q) - \sum_{j=0}^{k} q^{jn} \Phi_j(q) \right) = 0 \in \mathbb{Z}((q)).$$

We will denote by

$$F(x, q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]]$$

the corresponding series associated with the stable sequence $(f_n(q))$.

Thus, a 0-stable sequence $f_n(q) \in \mathbb{Z}[q]$ gives rise to a q-series $\lim_n f_n(q) \in \mathbb{Z}[[q]]$. The q-series that come from the colored Jones polynomial are q-hypergeometric series of a special shape, i.e., they are generalized Nahm sums. The latter are introduced in the next section.

6.2 Generalized Nahm sums

In [51], Nahm studied q-hypergeometric series $f(q) \in \mathbb{Z}[[q]]$ of the form

$$f(q) = \sum_{n_1,\dots,n_r \geq 0} \frac{q^{\frac{1}{2} n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}}$$

where A is a positive definite even integral symmetric matrix and $b \in \mathbb{Z}^r$. Nahm sums appear in character formulas in Conformal Field Theory and define analytic functions in the complex unit disk $|q| < 1$ with interesting asymptotics at complex roots of unity, and with sometimes modular behavior. Examples of Nahm sums are the famous list of seven mysterious q-series of Ramanujan that are nearly modular (in modern terms, mock modular). For a detailed discussion, see [64]. Nahm sums give rise to elements of the Bloch group, which governs the leading radial asymptotics of $f(q)$ as q approaches a complex root of unity. Nahm's Conjecture concerns the modularity of a Nahm sum $f(q)$ and was studied extensively by Zagier, Vlasenko-Zwegers and others [58,63].

The limit of the colored Jones function of an alternating link leads us to consider generalized Nahm sums of the form

$$\Phi(q) = \sum_{n \in C \cap \mathbb{N}^r} (-1)^{c \cdot n} \frac{q^{\frac{1}{2}n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}}, \tag{4}$$

where C is a rational polyhedral cone in $\mathbb{R}^r$, $b, c \in \mathbb{Z}^r$ and A is a symmetric (possibly indefinite) matrix. We will say that the generalized Nahm sum (4) is *regular* if the function

$$n \in C \cap \mathbb{N}^r \mapsto \frac{1}{2} n^t \cdot A \cdot n + b \cdot n$$

is proper and bounded below, where mindeg_q denotes the minimum degree with respect to q. Regularity ensures that the series (4) is a well-defined element of the ring $\mathbb{Z}((q))$. In the remaining of the paper, the term Nahm sum will refer to a regular generalized Nahm sum.

6.3 Stability for alternating links

Let K denote an alternating link. The lowest monomial of $J_{K,n}(q)$ has coefficient ± 1, and dividing $J_{K,n+1}(q)$ by its lowest monomial gives a polynomial $J^+_{K,n}(q) \in 1 + q\mathbb{Z}[q]$. We can now quote the main theorem of [27].

Theorem 6.2 [27] *For every alternating link K, the sequence $(J^+_{K,n}(q))$ is stable and the corresponding limit $F_K(x, q)$ can be effectively computed by a planar projection D of K. Moreover, $F_K(0, q) = \Phi_{K,0}(q)$ is given by an explicit Nahm sum computed by D.*

An illustration of the corresponding q-series $\Phi_{K,0}(q)$ the knots 3_1, 4_1 and 6_3 is given in Sect. 6.4.

6.4 Computation of the *q*-series of alternating links

Given the generalized Nahm sum for $\Phi_{K,0}(q)$, a multidimensional sum of as many variables as the number of crossings of K, one may try to identify the q-series $\Phi_{K,0}(q)$ with a known one. In joint work with Zagier, we computed the first few terms of the corresponding series (an interesting and non-trivial task in itself) and guessed the answer for knots with a small number of crossings. The guesses are presented in the following table

K	c_-	c_+	σ	$\Phi^*_{K,0}(q)$	$\Phi_{K,0}(q)$
$3_1 = -K_1$	3	0	2	h_3	h_2
$4_1 = K_{-1}$	2	2	0	h_3	h_3
5_1	5	0	4	h_5	h_2
$5_2 = K_2$	0	5	-2	h_4	h_3
$6_1 = K_{-2}$	4	2	0	h_3	h_5
6_2	4	2	2	h_3h_4	h_3
6_3	3	3	0	h_3^2	h_3^2
7_1	7	0	6	h_7	h_2
$7_2 = K_3$	0	7	-2	h_6	h_3
7_3	0	7	-4	h_4	h_5
7_4	0	7	-2	$(h_4)^2$	h_3
7_5	7	0	4	h_3h_4	h_4
7_6	5	2	2	h_3h_4	h_3^2
7_7	3	4	0	h_3^3	h_3^2
$8_1 = K_{-3}$	6	2	0	h_3	h_7
8_2	6	2	4	h_3h_6	h_3
8_3	4	4	0	h_5	h_5
8_4	4	4	2	h_4h_5	h_3
8_5	2	6	-4	h_3	???
$K_p, p > 0$	0	$2p+1$	-2	h^*_{2p}	h_3
$K_p, p < 0$	$2\|p\|$	2	0	h_3	$h_{2\|p\|+1}$
$T(2,p), p > 0$	$2p+1$	0	$2p$	h_{2p+1}	1

where σ denotes the signature of a knot [53] and for a positive natural number b, h_b are the unary theta and false theta series

$$h_b(q) = \sum_{n\in\mathbb{Z}} \varepsilon_b(n)\, q^{\frac{b}{2}n(n+1)-n},$$

where

$$\varepsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

Observe that

$$h_1(q) = 0, \qquad h_2(q) = 1, \qquad h_3(q) = (q)_\infty.$$

In the above table, c_+ (resp. c_-) denotes the number of positive (resp., negative) crossings of an alternating knot K, and $\Phi^*_{K,0}(q) = \Phi_{-K,0}(q)$ denotes the q-series of the mirror $-K$ of K, and $T(2,p)$ denotes the $(2,p)$ torus knot.

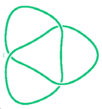
Concretely, the above table predicts the following identities

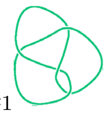
$$(q)_\infty^{-2} = \sum_{a,b,c\geq 0} (-1)^a \frac{q^{\frac{3}{2}a^2+ab+ac+bc+\frac{1}{2}a+b+c}}{(q)_a(q)_b(q)_c(q)_{a+b}(q)_{a+c}},$$

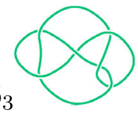
$$(q)_\infty^{-3} = \sum_{\substack{a,b,c,d,e\geq 0\\ a+b=d+e}} (-1)^{b+d} \frac{q^{\frac{b^2}{2}+\frac{d^2}{2}+bc+ac+ad+be+\frac{a}{2}+c+\frac{e}{2}}}{(q)_{b+c}(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{c+d}},$$

$$(q)_\infty^{-4} = \sum_{\substack{a,b,c,d,e,f\geq 0\\ a+e\geq b, b+f\geq a}} (-1)^{a-b+e} \times \frac{q^{\frac{a}{2}+\frac{3a^2}{2}+\frac{b}{2}+\frac{b^2}{2}+c+ac+d+ad+cd+\frac{e}{2}+2ae-2be+de+\frac{3e^2}{2}-af+bf+f^2}}{(q)_a(q)_b(q)_c(q)_{a+c}(q)_d(q)_{a+d}(q)_e(q)_{a-b+e}(q)_{a-b+d+e}(q)_f(q)_{-a+b+f}}$$

corresponding to the knots

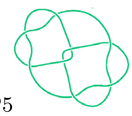

3_1


4_1


6_3

Some of the identities of the above table have been consequently proven [1]. In particular this settles the (mock)-modularity properties of the series $\Phi_{K,0}(q)$ for all but one knot. The q-series of the remaining knot 8_5 is given by an eight-dimensional Nahm sum

$$\Phi_{8_5,0}(q) = (q)_\infty^8 \sum_{\substack{a,b,c,d,e,f,g,h\geq 0\\ a+f\geq b}} S(a,b,c,d,e,f,g,h)$$

8_5


where $S = S(a, b, c, d, e, f, g, h)$ is given by

$$S = (-1)^{b+f} \frac{q^{2a+3a^2-\frac{b}{2}-2ab+\frac{3b^2}{2}+c+ac+d+ad+cd+e+ae+de+\frac{3f}{2}+4af-4bf+ef+\frac{5f^2}{2}+g+ag-bg+eg+fg+h+ah-bh+fh+gh}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a-b+f}(q)_{a-b+e+f}(q)_{a-b+f+g}(q)_{a-b+f+h}}.$$

The first few terms of the series $\Phi_{8_5,0}(q)$, which somewhat simplify when divided by $(q)_\infty$, are given by

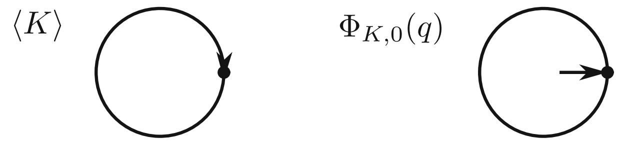
$$\begin{aligned}\Phi_{8_5,0}(q)/(q)_\infty &= 1-q+q^2-q^4+q^5+q^6-q^8+2q^{10}+q^{11}+q^{12}-q^{13}\\ &\quad -2q^{14}+2q^{16}+3q^{17}+2q^{18}+q^{19}-3q^{21}-2q^{22}+q^{23}+4q^{24}+4q^{25}\\ &\quad +5q^{26}+3q^{27}+q^{28}-2q^{29}-3q^{30}-3q^{31}+5q^{33}+8q^{34}+8q^{35}+8q^{36}+6q^{37}+3q^{38}\\ &\quad -2q^{39}-5q^{40}-6q^{41}-q^{42}+2q^{43}+9q^{44}+13q^{45}+17q^{46}+16q^{47}+14q^{48}+9q^{49}\\ &\quad +4q^{50}-3q^{51}-8q^{52}-8q^{53}-5q^{54}+3q^{55}+14q^{56}+21q^{57}+27q^{58}+32q^{59}+33q^{60}\\ &\quad +28q^{61}+21q^{62}+11q^{63}+q^{64}-9q^{65}-11q^{66}\\ &\quad -11q^{67}-2q^{68}+9q^{69}+27q^{70}+40q^{71}+56q^{72}+60q^{73}+65q^{74}+62q^{75}+54q^{76}\\ &\quad +39q^{77}+23q^{78}+4q^{79}-9q^{80}-16q^{81}-14q^{82}-3q^{83}+16q^{84}+40q^{85}+67q^{86}\\ &\quad +92q^{87}+114q^{88}+129q^{89}+135q^{90}+127q^{91}+115q^{92}+92q^{93}\\ &\quad +66q^{94}+35q^{95}+9q^{96}-12q^{97}-14q^{98}-11q^{99}+13q^{100}+O(q)^{101}.\end{aligned}$$

We were unable to identify $\Phi_{8_5,0}(q)$ with a known q-series. Nor were we able to decide whether it is a mock-modular form [64]. It seems to us that 8_5 is not an exception, and that the mock-modularity of the q-series $\Phi_{8_5,0}(q)$ is an open problem.

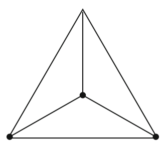
Question 6.3 Can one decide if a generalized Nahm sum is a mock-modular form?

7 Modularity and stability

Modularity and Stability are two important properties of quantum knot invariants. The Kashaev invariant $\langle K\rangle$ and the q-series $\Phi_{K,0}(q)$ of a knotted three-dimensional object have some common features, namely asymptotic expansions at roots of unity approached radially (for $\Phi_{K,0}(q)$) and on the unit circle (for $\langle K\rangle$), depicted in the following figure



The leading asymptotic expansions of $\langle K\rangle$ and $\Phi_{K,0}(q)$ are governed by elements of the Bloch group as is the case of the Kashaev invariant and also the case of the radial limits of Nahm sums [58]. In this section, we discuss a conjectural relation, discovered accidentally by Zagier and the author in the spring of 2011, between the asymptotics of $\langle 4_1\rangle$ and $\Phi_{6j,0}(q)$, where $6j$ is the q-6j symbol of the tetrahedron graph whose edges are colored with $2N$ [7,32]



The evaluation of the above tetrahedron graph $J^+_{6j,N}(q) \in 1 + q\mathbb{Z}[q]$ is given explicitly by [7,32]

$$J^+_{6j,N}(q) = \frac{1}{1-q}\sum_{n=0}^{N}(-1)^n \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\frac{(q)_{4N+1-n}}{(q)_n^3(q)_{N-n}^4}.$$

The sequence $(J^+_{6j,N}(q))$ is stable, and the corresponding series $F_{6j}(x,q)$ is given by

$$F_{6j}(x,q) = \frac{1}{(1-q)(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}\frac{(xq^{-n})_\infty^4}{(x^4q^{-n+1})_\infty} \in \mathbb{Z}((q))[[x]],$$

where as usual $(x)_\infty = \prod_{k=0}^{\infty}(1-xq^k)$ and $(q)_n = \prod_{k=1}^{n}(1-q^k)$. In particular,

$$\lim_N J^+_{6j,N}(q) = \Phi_{6j,0}(q) = \frac{1}{(1-q)(q)_\infty^3}\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}.$$

Let

$$\phi_{6j,0}(q) = \frac{(q)_\infty^4}{1-q}\Phi_{6j,0}(q) = (q)_\infty\sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{3}{2}n^2+\frac{1}{2}n}}{(q)_n^3}.$$

The first few terms of $\phi_{6j,0}(q)$ are given by

$$\begin{aligned}\phi_{6j,0}(q) &= 1 - q - 2q^2 - 2q^3 - 2q^4 + q^6 + 5q^7 + 7q^8 + 11q^9 + 13q^{10}\\ &\quad + 16q^{11} + 14q^{12} + 14q^{13} + 8q^{14} - 12q^{16} - 26q^{17} - 46q^{18} - 66q^{19} - 90q^{20} - 114q^{21}\\ &\quad - 135q^{22} - 155q^{23} - 169q^{24} - 174q^{25} - 165q^{26} - 147q^{27} - 105q^{28} - 48q^{29} + 37q^{30}\\ &\quad + 142q^{31} + 280q^{32} + 435q^{33} + 627q^{34} + 828q^{35} + 1060q^{36} + O(q)^{37}.\end{aligned}$$

The next conjecture which combines stability and modularity of two knotted objects has been numerically checked around complex roots of unity of order at most 3.

Conjecture 7.1 *As $X \longrightarrow +\infty$ with bounded denominator, we have*

$$\phi_{6j,0}(e^{-1/X}) = \phi_{4_1}(X)/X^{1/2} + \overline{\phi_{4_1}(-\bar{X})/(-\bar{X})^{1/2}}.$$

Acknowledgements
The author was supported in part by NSF.
To Don Zagier, with admiration.

Ethics approval and consent to participate
Not applicable.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 August 2017 Accepted: 13 November 2017 Published online: 8 February 2018

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