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RESEARCH

Peacock patterns and resurgence in complex Chern–Simons theory

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Abstract

The partition function of complex Chern–Simons theory on a 3-manifold with torus boundary reduces to a finite-dimensional state-integral which is a holomorphic function of a complexified Planck's constant τ in the complex cut plane and an entire function of a complex parameter u. This gives rise to a vector of factorially divergent perturbative formal power series whose Stokes rays form a peacock-like pattern in the complex plane. We conjecture that these perturbative series are resurgent, their trans-series involve two non-perturbative variables, their Stokes automorphism satisfies a unique factorization property and that it is given explicitly in terms of a fundamental matrix solution to a (dual) linear q-difference equation. We further conjecture that a distinguished entry of the Stokes automorphism matrix is the 3D-index of Dimofte–Gaiotto–Gukov. We provide proofs of our statements regarding the q-difference equations and their properties of their fundamental solutions and illustrate our conjectures regarding the Stokes matrices with numerical calculations for the two simplest hyperbolic $\mathbf{4}_1$ and $\mathbf{5}_2$ knots.

Keywords: Chern–Simons theory, Holomorphic blocks, State-integrals, Knots, 3-manifolds, Resurgence, Perturbative series, Borel resummation, Stokes automorphisms, Stokes constants, q-holonomic modules, q-difference equations, DT-invariants, BPS states, Peacocks, Meromorphic quantum Jacobi forms

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1 Introduction

1.1 Chern–Simons theory with compact and complex gauge group

Chern–Simons gauge theory, introduced by Witten in his seminal paper [68] as a quantum field theory proposal of the Jones polynomial [47], remains one of the most fascinating quantum field theories. It gives a powerful framework to study the quantum topology of knots and three-manifolds, and at the same time it provides a rich yet tractable model to explore general aspects of quantum field theories.

In [68], Witten analyzed in detail Chern–Simons gauge theory with a compact gauge group (such as SU(2)). Its partition function on a 3-manifold M with torus boundary components depends on a quantized version $k \in \mathbb{Z}$ of Planck's constant (or equivalently, on a complex root of unity $e^{2\pi i k}$), as well as a discrete color (a finite-dimensional irreducible representation of SU(2)) per boundary component of M. A more powerful Chern–Simons theory with complex gauge group (such as $SL(2, \mathbb{C})$) was introduced by Witten [69] and developed extensively by Gukov [42]. A key feature of complex Chern–Simons theory is that the partition function $Z_M(u; \tau)$ for a 3-manifold M with torus boundary components depends analytically on a complex parameter τ (where $\tau = 1/k$ in the Chern–Simons theory with compact gauge group) as well as on a complex parameter u per each boundary component of M that plays the role of the holonomy of a peripheral curve. The analytic dependence of $Z_M(u; \tau)$ on the parameters u and τ allows one to formulate questions of complex analysis and complex geometry which would be difficult, or impossible, to do in Chern–Simons theory with compact gauge group.

There is a key difference between Chern–Simons theory with compact versus complex gauge group: The former is an exactly solvable theory, meaning that the partition function can be computed by a finite state-sum, a consequence of the fact that it is a TQFT in 3 dimensions. On the other hand, the situation with complex Chern–Simons theory is more mysterious. For reasons that are not entirely understood, the partition function $Z_M(u; \tau)$ for manifolds with torus boundary components reduces to a finite-dimensional integral (the so-called state-integral) whose integrand is a product of Faddeev's quantum dilogarithm functions [24], assembled out of an ideal triangulation of the manifold. This was the approach taken by Andersen-Kashaev [3,4] and Dimofte [21] following prior

ideas of [19,46]. Focusing for simplicity on the case of a 3-manifold with a single torus boundary component (such as the complement of a hyperbolic knot in S^3), the state-integral $Z_M(u;\tau)$ is a holomorphic function of $\tau \in \mathbb{C}' = \mathbb{C}\backslash(-\infty, 0]$ and $u \in \mathbb{C}$ that satisfies a pair of linear q-difference equations. The existence of these equations for a state-integral follows from the closure properties of Zeilberger's theory of q-holonomic functions and the quasi-periodicity properties of Faddeev's quantum dilogarithm, in much the same way as the q-holonomicity of the colored Jones polynomial of a knot follows from a state-sum formula [34]. In fact, it is conjectured that the linear q-difference equation satisfied by the colored Jones polynomial of a knot coincides with the linear q-difference equations of $Z_M(u;\tau)$ (see, e.g., [5] and Sects. 5 and 6 for examples).

1.2 Resurgence and the Stokes automorphism

The global function $Z_M(u;\tau)$ gives rise to a vector $\Phi(x;\tau)$ of perturbative series in τ whose coefficients are meromorphic functions of u. These series are typically factorially divergent and a key question is a description of the analytic continuation of their Borel transform in Borel plane, their trans-series and their Stokes automorphisms. This is a typical question in perturbative quantum field theory where resurgence aims to reproduce analytic functions from factorially divergent series (for an introduction to resurgence, see, for instance, [1, 57,58,60]), and where Chern–Simons theory with a compact or complex gauge group is an excellent case to analyze. Some aspects of resurgence in Chern–Simons theory were studied in [12,27,30,40,44,45,57]. The multi-valuedness of the complex Chern–Simons action dictates that the trans-series are assembled out of monomials in $\tilde{x}$ and $\tilde{q}$ where $\tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\tau}$ and $\tilde{x} = \mathrm{e}^{u/\tau}$.

Our discoveries are summarized as follows:

- The singularities of the series $\Phi(x;\tau)$ in Borel plane are arranged in horizontal lines $2\pi\mathrm{i}$ apart, and within these lines in finitely many points $\log x$ apart. This defines a collection of Stokes lines in a peacock-like pattern (see Fig. 1) whose corresponding Stokes automorphisms satisfy a unique factorization property with integer Stokes constants.
- The Stokes automorphism $\mathsf{S}(\mathsf{x};\mathsf{q})$ along a half plane is a fundamental matrix solution to a (dual) linear q-difference equation, hence fully computable.
- The function $Z_M(u;\tau)$ is one entry of a matrix-valued collection of descendant partition functions which are a fundamental solution to a q-holonomic system in two variables.

The arrangement of the singularities in Borel plane is reminiscent of a "stability datum" of Kontsevich-Soibelman [53–55] where the corresponding integers are often called DT-invariants or BPS degeneracies. The Stokes automorphisms along half planes are analogous to the spectrum generators in Gaiotto-Moore-Neitzke [37–39]. Our integers are locally constant functions of a complex parameter x and their jumping along a wall-crossing will be the topic of a subsequent publication.

Our paper gives a concrete realization of these abstract ideas of perturbative series and their resurgence, Stokes automorphisms and their wall-crossing formulas for the case of complex Chern–Simons theory and illustrates our results with the 3-manifolds of the two simplest hyperbolic knot complements, the complements of the $\mathbf{4}_1$ and the $\mathbf{5}_2$ knots.

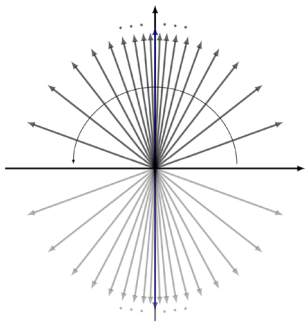

Fig. 1 A peacock arrangement of Stokes rays in the complex Borel plane

1.3 A q-holonomic module of the partition function and its descendants

In this section, we discuss a q-holonomic module associated with the partition function and its descendants. This module and its fundamental solutions are crucial to our exact computation of the Stokes matrices in Sect. 1.4. One advantage of introducing this module before we discuss resurgence of perturbative series is that the former has been established mathematically in many cases, whereas the latter remains a mathematical challenge.

We begin our discussion with a factorization of the state-integral

$$Z_M(u_{\mathsf{b}};\tau) = B(\tilde{x},\tilde{q}^{-1})^T \Delta(\tau) B(x;q), \qquad (\tau \in \mathbb{C}\setminus\mathbb{R}) \tag{1}$$

where

$$u_{\mathsf{b}} = \frac{u}{2\pi\mathsf{b}} \tag{2}$$

(this rescaling is dictated by the asymptotics of Faddeev's quantum dilogarithm), $\Delta(\tau)$ is a diagonal matrix with diagonal entries a 24-th root of unity times an integer power of $\mathrm{e}^{\frac{\pi\mathrm{i}}{24}(\tau+\tau^{-1})}$, $B(x;q) = (B^{(j)}(x;q))_{j=1}^r$ is a vector of holomorphic blocks, and

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \qquad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\tau}, \qquad x = \mathrm{e}^{u}, \qquad \tilde{x} = \mathrm{e}^{u/\tau}, \qquad \tau = \mathsf{b}^2. \tag{3}$$

The above notation is consistent with the literature in modular forms and Jacobi forms [23] and indicates that $u \in \mathbb{C}$ can be thought of as a Jacobi variable. The factorization (1) was first noted in a related context in [62] and further developed in complex Chern–Simons theory in [7,20]. We find that this factorization persists to descendant state-integrals parameterized by a pair of (Jacobi-like) variables m and μ (see Equations (125) and (240) for the definition of descendant state-integrals for the knots $\mathbf{4}_1$, $\mathbf{5}_2$)

$$Z_{M,m,\mu}(u_{\mathsf{b}};\tau) = (-1)^{m+\mu} q^{m/2} \tilde{q}^{\mu/2} B_{-\mu}(\tilde{x};\tilde{q}^{-1})^T \Delta(\tau) B_m(x;q), \qquad (m,\mu \in \mathbb{Z}) \tag{4}$$

where $Z_{M,0,0} = Z_M$. The holomorphic blocks determine a matrix $W_m(x;q)$ defined by

$$W_m(x;q) = \begin{pmatrix} B_m^{(1)}(x;q) & \dots & B_m^{(r)}(x;q) \\ \vdots & & \vdots \\ B_{m+r-1}^{(1)}(x;q) & \dots & B_{m+r-1}^{(r)}(x;q) \end{pmatrix} \tag{5}$$

with the following properties

(a) The entries of $W_m(x;q)$ are holomorphic functions of $|q| \neq 1$, meromorphic functions of $x \in \mathbb{C}^*$ with poles in $x \in q^{\mathbb{Z}}$ of order at most r, and have Taylor series expansions in $(1-x)^{-r}\mathbb{Z}[x^{\pm}][[q^{\frac{1}{2}}]]$ whose monomials $x^k q^{n/2}$ satisfy $n = O(k^2)$.

In other words, the support of the monomials in x and q in $(1-x)^r W_m(x;q)$ is similar to the one of Jacobi forms (in their holomorphic version of Eichler-Zagier [23, Eqn.(3)] or the meromorphic version from Zwegers thesis [74, Chpt.3]) and of the admissible functions of Kontsevich-Soibelman [54].

(b) The matrix

$$W_{m,\mu}(u;\tau) = W_{-\mu}(\tilde{x};\tilde{q}^{-1})\Delta(\tau)W_m(x;q)^T \tag{6}$$

defined for $\tau = \mathbb{C} \setminus \mathbb{R}$, extends to a holomorphic function of $\tau \in \mathbb{C}'$ and $u \in \mathbb{C}$ for all integers m and μ.

More precisely, if we define the normalized descendant integral

$$z_{M,m,\mu}(u;\tau) = (-1)^{m+\mu} q^{-m/2} \tilde{q}^{-\mu/2} Z_{M,m,\mu}(u;\tau), \tag{7}$$

then $W_{m,\mu}(u;\tau) = (z_{m+i,\mu+j}(u_{\mathsf{b}};\tau))_{i,j=0}^{r-1}$. The above statement is remarkable in two ways: (i) $W_m(x;q)$ is a holomorphic function of $\tau \in \mathbb{C} \setminus \mathbb{R}$ that cannot be extended holomorphically over the positive reals, yet $W_{m,\mu}(u;\tau)$ holomorphically extends over the positive reals, and (ii) $W_m(x;q)$ is a meromorphic function of u with singularities, yet $W_{m,\mu}(u;\tau)$ is an entire function of u. Property (ii) is common in quantum mechanics, where the wave function is often entire, whereas its WKB expansion is singular at the turning points. The same behavior is also observed in the case of open topological strings in [59].

(c) We have an orthogonality relation

$$W_{-1}(x;q)\, W_{-1}(x;q^{-1})^T \in \mathrm{GL}(r, \mathbb{Z}[x^{\pm}]). \tag{8}$$

(d) The columns of $W_m(x;q)$, as functions of (x, m), form a q-holonomic module of rank r.

The factorization (4) and (d) implies that the annihilator $\mathcal{I}_M$ of $z_{M,m,\mu}(u_{\mathsf{b}};\tau)$ as a function of (x, m) coincides with the annihilator of $W_m(x;q)$. The latter is a left ideal in the Weyl algebra $\mathcal{W}$ over $\mathbb{Q}[q^{\pm}]$ generated by the pairs (S_x, x) and (S_m, q^m) of q-commuting operators which act on a function $f(x, m; q)$ by

$$(S_x f)(x,m;q) = f(qx,m;q) \qquad (xf)(x,m;q) = xf(x,m;q) \tag{9}$$

$$(S_m f)(x,m;q) = f(x,m+1;q) \qquad (q^m f)(x,m;q) = q^m f(x,m;q). \tag{10}$$

Properties (a)-(d) above define meromorphic quantum Jacobi forms, a concept which is further studied in [41]. Although the above statements are largely conjectural for the partition function of complex Chern–Simons theory, we have the following result (see Theorems 14, 16, 22, 24).

Theorem 1 *The above statements hold for the* $\mathbf{4}_1$ *and* $\mathbf{5}_2$ *knots.*

We also study the Taylor series expansion of Equations (4), (6) and (8) at $u = 0$ noting that the left-hand side of the above equations are entire functions of u, whereas the right-hand side are a priori meromorphic functions of u with a pole of order r at zero. More precisely, in Sects. 3.1 and 3.2 we prove the following.

Theorem 2 *For the* $\mathbf{4}_1$ *and* $\mathbf{5}_2$ *knots, Equations* (4), (6) *and* (8) *can be expanded in Taylor series at* $u = 0$ *whose constant terms are expressed in terms of the q-series of [29].*

1.4 Perturbative series and their resurgence

We now discuss the resurgence properties of the asymptotic expansion of the state-integral $Z_M(u;\tau)$. Once we fix an integral presentation of $Z_M(u;\tau)$, the critical points of the integrand are described by an affine curve S defined by a polynomial equation

$$S : p(x, y) = 0\,. \tag{11}$$

We denote by $\mathcal{P}$ the labeling set of the branches $y = y_\sigma(x)$ of S. The perturbative expansion of the state-integral has the form

$$\Phi(x, y;\tau) = \mathrm{e}^{\frac{V(u,v)}{2\pi\mathrm{i}\tau}}\varphi(x, y;\tau), \qquad \varphi(x, y;\tau) \in \frac{1}{\sqrt{\mathrm{i}\delta}}\mathbb{Q}[x^{\pm}, y^{\pm}, \delta^{-1}][[2\pi\mathrm{i}\tau]] \tag{12}$$

where $V : S^* \to \mathbb{C}/(2\pi\mathrm{i})$, S^* is the exponentiated defined by $p(x, y) = 0$ with $x = \mathrm{e}^u$, $y = -\mathrm{e}^v$ and $\delta \in \mathbb{Q}(x, y)$ is the so-called 1-loop invariant. The asymptotic series $\varphi(x, y;\tau)$ satisfies $\varphi(x, y; 0) = 1$. The 8-th root of unity that appears as a prefactor in $\phi(x, y;\tau)$ exactly matches with one appearing in the asymptotics of the Kashaev invariant noticed in [45, Sec.1]. After choosing local branches, we define the vector $\varphi(x;\tau) = (\varphi_\sigma(x;\tau))_{\sigma\in\mathcal{P}} = (\varphi(x, y_\sigma(x);\tau))_{\sigma\in\mathcal{P}}$ of asymptotic series. Recall the vector of holomorphic blocks $B(x; q)$ from (4). We now discuss the relation between the asymptotics of $B(x; q)$ when $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and τ approaches zero (in sectors) and the Borel resummation $s(\Phi)$ of the vector of power series $\Phi(x;\tau)$.

The next conjecture summarizes the singularities of $\Phi(x;\tau)$ in the Borel plane, the relation between the asymptotics of the holomorphic blocks with the Borel resummation $s_\theta(\Phi)(x;\tau)$ as well as the properties of the Stokes automorphism matrices S, whose detailed definition is given in Sect. 2.

Conjecture 3 (a) The singularities of $\Phi_\sigma(x;\tau)$ in the Borel plane are a subset of

$$\{\iota^{(\ell,k)}_{\sigma,\sigma'} \mid \sigma' \in \mathcal{P},\ k, \ell \in \mathbb{Z},\ k = O(\ell^2)\} \tag{13}$$

where

$$\iota^{(\ell,k)}_{\sigma,\sigma'} = \frac{V(\sigma) - V(\sigma')}{2\pi\mathrm{i}} + 2\pi\mathrm{i}k + \ell\log x \qquad (\sigma, \sigma' \in \mathcal{P},\ k, \ell \in \mathbb{Z})\,. \tag{14}$$

In particular, the set of trans-series is labeled by three indices, $\sigma \in \mathcal{P}$ and $k, \ell \in \mathbb{Z}$, and they are of the form $\Phi_\sigma(\tau)\tilde{q}^k\tilde{x}^\ell$.

(b) On each ray ρ in the complement of the singularities of $\Phi(x;\tau)$ in Borel plane, there exist a matrix $M_\rho(\tilde{x};\tilde{q})$ with entries in $\mathbb{Z}[\tilde{x}^{\pm}][[\tilde{q}]]$ such that

$$\Delta(\tau)B(x;\tau) = M_\rho(\tilde{x};\tilde{q})s_\rho(\Phi)(x;\tau)\,. \tag{15}$$

(c) The Stokes matrices $\mathsf{S}^+(x,q)$, $\mathsf{S}^-(x,q^{-1})$ are given by

$$\mathsf{S}^+(x;q) \doteq W_{-1}(x^{-1};q^{-1})\cdot W_{-1}(x;q)^T, \qquad \mathsf{S}^-(x;q^{-1}) \doteq W_{-1}(x;q)\cdot W_{-1}(x^{-1};q^{-1})^T \tag{16}$$

where $\doteq$ means equality up to multiplication on the left and on the right by a matrix in $\mathrm{GL}(r,\mathbb{Z}[x^{\pm}])$.
(d) The Stokes matrix S uniquely determines the Stokes matrices at each Stokes ray, and the Stokes constants are integers corresponding to the Donaldson–Thomas invariants in [53–55] and the BPS degeneracies in [37,39].
(e) The Stokes matrices satisfy the inversion relation

$$\mathsf{S}^+(x;q)^T\mathsf{S}^-(x^{-1};q) = \mathbf{1}. \tag{17}$$

The Stokes automorphism matrix has an interesting connection to physics which we now discuss. Given a hyperbolic knot K in S^3, one can construct a three-dimensional $\mathcal{N}=2$ supersymmetric theory T_M associated with the knot complement $M = S^3\backslash K$ [18] (see also [66]), whose BPS invariants are conjectured to coincide with the Stokes constants of $s(\Phi)(x;\tau)$. This conjecture can be made more precise in the following manner. The BPS invariants of T_M are encoded in the 3D-index $\mathcal{I}_K(m,e)(q)$ labeled by two integers (m,e) called magnetic and electric fluxes, respectively [18]. One can further define the 3D-index in the fugacity basis (also known as the rotated index) by [17]

$$\mathrm{Ind}_K^{\mathrm{rot}}(m,\zeta;q) = \sum_{e\in\mathbb{Z}} \mathcal{I}_K(m,e)(q)\zeta^e. \tag{18}$$

The 3D-index is a topological invariant of hyperbolic 3-manifolds with at least one cusp (see [31]). And it can be evaluated using holomorphic blocks $B_K^{\alpha}(x;q)$ by [7]

$$\mathrm{Ind}_K^{\mathrm{rot}}(m,\zeta;q) = \sum_{\alpha} B_K^{\alpha}(q^{m/2}\zeta;q)B_K^{\alpha}(q^{m/2}\zeta^{-1};q^{-1}). \tag{19}$$

We have observed the following relation between the Stokes matrix and the rotated 3D-index and have proven it for the case of the $\mathbf{4}_1$ and $\mathbf{5}_2$ knots using the explicit formulas for the Stokes matrices.

Conjecture 4 For every hyperbolic knot K, we have

$$\mathsf{S}^+(x;q) \doteqdot \left(\mathrm{Ind}_K^{\mathrm{rot}}(j-i, q^{\frac{j+i}{2}}x;q)\right)_{i,j=0,1,\dots} \tag{20}$$

where $\doteqdot$ means equality up to multiplication on the left and on the right by a matrix in $\mathrm{GL}(r,\mathbb{Z}(x,q))$. In particular, we find

$$\mathsf{S}^+_{\sigma_1\sigma_1}(x;q) = \mathrm{Ind}_K^{\mathrm{rot}}(0,x;q), \tag{21}$$

where the equality is exact. This holds true for the $\mathbf{4}_1$ and the $\mathbf{5}_2$ knots.

We comment that the 3D-index itself also has asymptotic expansions which are studied in [43].

One consequence of (15) is that (after multiplying both terms of (15) by the inverse of $M_R(\tilde{x};\tilde{q})$), we can express the Borel resummation of the factorially divergent series $\Phi(\tau)$ in terms of descendant state-integrals which are holomorphic functions in the cut plane $\mathbb{C}' = \mathbb{C}\backslash(-\infty, 0]$.

Another consequence of the q-holonomic module defined by the annihilation ideal $\mathcal{I}_M$ is a refinement of the $\widehat{A}$-polynomial of a knot as well as a new $\widehat{B}$-polynomial whose classical limit is new. The refinement comes in the form of a new variable q^m where m is the descendant variable, whose geometric meaning is not understood but might be related to some kind of quantum K-theory, or perhaps related to the Weil–Gelfand–Zak transform of [2]. This refinement does not seem to be directly related to other refinements of the $\widehat{A}$-polynomial, as those considered in [6,25,36]. At any rate, the q-holonomic ideal $\mathcal{I}_M$ contains unique polynomials $\widehat{A}_M(S_x, x, q^m, q) \in \mathcal{W}$ and $\widehat{B}_M(S_m, q^m, x, q)$ (of lowest degree, content-free) that annihilate the functions $z_{M,m,k}(u;\tau)$ in the variables (m, x).

Conjecture 5 When $M = S^3 \setminus K$ is the complement of a knot K, then

(a) $\widehat{A}_M(S_x, x, 1, q)$ is the homogeneous $\widehat{A}$-polynomial of the knot [26] and $\widehat{A}_M(S_x, x, 1, 1)$ is the A-polynomial of the knot with meridian variable x^2 and longitude variable S_x [11]

(b) $\widehat{B}_M(y, x, 1, 1)$ is the defining polynomial of the curve S.

In Theorems 17 and 25, we prove the following.

Theorem 6 *Conjecture 5 holds for the* $\mathbf{4}_1$ *and* $\mathbf{5}_2$ *knots.*

1.5 Disclaimers

We end this introduction with some comments and disclaimers.

The first is that that there is no canonical labeling of holomorphic blocks by $\mathcal{P}$. Instead, the holomorphic blocks $B(x;q)$ is an $r \times 1$ vector, M_R are $r \times \mathcal{P}$ matrices for all R, $W_m(x;q)$ are $r \times r$ matrices and S are $\mathcal{P} \times \mathcal{P}$ matrices and where r is the cardinality of $\mathcal{P}$.

The second is that the entries of the matrix $W_m(x;q)$ are holomorphic functions of $q^{1/N}$ for $|q| \neq 1$, where N is a natural number (the "level" of the knot) being one for the $\mathbf{4}_1$ and $\mathbf{5}_2$ knots, but being 2 for the $(-2, 3, 7)$-pretzel knot. For instance, the entries of the matrix $W_0(x;q)$ are power series in $q^{1/2}$ [45]. This phenomenon was observed first in [45] in a related matrix-valued Kashaev invariant of the knot as well as in [44] in a matrix of q-series associated with the three simplest hyperbolic knots and replaces the modular group $\mathrm{SL}(2,\mathbb{Z})$ by its congruence subgroup $\mathrm{SL}(2, N)$. In our current paper, we will assume that $N = 1$.

The third comment involves the crucial question of topological invariance. Strictly speaking, the curve S in Equation (11) and the vector of power series $\Phi(x;\tau)$ depend on an integral representation of $Z_M(u;\tau)$, determined, for instance, by a suitable ideal triangulation of M as was done in [3]. On the other hand, the vector of power series $\Phi(x;\tau)$, its Stokes matrix $\mathsf{S}(x;q)$ and the q-holonomic module generated by the matrix $W_m(x;\tau)$ are expected to be topological invariants of M. Even if we fix an ideal triangulation, and

we fix the q-holonomic module, the fundamental solution matrix $W_m(x;\tau)$ in general has a potential ambiguity, which we now discuss.

Lemma 7 *Suppose that a matrix $W_m(x;q)$ satisfies the following properties:*

- *It factorizes the state-integral* (4)*,*
- *It is a fundamental solution matrix to a q-holonomic module,*
- *It satisfies the orthogonality equation* (8)
- *It satisfies the analytic conditions of (a) above.*

Then, $W_m(x;q)$ is uniquely determined up to right multiplication by a diagonal matrix of signs.

Proof Any two fundamental solutions of a q-holonomic system differ by multiplication by a diagonal matrix $\mathrm{diag}(E(x;q))$. If both fundamental solutions satisfy (4) and (8), it follows that each $E(x;q)$ satisfies

$$E(\tilde{x};\tilde{q}^{-1})E(x;q)=1, \qquad E(x;q)E(x;q^{-1})=1\,. \tag{22}$$

Thus, $E(x;q^{-1})=E(\tilde{x};\tilde{q}^{-1})$ and after replacing b by b^{-1}, it implies that $E(x;q)=E(\tilde{x};\tilde{q})$. It follows that $E(qx;q)=E(x;q)$ and $E(\tilde{q}\tilde{x};\tilde{q})=E(\tilde{x};\tilde{q})$. In other words, E is elliptic. Condition (a) implies that the poles of $E(x;q)$ are a subset of $\mathsf{i}\mathsf{b}\mathbb{Z}+\mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$ for $|q|\neq 1$. It follows from $E(x;q)E(x;q^{-1})=1$ that both the poles and the zeros of $E(x;q)$ are a subset of $\mathsf{i}\mathsf{b}\mathbb{Z}+\mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$ and each pole and zero has order at most r. Thus, $E(x;q)$ and $1/E(x;q)$ are a polynomial in the Weierstrass polynomial $p(x;q)$ with coefficients independent of x, and this implies that $E(x;q)=g(q)$ is independent of x, where g is a modular function with no zeros in the upper half plane, and hence, g is a modular unit [48]. There is none for $\mathrm{SL}(2,\mathbb{Z})$ (see [48]), and hence, $g(q)=\pm 1$. Hence, $W_m(q;x)$ is well defined up to right multiplication by a diagonal matrix of signs. □

1.6 Further directions

In this short section, we make some comments about future directions. The factorization of the state-integral (1) and its descendant version (4) into a matrix points toward a TQFT in 4 dimensions where the vector space associated with a 3-manifold is labeled by $\mathcal{P}$.

In another direction, as shown in [39,49], in $\mathcal{N}=2$ theories in four dimensions, the BPS invariants can be studied by applying WKB methods to their Seiberg–Witten curve. Since, in complex Chern–Simons theory, the A-polynomial curve plays in a sense the role of a Seiberg–Witten curve [42], one could study it with the techniques of [39,49], further extended in [8,9,22] to curves in exponentiated variables. It would be interesting to see one can obtain in this way the BPS invariants directly from the A-polynomial of the hyperbolic knot.

Peacock patterns of Borel singularities, with integer Stokes constants, are likely to appear in problems controlled by a quantum curve in exponentiated variables. An important example is topological string theory on Calabi–Yau threefold, and indeed, peacock patterns can be observed in, e.g., [15]. It would be very interesting to understand the resurgent structure in these examples and work along this direction is in progress.

2 Borel resummation and Stokes automorphisms

2.1 Borel resummation

In this section, we briefly review the process of Borel resummation of a factorially divergent series, its Laplace integral along rays and the corresponding Stokes automorphism across a Stokes ray. The material in this section is classical and well known and is explained in detail in the books [14,56,60], and in the references therein. We will be following the physics convention of Borel resummation as found, for example, in [58, Sec. 3.2] and [73, Sec.42.5], which differs by a factor of τ from the Borel resummation found in the math literature.

Borel resummation is a 2-step process to pass from a factorially divergent series $F(\tau)$ to the analytic function $s(F)(\tau)$ defined in the right half plane $\mathrm{Re}(\tau) > 0$ summarized in the following diagram

$$F(\tau) \rightsquigarrow \widehat{F}(\zeta) \rightsquigarrow s(F)(\tau)\,. \tag{23}$$

Here, one starts with a Gevrey-1 a formal power series $F(\tau)$

$$F(\tau) = \sum_{n=0}^{\infty} f_n \tau^n, \quad f_n = O(C^n\, n!) \tag{24}$$

and defines its Borel transform $\widehat{F}(\zeta)$ by

$$\widehat{F}(\zeta) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \zeta^n\,. \tag{25}$$

It follows that $\widehat{F}$ is the germ of an analytic function at $\zeta = 0$. If it analytically continues to an L^1-analytic function along the ray $\rho_\theta := \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ where $\theta = \arg\tau$, we define its Laplace transform by

$$s_\theta(F)(\tau) = \int_0^\infty \widehat{F}(\tau\zeta)\mathrm{e}^{-\zeta}\mathrm{d}\zeta = \frac{1}{\tau}\int_{\rho_\theta} \widehat{F}(\zeta)\mathrm{e}^{-\zeta/\tau}\mathrm{d}\zeta \tag{26}$$

The function $s(F)(\tau)$ is often called the Borel resummation of the formal power series F, and we often suppress the subscript $\theta = 0$ when τ is real and positive. If we vary $\theta = \arg\tau$ and we do not encounter singularities of $\widehat{F}$, the function $s_\theta(F)(\tau)$ is locally analytic. Thus, the problem is to understand the analytic continuation of $\widehat{F}$ and to analyze what happens to the Borel resummation $s_\theta(F)(\tau)$ when $\theta = \arg(\tau)$ crosses a Stokes ray, i.e., a ray in Borel plane that contains one or more singularities of $\widehat{F}$. This is described by a Stokes automorphism.

2.2 Stokes automorphism

We will specialize our discussion to the series of interest, namely to the Borel transform $\widehat{\Phi}(x;\zeta)$ of the vector of series $\Phi(x;\tau)$. The singularities of $\Phi(x;\tau)$ are conjectured to be in the set (13) that generates a set of Stokes rays whose complement is a countable union of open cones in Borel plane. When θ is in a fixed such cone C, the Laplace transform $s_\theta(\Phi)(x;\tau)$ depends on C but not on θ. To compare two adjacent such cones, let $\iota^{(\ell,k)}_{\sigma,\sigma'}$ denote one of the singularities of $\Phi_\sigma(x;\tau)$, θ denote its argument and $\rho = \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+$ denote

the corresponding Stokes ray. When x is generic, a Stokes ray contains a single singularity and the Laplace integrals to the right and the left of ρ are related by

$$s_{\theta^+}(\Phi_\sigma)(x;\tau) = s_{\theta^-}(\Phi_\sigma)(x;\tau) + \mathcal{S}^{(\ell,k)}_{\sigma,\sigma'} \tilde{x}^\ell \tilde{q}^k s_{\theta^-}(\Phi_{\sigma'})(x;\tau), \tag{27}$$

where $\mathcal{S}^{(\ell,k)}_{\sigma,\sigma'}$ is the Stokes constant. In matrix form, the above formula reads

$$s_{\theta^+}(\Phi)(x;\tau) = \mathfrak{S}_\theta(\tilde{x};\tilde{q}) s_{\theta^-}(\Phi)(x;\tau) \tag{28}$$

where

$$\mathfrak{S}_\theta(\tilde{x};\tilde{q}) = I + \mathcal{S}^{(\ell,k)}_{\sigma,\sigma'} \tilde{x}^\ell \tilde{q}^k E_{\sigma,\sigma'} \tag{29}$$

where $E_{\sigma,\sigma'}$ is the elementary matrix with (σ,σ')-entry 1 and all other entries zero.

More generally, consider two non-Stokes rays ρ_{θ^+} and ρ_{θ^-} whose arguments satisfy $0 \leq \theta^+ - \theta^- \leq \pi$. Then, the Laplace integrals are related by

$$s_{\theta^+}(\Phi)(x;\tau) = \mathfrak{S}_{\theta^- \to \theta^+}(\tilde{x};\tilde{q}) s_{\theta^-}(\Phi)(x;\tau) \tag{30}$$

where the Stokes matrices satisfy the factorization property

$$\mathfrak{S}_{\theta^- \to \theta^+}(\tilde{x},\tilde{q}) = \overleftarrow{\prod_{\theta^- < \theta < \theta^+}} \mathfrak{S}_\theta(\tilde{x},\tilde{q}). \tag{31}$$

where the ordered product is taken over the Stokes rays in the cone generated by ρ_{θ^-} and ρ_{θ^+}. This factorization is well known in the classical literature on WKB (see, for instance, Voros [67, p.228] who called it the "radar method" for obvious visual reasons). In our case, there are four special non-Stokes rays denoted by

$$I = \mathrm{e}^{\mathrm{i}\epsilon}\mathbb{R}_+, \qquad II = \mathrm{e}^{\mathrm{i}(\pi-\epsilon)}\mathbb{R}_+, \qquad III = \mathrm{e}^{\mathrm{i}(\pi+\epsilon)}\mathbb{R}_+, \qquad IV = \mathrm{e}^{\mathrm{i}(2\pi-\epsilon)}\mathbb{R}_+ \tag{32}$$

(for $\epsilon > 0$ and sufficiently small) that belong to the four distinguished cones (labeled *I, II, III, IV*) adjacent to the real axis and free of Stokes lines. The corresponding Stokes matrices

$$\mathsf{S}^+(\tilde{x};\tilde{q}) = \mathfrak{S}_{I\to II}(\tilde{x};\tilde{q})\,\mathfrak{S}_{IV\to I}(\tilde{x};\tilde{q}), \qquad \mathsf{S}^-(\tilde{x};\tilde{q}) = \mathfrak{S}_{III\to IV}(\tilde{x};\tilde{q})\,\mathfrak{S}_{II\to III}(\tilde{x};\tilde{q}) \tag{33}$$

that swap two complementary and nearly horizontal half planes separated by a line L are the ones that appear in Conjecture 3. They are related to the matrices M_ρ in the second part of that conjecture by

$$\begin{aligned}
\mathfrak{S}_{I\to II}(\tilde{x};\tilde{q}) &= (M_{II}(\tilde{x};\tilde{q}))^{-1} \cdot M_I(\tilde{x};\tilde{q}), \quad |\tilde{q}| < 1 \\
\mathfrak{S}_{III\to IV}(\tilde{x};\tilde{q}) &= (M_{IV}(\tilde{x};\tilde{q}^{-1}))^{-1} \cdot M_{III}(\tilde{x};\tilde{q}^{-1}), \quad |\tilde{q}| < 1 \\
\mathfrak{S}_{IV\to I}(\tilde{x};\tilde{q}) &= (M_I(\tilde{x};\tilde{q}))^{-1} \cdot M_{IV}(\tilde{x};\tilde{q}), \\
\mathfrak{S}_{II\to III}(\tilde{x};\tilde{q}) &= (M_{III}(\tilde{x};\tilde{q}))^{-1} \cdot M_{II}(\tilde{x};\tilde{q}).
\end{aligned} \tag{34}$$

We now come to an important feature of our resurgent series, a unique factorization property for the Stokes matrices reminiscent of the "stability data" description of DT-invariants in Kontsevich-Soibelman [53–55]), and of the properties of BPS spectrum generators in Gaiotto-Moore-Neitzke [37–39].

Lemma 8 S *uniquely determines* $\mathfrak{S}_\theta$ *for all* θ.

Proof Without loss of generality, we will show that S^+ uniquely determines the Stokes matrices $\mathfrak{S}_\theta$ for all θ such that $-\varepsilon < \theta < \pi - \varepsilon$ for $\varepsilon > 0$ and sufficiently small. We have

$$\mathsf{S}^+(\tilde{x};\tilde{q}) = \prod_{\sigma,\sigma',k,\ell}^{\longleftarrow} (I + \mathcal{S}^{(\ell,k)}_{\sigma,\sigma'} \tilde{x}^\ell \, \tilde{q}^k E_{\sigma,\sigma'}) \tag{35}$$

where the product is over all the singularities above the line L. The entries of the above matrices are in the ring $\mathbb{Z}[\tilde{x}^{\pm}][[\tilde{q}]]$. For each fixed natural number N, there are only finitely many horizontal lines of singularities in Borel plane, of height at most N and within those, there are finitely many $\tilde{x}$-dots. It follows by induction on k that the finite collection $\{\mathcal{S}^{(\ell,k)}_{\sigma,\sigma'}\}_\ell$ is uniquely determined from $\mathsf{S}^+(\tilde{x};\tilde{q})$. □

It follows that we can repackage the information of the Stokes constants in two matrices $\mathcal{S}^+$ and $\mathcal{S}^-$ defined by

$$\mathcal{S}^{\pm}(\tilde{x};\tilde{q}) = \sum_{\ell,k} \mathcal{S}^{(\ell,k)}_{\sigma,\sigma'} \tilde{x}^\ell \, \tilde{q}^k E_{\sigma,\sigma'} \tag{36}$$

where the sum in $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) is over the singularities above (resp., below) L. The matrices $\mathcal{S}^{\pm}(\tilde{x};\tilde{q})$ appear to have some positivity properties; see Sects. 5.4 and 6.4 for the $\mathbf{4}_1$ and the $\mathbf{5}_2$ knots.

3 A summary of the story when $u = 0$

In this section, we recall briefly the results from [29] for our two sample hyperbolic knots, the $\mathbf{4}_1$ and the $\mathbf{5}_2$ knot.

3.1 The 4_1 knot when $u = 0$

The state-integral of the $\mathbf{4}_1$ at $u = 0$ is given by

$$Z_{\mathbf{4}_1}(0;\tau) = \int_{\mathbb{R}+\mathrm{i}0} \Phi_{\mathsf{b}}(\nu)^2 \, \mathrm{e}^{-\pi \mathrm{i} \nu^2} \mathrm{d}\nu \,. \tag{37}$$

The critical points of the integrand are the logarithms of the solutions $\xi_1 = \mathrm{e}^{2\pi\mathrm{i}/6}$ and $\xi_2 = \mathrm{e}^{-2\pi\mathrm{i}/6}$ of the polynomial equation

$$(1-y)(1-y^{-1}) = 1 \,. \tag{38}$$

The labeling set $\mathcal{P} = \{\sigma_1, \sigma_2\}$, where σ_1 corresponds to the geometric representation of the $\mathbf{4}_1$ and σ_2 to the complex conjugate of the geometric representation. Observe that ξ_1 (resp., ξ_2) lie in the trace field $\mathbb{Q}(\sqrt{-3})$ (resp., its complex conjugate) of the $\mathbf{4}_1$ knot, where $\mathbb{Q}(\sqrt{-3})$ is a subfield of the complex numbers with $\sqrt{-3}$ taken to have positive imaginary part.

The first ingredient is a vector of formal power series

$$\Phi(\tau) = \begin{pmatrix} \Phi_{\sigma_1}(\tau) \\ \Phi_{\sigma_2}(\tau) \end{pmatrix} \tag{39}$$

defined by the asymptotic expansion of the state-integral (37) at each of the two critical points, and which has the form

$$\Phi_\sigma(\tau) = \exp\left(\frac{V(\sigma)}{2\pi \mathrm{i}\tau}\right)\varphi_\sigma(\tau), \tag{40}$$

satisfies the symmetry $\Phi_{\sigma_2}(\tau) = \mathrm{i}\Phi_{\sigma_1}(-\tau)$, where

$$V(\sigma_1) = \mathrm{i}\mathrm{Vol}(\mathbf{4}_1) = \mathrm{i}\, 2\mathrm{Im}\,\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\pi/3}) = \mathrm{i}\, 2.029883\ldots. \tag{41}$$

with $\mathrm{Vol}(\mathbf{4}_1)$ being the hyperbolic volume of $\mathbf{4}_1$ and the first few terms of $\varphi_{\sigma_1}(\tau/(2\pi\mathrm{i})) \in 3^{-1/4}\mathbb{Q}(\sqrt{-3})[[\tau]]$ are given by

$$\varphi_{\sigma_1}\left(\frac{\tau}{2\pi\mathrm{i}}\right) = \frac{1}{\sqrt[4]{3}}\left(1 + \frac{11\tau}{72\sqrt{-3}} + \frac{697\tau^2}{2\,(72\sqrt{-3})^2} + \frac{724351\tau^3}{30\,(72\sqrt{-3})^3} + \cdots\right). \tag{42}$$

The second ingredient is a vector $G(q) = \binom{G^0(q)}{G^1(q)}$ of q-series defined for $|q| < 1$ by

$$G^0(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} \tag{43a}$$

$$G^1(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}\left(E_1(q) + 2\sum_{j=1}^{n}\frac{1+q^j}{1-q^j}\right), \tag{43b}$$

where $E_1(q) = 1 - 4\sum_{n=1}^{\infty} q^n/(1-q^n)$ is the Eisenstein series, and extended to $|q| > 1$ by $G^0(q^{-1}) = G^0(q)$ and $G^1(q^{-1}) = -G^1(q)$. These series are motivated by, and appear in, the factorization of the state-integral of the $\mathbf{4}_1$ knot given in [33, Cor.1.7]

$$Z_{\mathbf{4}_1}(0;\tau) = -\frac{\mathrm{i}}{2}\left(\frac{q}{\tilde{q}}\right)^{\frac{1}{24}}\left(\sqrt{\tau}G^0(\tilde{q})G^1(q) - \frac{1}{\sqrt{\tau}}G^0(q)G^1(\tilde{q})\right), \qquad (\tau \in \mathbb{C}\setminus\mathbb{R}) \tag{44}$$

where

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau}, \qquad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\tau}. \tag{45}$$

The above factorization follows by applying the residue theorem to the integrand of (37), a meromorphic function of ν with prescribed zeros and poles. In particular, the integrand of (37) determines the q-hypergeometric formula for the vector $G(q)$ of q-series. Below, given a q-series $H(q)$ defined on $|q| \neq 1$, we denote by $h(\tau) = H(\mathrm{e}^{2\pi\mathrm{i}\tau})$ the corresponding holomorphic function in $\mathbb{C}\setminus\mathbb{R}$.

The vector $G(q)$ of q-series and the vector of asymptotic series $\Phi(\tau)$ come together when we consider the asymptotics of $\mathrm{diag}(\frac{1}{\sqrt{\tau}}, \sqrt{\tau})g(\tau)$ in the τ-plane (as was studied in [44]) and compare them with the Borel summed vector Φ. Recall that when the Borel transform of an asymptotic series has singular points ι_i in the Borel plane, the rays (Stokes rays) emanating from the origin with angle $\theta = \arg\iota_i$ divide the complex plane into different sectors. When one crosses into neighboring sectors, the Borel sum of the asymptotic series undergoes Stokes automorphism. In the case of the vector of asymptotic series $\Phi(\tau)$, the

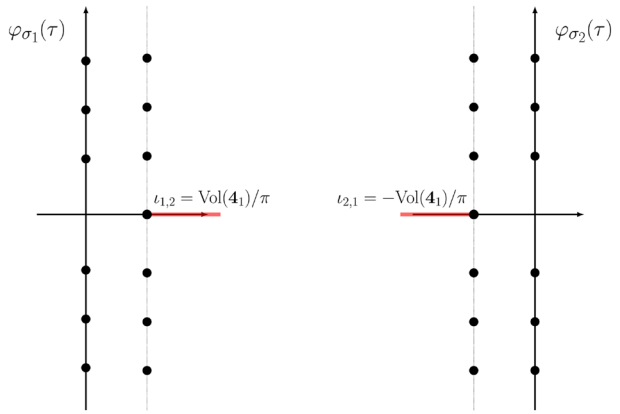

Fig. 2 The singularities in the Borel plane for the series $\varphi_{\sigma_j}(0;\tau)$ for $j = 1, 2$ of knot $\mathbf{4}_1$

singularities of the Borel transforms of its two component asymptotic series are located at

$$\iota_{i,j} = \frac{V(\sigma_i) - V(\sigma_j)}{2\pi \mathrm{i}}, \quad i, j = 1, 2, \ i \neq j, \tag{46}$$

as well as

$$2\pi \mathrm{i} k, \ \iota_{i,j} + 2\pi \mathrm{i} k, \quad k \in \mathbb{Z}_{\neq 0}, \tag{47}$$

forming vertical towers as illustrated in Fig. 2. In particular, the two singularities $\iota_{1,2}, \iota_{2,1}$ are on the positive and the negative real axis. We pick out four sectors which separate the two singularities on the real axis and all the others, and label them by *I, II, III, IV*, as illustrated in Fig. 3. The relation between the vector $G(q)$ and the Borel summed vector $\Phi(\tau)$ depend on the sector R. In [29], we found out that we do not get an agreement, but rather both sides agree up to powers of the exponentially small quantity $\tilde{q}$, and what is more, several coefficients of those powers were numerically recognized to be integers. In other words, we found that

$$\mathrm{diag}(\frac{1}{\sqrt{\tau}}, \sqrt{\tau}) g(\tau) = M_R(\tilde{q})\, s_R(\Phi)(\tau). \tag{48}$$

where $\mathrm{diag}(\nu)$ denotes the diagonal matrix with diagonal given by ν and $M_R(q)$ is a matrix of q-series with integer coefficients.

To identify the matrices M_R, we used the third ingredient, namely the linear q-difference equation

$$y_{m+1}(q) - (2 - q^m) y_m(q) + y_{m-1}(q) = 0 \qquad (m \in \mathbb{Z}). \tag{49}$$

It has a fundamental solution set given by the columns of the following matrix

$$W_m(q) = \begin{pmatrix} G^0_m(q) & G^1_m(q) \\ G^0_{m+1}(q) & G^1_{m+1}(q) \end{pmatrix}, \qquad (|q| \neq 1) \tag{50}$$

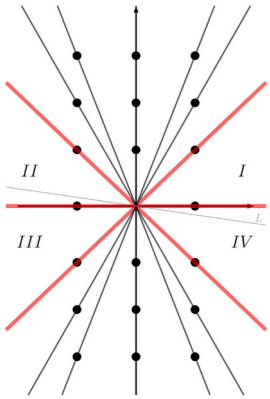

Fig. 3 Four different sectors in the τ-plane for $\Phi(0;\tau)$ of knot $\mathbf{4}_1$

where $G_m(q) = \begin{pmatrix} G^0_m(q) \\ G^1_m(q) \end{pmatrix}$, and $G^0_m(q)$ and $G^1_m(q)$ are defined by

$$G^0_m(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2+mn}}{(q;q)_n^2} \tag{51a}$$

$$G^1_m(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2+mn}}{(q;q)_n^2} \left(2m + E_1(q) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j} \right), \tag{51b}$$

for $|q| < 1$ and extended to $|q| > 1$ by $G^j_m(q^{-1}) = (-1)^j G^j_m(q)$. Observe that $G_0(q) = G(q)$, the vector that appears in the factorization (44) of the state-integral $Z_{\mathbf{4}_1}(0;\tau)$. The matrix $W_m(q)$ of holomorphic functions in $|q| \neq 1$ satisfies several properties summarized in the following theorem.

Theorem 9 *$W_m(q)$ is a fundamental solution of the linear q-difference equation (49) that has constant determinant*

$$\det(W_m(q)) = 2, \tag{52}$$

satisfying the symmetry

$$W_m(q^{-1}) = W_{-m}(q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{53}$$

the orthogonality property

$$\frac{1}{2} W_m(q) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_m(q)^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{54}$$

as well as

$$\frac{1}{2} W_m(q) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_\ell(q)^T \in \mathrm{SL}(2, \mathbb{Z}[q^{\pm}]) \tag{55}$$

for all integers m, ℓ and for $|q| \neq 1$.

Conjecture 10 Equation (48) holds where the matrices $M_R(q)$ are given in terms of $W_{-1}(q)$ as follows

$$M_I(q) = W_{-1}(q)^T \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad |q| < 1, \tag{56a}$$

$$M_{II}(q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad |q| < 1, \tag{56b}$$

$$M_{III}(q) = W_{-1}(q^{-1})^T \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad |q| > 1, \tag{56c}$$

$$M_{IV}(q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_{-1}(q^{-1})^T \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad |q| > 1. \tag{56d}$$

Assuming the above conjecture, we can now describe completely the resurgent structure of $\Phi(\tau)$. The Stokes matrices are given by

$$\mathsf{S}^+(q) = \mathfrak{S}_{I\to II}(q)\mathfrak{S}_{IV\to I}, \qquad \mathsf{S}^-(q) = \mathfrak{S}_{III\to IV}(q)\mathfrak{S}_{II\to III}, \tag{57}$$

where

$$\mathfrak{S}_{I\to II}(q) = M_{II}(q)^{-1}M_I(q) \qquad \mathfrak{S}_{III\to IV}(q) = M_{IV}(q^{-1})^{-1}M_{III}(q^{-1}) \tag{58a}$$

$$\mathfrak{S}_{IV\to I} = M_I(q)^{-1}M_{IV}(q) \qquad \mathfrak{S}_{II\to III} = M_{III}(q)^{-1}M_{II}(q). \tag{58b}$$

(Compare with Equations (33) and (34) after we set $\tilde{x} = 1$ and replace $\tilde{q}$ by q). Note that since $M_I(q), M_{II}(q)$ and $M_{III}(q), M_{IV}(q)$ are given, respectively, as q-series and q^{-1}-series in (56a),(56b) and (56c),(56d), analytic continuation as discussed below (51b) is needed when one computes $\mathfrak{S}_{IV\to I}, \mathfrak{S}_{II\to II}$ in (58b). Using (52)–(55) we can express the answer in terms of $W_{-1}(q)$. Explicitly, the Stokes matrices are given by

$$\mathsf{S}^+(q) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} W_{-1}(q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \qquad |q| < 1, \tag{59a}$$

$$\mathsf{S}^-(q) = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} W_{-1}(q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad |q| < 1. \tag{59b}$$

In the $q \to 0$ limit,

$$\mathsf{S}^+(0) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{S}^-(0) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \tag{60}$$

whose off-diagonal entries $-3, +3$ are Stokes constants associated with the singularities $\iota_{2,1}$ and $\iota_{1,2}$ on the negative and positive real axis, respectively, and they agree with the matrix of integers obtained numerically in [30,45]. In addition, we can assemble the Stokes constants into the matrix $\mathcal{S}$ of Equation (36) (after we set $\tilde{x} = 1$ and replace $\tilde{q}$ by q). The resulting matrix $\mathcal{S}^+(q)$ has entries in $q\mathbb{Z}[[q]]$, and we find

$$\begin{aligned} \mathcal{S}^+_{\sigma_1,\sigma_1}(q) =& \mathsf{S}^+(q)_{1,1} - 1 \\ =& -8q - 9q^2 + 18q^3 + 46q^4 + 90q^5 + 62q^6 + O(q^7), \end{aligned} \tag{61}$$

$$\begin{aligned}
\mathcal{S}^{+}_{\sigma_1,\sigma_2}(q) =& \mathsf{S}^{+}(q)_{1,2}/\mathsf{S}^{+}(q)_{1,1} - \mathcal{S}^{(0)}_{\sigma_1,\sigma_2} \\
=& 9q + 75q^2 + 642q^3 + 5580q^4 + 48558q^5 + 422865q^6 + O(q^7), \qquad (62)\\
\mathcal{S}^{+}_{\sigma_2,\sigma_1}(q) =& \mathsf{S}^{+}(q)_{2,1}/\mathsf{S}^{+}(q)_{1,1} \\
=& -9q - 75q^2 - 642q^3 - 5580q^4 - 48558q^5 - 422865q^6 + O(q^7), \qquad (63)\\
\mathcal{S}^{+}_{\sigma_2,\sigma_2}(q) =& \mathsf{S}^{+}(q)_{2,2} - 1 - \mathsf{S}^{+}(q)_{1,2}\mathsf{S}^{+}(q)_{2,1}/\mathsf{S}^{+}(q)_{1,1} \\
=& 8q + 73q^2 + 638q^3 + 5571q^4 + 48538q^5 + 422819q^6 + O(q^7). \qquad (64)
\end{aligned}$$

We notice the symmetry

$$\mathcal{S}^{(k)}_{1,2} = -\mathcal{S}^{(k)}_{2,1}, \quad \text{for } k \in \mathbb{Z}_{>0}, \tag{65}$$

which is due to the reflection property $\varphi_{\sigma_1}(-\tau^*) = \varphi_{\sigma_2}(\tau)^*$ of the asymptotic series. Also experimentally it appears that the entries of the matrix $\mathcal{S}^{+}(q) = (\mathcal{S}^{+}_{\sigma_i,\sigma_j}(q))$ (except the upper-left one) are (up to a sign) in $\mathbb{N}[[q]]$. Similarly, we can extract the Stokes constants $\mathcal{S}^{(-k)}_{\sigma_i,\sigma_j}$ associated with the singularities in the lower half planes and collect them in q^{-1}-series $\mathcal{S}^{-}_{\sigma_i,\sigma_j}(q^{-1})$ accordingly, and we find

$$\mathcal{S}^{(-k)}_{\sigma_i,\sigma_j} = -\mathcal{S}^{(+k)}_{\sigma_j,\sigma_i},\ i \neq j, \quad \text{and} \quad \mathcal{S}^{(-k)}_{\sigma_1,\sigma_1} = \mathcal{S}^{(+k)}_{\sigma_2,\sigma_2},\ \ \mathcal{S}^{(-k)}_{\sigma_2,\sigma_2} = \mathcal{S}^{(+k)}_{\sigma_1,\sigma_1}, \quad \text{for } k \in \mathbb{Z}_{>0}. \tag{66}$$

A non-trivial consistency check in the above calculation is that the matrices $\mathfrak{S}_{IV\to I}(q)$ and $\mathfrak{S}_{II\to III}(q)$ should come out to be independent of q and coincide with $\mathsf{S}^{-}(0)$ and $\mathsf{S}^{+}(0)$. That is exactly what we find.

The fourth and last ingredient, which makes a full circle of ideas, is the descendant state-integral of the $\mathbf{4}_1$ knot

$$Z_{\mathbf{4}_1,m,\mu}(0;\tau) = \int_{\mathcal{D}} \Phi_{\mathsf{b}}(\nu)^2\, \mathrm{e}^{-\pi \mathrm{i}\nu^2 + 2\pi(m\mathsf{b} - \mu\mathsf{b}^{-1})\nu}\, \mathrm{d}\nu \qquad (m, \mu \in \mathbb{Z}). \tag{67}$$

The integration contour $\mathcal{D}$ asymptotes at infinity to the horizontal line $\mathrm{Im}\nu = \nu_0$ with $\nu_0 > |\mathrm{Re}(m\mathsf{b} - \mu\mathsf{b}^{-1})|$ but is deformed near the origin so that all the poles of the quantum dilogarithm located at

$$c_{\mathsf{b}} + \mathrm{i}\mathsf{b}r + \mathrm{i}\mathsf{b}^{-1}s, \quad r, s \in \mathbb{Z}_{\geq 0}, \tag{68}$$

are above the contour. The integral $Z_{\mathbf{4}_1,m,\mu}(0;\tau)$ is a holomorphic function of $\tau \in \mathbb{C}'$ that coincides with $Z_{\mathbf{4}_1}(0;\tau)$ when $m = \mu = 0$ and can be expressed bilinearly in $G_m(q)$ and $G_\mu(\tilde{q})$ as follows

$$Z_{\mathbf{4}_1,m,\mu}(0;\tau) = (-1)^{m-\mu+1} \mathrm{i} q^{\frac{m}{2}} \tilde{q}^{\frac{\mu}{2}} \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{24}} \frac{1}{2} \left(\sqrt{\tau}\, G^0_\mu(\tilde{q}) G^1_m(q) - \frac{1}{\sqrt{\tau}} G^1_\mu(\tilde{q}) G^0_m(q) \right). \tag{69}$$

It follows that the matrix-valued function

$$W_{m,\mu}(\tau) = (W_\mu(\tilde{q})^T)^{-1} \begin{pmatrix} 1/\sqrt{\tau} & 0 \\ 0 & \sqrt{\tau} \end{pmatrix} W_m(q)^T \tag{70}$$

defined for $\tau = \mathbb{C} \setminus \mathbb{R}$ has entries given by the descendant state-integrals (up to multiplication by a prefactor of (69)) and hence extends to a holomorphic function of $\tau \in \mathbb{C}'$ for all integers m and μ. Using this for $m = -1$ and $\mu = 0$ and the orthogonality relation (54), it follows that we can express the Borel sums of $\Phi(\tau)$ in a region R in terms of descendant state-integrals and hence, as holomorphic functions of $\tau \in \mathbb{C}'$ as follows. For instance, in the region I we have

$$s_I(\Phi)(\tau) = M_I(\tilde{q})^{-1} \begin{pmatrix} \frac{1}{\sqrt{\tau}} g^0_0(\tau) \\ \sqrt{\tau} g^1_0(\tau) \end{pmatrix} = \mathrm{i} \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{24}} \begin{pmatrix} Z_{\mathbf{4}_1,0,0}(0;\tau) - \tilde{q}^{1/2} Z_{\mathbf{4}_1,0,-1}(0;\tau) \\ Z_{\mathbf{4}_1,0,0}(0;\tau) \end{pmatrix} \tag{71}$$

This completes the discussion of $u = 0$ for the $\mathbf{4}_1$ knot.

3.2 The 5_2 knot when $u = 0$

The state-integral of the $\mathbf{5}_2$ at $u = 0$ is given by

$$Z_{\mathbf{5}_2}(0;\tau) = \int_{\mathbb{R}+\mathrm{i}0} \Phi_{\mathsf{b}}(\nu)^3 \, \mathrm{e}^{-2\pi \mathrm{i} \nu^2} \mathrm{d}\nu. \tag{72}$$

The critical points of the integrand are the logarithms of the solutions $\xi_1 \approx 0.78492 + 1.30714\mathrm{i}$, $\xi_2 \approx 0.78492 - 1.30714\mathrm{i}$ and $\xi_3 \approx 0.43016$ of the polynomial equation

$$(1-y)^3 = y^2. \tag{73}$$

The trace field of the $\mathbf{5}_2$ knot is $\mathbb{Q}(\xi_1)$, the cubic field of discriminant -23, which has three complex embeddings labeled by σ_j for $j = 1, 2, 3$ corresponding to the x_j, and the labeling set is $\mathcal{P} = \{\sigma_1, \sigma_2, \sigma_3\}$, where σ_1 corresponds to the geometric representation of the $\mathbf{5}_2$ knot, σ_2 to the complex conjugate of the geometric representation and σ_3 for the corresponding real representation.

The first ingredient is a vector of formal power series

$$\Phi(\tau) = \begin{pmatrix} \Phi_{\sigma_1}(\tau) \\ \Phi_{\sigma_2}(\tau) \\ \Phi_{\sigma_3}(\tau) \end{pmatrix} \tag{74}$$

where

$$\mathrm{Im}\, V(\sigma_1) = -\mathrm{Im}\, V(\sigma_2) = \mathrm{Vol}(\mathbf{5}_2) = 2.82812\ldots \tag{75}$$

is the hyperbolic volume of the knot $\mathbf{5}_2$ and the first few terms of $\varphi_{\sigma_j}(\tau/(2\pi\mathrm{i})) \in \delta_j^{-1/2}\mathbb{Q}(\xi_j)[[\tau]]$ are given by

$$\varphi_{\sigma_j}\left(\frac{\tau}{2\pi\mathrm{i}}\right) = \left(\frac{-3\xi_j^2+3\xi_j-2}{23}\right)^{1/4} \times \left(1+\frac{-242\xi^2+209\xi-454}{2^2\cdot 23^2}\tau+\frac{12643\xi^2-22668\xi+25400}{2^5\cdot 23^3}\tau^2 + \frac{-35443870\xi^2+85642761\xi-164659509}{2^7\cdot 3\cdot 5\cdot 23^5}\tau^3+\dots\right). \tag{76}$$

The second ingredient is two vectors $H^+(q) = (H_0^+(q), H_1^+(q), H_2^+(q))^T$ and $H^-(q) = (H_0^-(q), H_1^-(q), H_2^-(q))^T$ of q-series defined for $|q|<1$ by

$$H_0^+(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n^3}, \tag{77a}$$

$$H_1^+(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n^3}\left(1+2n-3E_1^{(n)}(q)\right), \tag{77b}$$

$$H_2^+(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n^3}\left((1+2n-3E_1^{(n)}(q))^2-3E_2^{(n)}(q)-\frac{1}{6}E_2(q)\right), \tag{77c}$$

and

$$H_0^-(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q;q)_n^3}, \tag{78a}$$

$$H_1^-(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q;q)_n^3}\left(\frac{1}{2}+n-3E_1^{(n)}(q)\right), \tag{78b}$$

$$H_2^-(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q;q)_n^3}\left(\left(\frac{1}{2}+n-3E_1^{(n)}(q)\right)^2-3E_2^{(n)}(q)-\frac{1}{12}E_2(q)\right), \tag{78c}$$

where

$$E_l^{(n)}(q) = \sum_{s=1}^{\infty} \frac{s^{l-1}q^{s(n+1)}}{1-q^s}. \tag{79}$$

The two sets of q-series can be extended to $|q|>1$ and are in fact related by $H_k^+(q^{-1}) = (-1)^k H_k^-(q)$, and define a q-series $H_k(q)$ for $|q|\neq 1$ by

$$H_k(q) = \begin{cases} H_k^+(q) & |q|<1 \\ (-1)^k H_k^-(q^{-1}) & |q|>1. \end{cases} \tag{80}$$

Likewise, we define holomorphic functions $h_k(\tau)$ in $\mathbb{C}\setminus\mathbb{R}$ by

$$h_k(\tau) = \begin{cases} H_k^+(\mathrm{e}^{2\pi\mathrm{i}\tau}), & \mathrm{Im}(\tau)>0 \\ (-1)^k H_k^-(\mathrm{e}^{-2\pi\mathrm{i}\tau}), & \mathrm{Im}(\tau)<0 \end{cases}, \quad k=0,1,2. \tag{81}$$

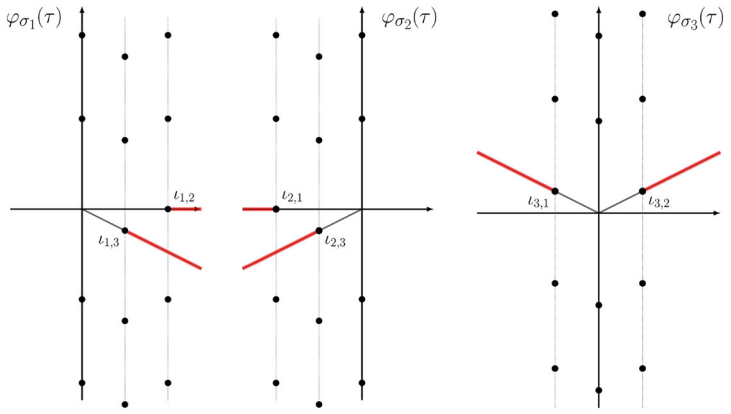

Fig. 4 The singularities in the Borel plane for the series $\varphi_{\sigma_j}(0;\tau)$ for $j=1,2,3$ of knot $\mathbf{5}_2$

These series appear in the factorization of the state-integral of the $\mathbf{5}_2$ knot given in [33, Cor.1.8]

$$Z_{\mathbf{5}_2}(0;\tau) = -\frac{\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}}{2}\left(\frac{q}{\tilde{q}}\right)^{\frac{1}{8}}\big(\tau H_0^-(\tilde{q})H_2^+(q) - 2H_1^-(\tilde{q})H_1^+(q) \\ +\tau^{-1}H_2^-(\tilde{q})H_0^+(q)\big), \qquad (\tau \in \mathbb{C}\setminus\mathbb{R}). \tag{82}$$

The above factorization follows by applying the residue theorem to the integrand of (72), a meromorphic function of ν with prescribed zeros and poles. In particular, the integrand of (72) determines the q-hypergeometric formula for the vectors $H^+(q), H^-(q)$ of q-series.

As in the case of the $\mathbf{4}_1$ knot, multiplying the vector $h(\tau)^T = (h_0(\tau), h_1(\tau), h_2(\tau))^T$ by the automorphy factors $\mathrm{diag}(\tau^{-1}, 1, \tau)$ (dictated by (82)), and looking at the asymptotics as τ approaches zero in sectors, we found that

$$\mathrm{e}^{\frac{3\pi \mathrm{i}}{4}}\,\mathrm{diag}(\tau^{-1}, 1, \tau)h(\tau) = M_R(\tilde{q})\, s_R(\Phi)(\tau), \tag{83}$$

where the right-hand side depends on the sectors of Borel resummation. The Borel plane singularities of the component series of the vector $\Phi(\tau)$ are similarly located at

$$\iota_{i,j} = \frac{V(\sigma_i) - V(\sigma_j)}{2\pi \mathrm{i}}, \quad i,j = 1,2,3, i \neq j, \tag{84}$$

as well as

$$2\pi \mathrm{i}k,\ \iota_{i,j} + 2\pi \mathrm{i}k, \quad k \in \mathbb{Z}_{\neq 0}, \tag{85}$$

which form vertical towers as illustrated in Fig. 4. In particular, the two singularities $\iota_{1,2}, \iota_{2,1}$ are on the positive and negative real axis. We pick out the four sectors which separate the two singularities on the real axis and all the others and label them by *I, II, III, IV*, as illustrated in Fig. 5.

To identify the matrices M_R, we consider the third ingredient, the linear q-difference equation

$$y_m(q) - 3y_{m+1}(q) + (3 - q^{2+m})y_{m+2}(q) - y_{m+3}(q) = 0 \qquad (m \in \mathbb{Z}), \tag{86}$$

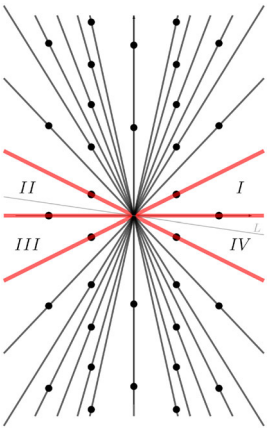

Fig. 5 Four different sectors in the τ-plane for $\Phi(0;\tau)$ of knot $\mathbf{5}_2$ and the line L that divides the τ-plane

They have fundamental solution sets given by the columns of the following matrix

$$W_m(q) = \begin{cases} W_m^+(q), & |q|<1 \\ \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix} W_{-m-2}^-(q^{-1}) \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&1 \end{pmatrix}, & |q|>1. \end{cases} \tag{87}$$

where the matrices $W_m^\varepsilon(q)$ with $\varepsilon = \pm$ are, respectively,

$$W_m^\varepsilon(q) = \begin{pmatrix} H_{0,m}^\varepsilon(q) & H_{1,m}^\varepsilon(q) & H_{2,m}^\varepsilon(q) \\ H_{0,m+1}^\varepsilon(q) & H_{1,m+1}^\varepsilon(q) & H_{2,m+1}^\varepsilon(q) \\ H_{0,m+2}^\varepsilon(q) & H_{1,m+2}^\varepsilon(q) & H_{2,m+2}^\varepsilon(q) \end{pmatrix}, \qquad (|q|\neq 1) \tag{88}$$

with entries the q-series

$$H_{0,m}^+(q) = \sum_{n=0}^\infty \frac{q^{n(n+1)+nm}}{(q;q)_n^3}, \tag{89a}$$

$$H_{1,m}^+(q) = \sum_{n=0}^\infty \frac{q^{n(n+1)+nm}}{(q;q)_n^3} \left(1+2n+m-3E_1^{(n)}(q)\right), \tag{89b}$$

$$H_{2,m}^+(q) = \sum_{n=0}^\infty \frac{q^{n(n+1)+nm}}{(q;q)_n^3} \left((1+2n+m-3E_1^{(n)}(q))^2 - 3E_2^{(n)}(q) - \frac{1}{6}E_2(q)\right), \tag{89c}$$

and

$$H_{0,m}^-(q) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{1}{2}n(n+1)+nm}}{(q;q)_n^3}, \tag{90a}$$

$$H_{1,m}^-(q) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{1}{2}n(n+1)+nm}}{(q;q)_n^3} \left(\frac{1}{2}+n+m-3E_1^{(n)}(q)\right), \tag{90b}$$

$$H_{2,m}^-(q) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{1}{2}n(n+1)+nm}}{(q;q)_n^3} \left(\left(\frac{1}{2}+n+m-3E_1^{(n)}(q)\right)^2 - 3E_2^{(n)}(q) - \frac{1}{12}E_2(q)\right), \tag{90c}$$

for $|q|<1$ and extended to $|q|>1$ by the relation $H^{+}_{k,m}(q^{-1}) = (-1)^k H^{-}_{k,-m}(q)$. Observe that $H^{\varepsilon}_{k,0}(q) = H^{\varepsilon}_{k}(q)$. The matrix $W_m(q)$ of holomorphic functions in $|q| \neq 1$ satisfies several properties summarized in the following theorem.

Theorem 11 *$W_m(q)$ is a fundamental solution of the linear q-difference equation (86) that has constant determinant*

$$\det(W_m(q)) = 2, \tag{91}$$

satisfying the orthogonality property

$$\frac{1}{2} W_{m-1}(q) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} W_{-m-1}(q^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3-q^m \end{pmatrix}. \tag{92}$$

as well as

$$\frac{1}{2} W_{m}(q) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} W_{\ell}(q^{-1})^T \in \mathrm{SL}(3, \mathbb{Z}[q^{\pm}]) \tag{93}$$

for all integers m, ℓ and for $|q| \neq 1$.

Conjecture 12 Equation (83) holds where the matrices $M_R(q)$ are given in terms of $W_{-1}(q)$ as follows

$$M_I(q) = W_{-1}(q)^T \begin{pmatrix} 0 & 0 & 1 \\ -1 & 3 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad |q|<1, \tag{94a}$$

$$M_{II}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 0 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad |q|<1, \tag{94b}$$

$$M_{III}(q) = W_{-1}(q)^T \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad |q|>1, \tag{94c}$$

$$M_{IV}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad |q|>1. \tag{94d}$$

Assuming the above conjecture, we can now describe completely the resurgent structure of $\Phi(\tau)$, following the same computation as in the case of the $\mathbf{4}_1$ knot. The Stokes matrices are given by (57)–(58b). Note that since $M_I(q), M_{II}(q)$ and $M_{III}(q), M_{IV}(q)$ are given, respectively, as q-series and q^{-1}-series in (94a),(94b) and (94c),(94d), analytic continuation as discussed below (79) is needed when one computes $\mathfrak{S}_{IV\to I}$, $\mathfrak{S}_{II\to II}$ in (58b). Using (91)–(93), we can express the answer in terms of $W_{-1}(q)$. Once again, we find

that the Stokes matrices $\mathfrak{S}_{IV\to I}(q)$ and $\mathfrak{S}_{II\to III}(q)$ are independent of q, consistent with semiclassical asymptotics. The Stokes matrices are given by

$$\mathsf{S}^+(q) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0\\ 0 & 1 & 1\\ -1 & 0 & 0\end{pmatrix} W_{-1}(q^{-1}) \begin{pmatrix} 0 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 0\end{pmatrix} W_{-1}(q)^T \begin{pmatrix} 0 & 0 & -1\\ 1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}, \quad |q|<1 \tag{95a}$$

$$\mathsf{S}^-(q) = \frac{1}{2}\begin{pmatrix} 0 & 3 & -1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix} W_{-1}(q) \begin{pmatrix} 0 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 0\end{pmatrix} W_{-1}(q^{-1})^T \begin{pmatrix} 0 & 0 & 1\\ 3 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}, \quad |q|<1. \tag{95b}$$

These Stokes matrices completely describe the resurgent structure of $\Phi(\tau)$. They also satisfy other statements in Conjectures 3 and 4 when $x=1$. The $q\to 0$ limit of the Stokes matrices factorizes

$$\mathsf{S}^+(0) = \mathfrak{S}_{\sigma_3,\sigma_1}\mathfrak{S}_{\sigma_3,\sigma_2}\mathfrak{S}_{\sigma_1,\sigma_2} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix} 1 & 4 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{96}$$

$$\mathsf{S}^-(0) = \mathfrak{S}_{\sigma_1,\sigma_3}\mathfrak{S}_{\sigma_2,\sigma_3}\mathfrak{S}_{\sigma_2,\sigma_1} = \begin{pmatrix} 1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ -4 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{97}$$

where the non-vanishing off-diagonal entry of $\mathfrak{S}_{\sigma_i,\sigma_j}$ is the Stokes constant associated with the Borel singularity $\iota_{i,j}$. Assembling these off-diagonal entries in a matrix, we obtain the matrix

$$\begin{pmatrix} 0 & 4 & 3\\ -4 & 0 & -3\\ -3 & 3 & 0\end{pmatrix} \tag{98}$$

that was found numerically in [45, Sec.3.3]. In addition, we can assemble the Stokes constants into the matrix $\mathcal{S}$ of Equation (36) (after we set $\tilde{x}=1$ and replace $\tilde{q}$ by q). The resulting matrix $\mathcal{S}^+(q)$ has entries in $q\mathbb{Z}[[q]]$, and we find

$$\begin{aligned}
\mathcal{S}^+_{\sigma_1,\sigma_1} &= \mathsf{S}^+(q)_{1,1} - 1\\
&= -12q + 3q^2 + 74q^3 + 90q^4 + 33q^5 + O(q^6), && (99a)\\
\mathcal{S}^+_{\sigma_1,\sigma_2} &= 12q + 141q^2 + 1520q^3 + 17397q^4 + 191970q^5 + O(q^6), && (99b)\\
\mathcal{S}^+_{\sigma_1,\sigma_3} &= q + 3q^2 + 9q^3 + 30q^4 + 99q^5 + O(q^6), && (99c)\\
\mathcal{S}^+_{\sigma_2,\sigma_1} &= -12q - 141q^2 - 1520q^3 - 17397q^4 - 191970q^5 + O(q^6), && (99d)\\
\mathcal{S}^+_{\sigma_2,\sigma_2} &= 12q + 141q^2 + 1582q^3 + 17583q^4 + 194703q^5 + O(q^6), && (99e)\\
\mathcal{S}^+_{\sigma_2,\sigma_3} &= -21q - 235q^2 - 2586q^3 - 28593q^4 - 316104q^5 + O(q^6), && (99f)\\
\mathcal{S}^+_{\sigma_3,\sigma_1} &= -q - 3q^2 - 9q^3 - 30q^4 - 99q^5 + O(q^6), && (99g)\\
\mathcal{S}^+_{\sigma_3,\sigma_2} &= 21q + 235q^2 + 2586q^3 + 28593q^4 + 316104q^5 + O(q^6), && (99h)\\
\mathcal{S}^+_{\sigma_3,\sigma_3} &= 0. && (99i)
\end{aligned}$$

The Stokes constants enjoy the symmetry

$$\mathcal{S}^{(k)}_{\sigma_i,\sigma_j} = -\mathcal{S}^{(k)}_{\sigma_{\varphi(i)},\sigma_{\varphi(j)}}, \quad i \neq j, \quad \text{for } k \in \mathbb{Z}_{>0}\,, \tag{100}$$

with $\varphi(1) = 2, \varphi(2) = 1, \varphi(3) = 3$. We notice that the entries of the matrix $\mathcal{S}^+(q) = (\mathcal{S}^+_{\sigma_i,\sigma_j}(q))$ (except the upper-left one) are (up to a sign) in $\mathbb{N}[[q]]$. Similarly, we can extract the Stokes constants $\mathcal{S}^{(-k)}_{\sigma_i,\sigma_j}$ associated with the singularities in the lower half plane, and we find

$$\mathcal{S}^{(-k)}_{\sigma_i,\sigma_j} = -\mathcal{S}^{(+k)}_{\sigma_j,\sigma_i}, \ i \neq j, \quad \text{and} \quad \mathcal{S}^{(-k)}_{\sigma_i,\sigma_i} = \mathcal{S}^{(+k)}_{\sigma_{\varphi(i)},\sigma_{\varphi(j)}}, \quad \text{for } k \in \mathbb{Z}_{>0}\,. \tag{101}$$

The fourth and last ingredient, which makes a full circle of ideas, is the descendant state-integral of the $\mathbf{5}_2$ knot

$$Z_{\mathbf{5}_2,m,\mu}(0;\tau) = \int_{\mathcal{D}} \Phi_{\mathsf{b}}(\nu)^3\, \mathrm{e}^{-2\pi \mathrm{i}\nu^2 + 2\pi(m\mathsf{b} - \mu\mathsf{b}^{-1})\nu}\, \mathrm{d}\nu \qquad (m, \mu \in \mathbb{Z})\,. \tag{102}$$

Here, the same contour $\mathcal{D}$ as in (67) is used. It is a holomorphic function of $\tau \in \mathbb{C}'$ that coincides with $Z_{\mathbf{5}_2}(0;\tau)$ when $m = \mu = 0$ and can be expressed bilinearly in $H^+_{k,m}(q)$ and $H^-_{k,\mu}(\tilde{q})$ as follows

$$\begin{aligned} Z_{\mathbf{5}_2,m,\mu}(0;\tau) =& (-1)^{m-\mu+1} \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{2} q^{\frac{m}{2}} \tilde{q}^{\frac{\mu}{2}} \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{8}} \\ & \left(\tau\, H^-_{0,\mu}(\tilde{q}) H^+_{2,m}(q) - 2H^-_{1,\mu}(\tilde{q}) H^+_{1,m}(q) + \tau^{-1} H^-_{2,\mu}(\tilde{q}) H^+_{0,m}(q)\right). \end{aligned} \tag{103}$$

It follows that the matrix-valued function

$$W_{m,\mu}(\tau) = (W_\mu(\tilde{q})^T)^{-1} \begin{pmatrix} \tau^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix} W_m(q)^T \tag{104}$$

defined for $\tau = \mathbb{C} \setminus \mathbb{R}$ has entries given by the descendant state-integrals (up to multiplication by a prefactor of (103)) and hence extends to a holomorphic function of $\tau \in \mathbb{C}'$ for all integers m and μ. Using this for $m = -1$ and $\mu = 0$ and the orthogonality relation (92), it follows that we can express the Borel sums of $\Phi(\tau)$ in a region R in terms of descendant state-integrals and hence, as holomorphic functions of $\tau \in \mathbb{C}'$ as follows. For instance, in the region I we have

$$s_I(\Phi)(\tau) = \mathrm{e}^{\frac{3\pi\mathrm{i}}{4}} M_I(\tilde{q})^{-1} \begin{pmatrix} \tau^{-1} h_0(\tau) \\ h_1(\tau) \\ \tau h_2(\tau) \end{pmatrix} = \mathrm{i} \left(\frac{q}{\tilde{q}}\right)^{-\frac{1}{8}} \begin{pmatrix} Z_{\mathbf{5}_2,0,0}(0;\tau) - \tilde{q}^{1/2} Z_{\mathbf{5}_2,0,-1}(0;\tau) \\ Z_{\mathbf{5}_2,0,0}(0;\tau) \\ \tilde{q}^{-1/2} Z_{\mathbf{5}_2,0,1}(0;\tau) \end{pmatrix} \tag{105}$$

This completes the discussion of $u = 0$ for the $\mathbf{5}_2$ knot.

4 Holonomic and elliptic functions inside and outside the unit disk

An important property for the functions of a complex variable q in our paper (such as the holomorphic blocks considered below) is that they can be defined both inside ($|q|<1$) and outside ($|q|>1$) the unit disk, in such a way that they have the same annihilator ideal. Recall that if L and M denote the operators that act on functions $f(x;q)$ by $(Lf)(x;q) = f(qx;q)$ and $(Mf)(x;q) = xf(x;q)$, then $LM = qML$, and hence, $L^{-1}M = q^{-1}ML^{-1}$. It follows that if $P(L,M,q)f(x;q) = 0$, then $P(L^{-1},M,q^{-1})f(x;q^{-1}) = 0$ where $P(L,M,q)$ denotes a polynomial in L with coefficients polynomials in M and q.

A first example of a function to consider is $(x;q)_\infty = \prod_{j=0}^{\infty}(1-q^j x)$, which is well defined for $|q|<1$ and $x\in\mathbb{C}$ and satisfies the linear q-difference equation

$$(1-x)(qx;q)_\infty = (x;q)_\infty \qquad (|q|<1)\,. \tag{106}$$

We can extend it to a meromorphic function of x when $|q|>1$ (by a slight abuse of notation) by defining

$$(x;q^{-1})_\infty := (qx;q)_\infty^{-1} \qquad (|q|<1)\,, \tag{107}$$

so that Equation (106) holds for $|q|\neq 1$. Our second example is the theta function

$$\theta(x;q) = (-q^{\frac{1}{2}}x;q)_\infty(-q^{\frac{1}{2}}x^{-1};q)_\infty \qquad (|q|<1) \tag{108}$$

which satisfies the linear q-difference equation

$$\theta(qx;q) = q^{-\frac{1}{2}}x^{-1}\theta(x;q) \qquad (|q|<1) \tag{109}$$

and can be extended to $\theta(x;q^{-1}) = \theta(x;q)^{-1}$ when $|q|>1$ so that Equation (109) holds for $|q|\neq 1$. $\theta(x;q)$ is a meromorphic function of $x\in\mathbb{C}^*$ with the following (simple) zeros and (simple) poles

$$\begin{array}{lll} |q|<1 & \text{zeros}(\theta) = -q^{\frac{1}{2}+\mathbb{Z}} & \text{poles}(\theta) = \emptyset \\ |q|>1 & \text{zeros}(\theta) = \emptyset & \text{poles}(\theta) = -q^{\frac{1}{2}+\mathbb{Z}}\,. \end{array} \tag{110}$$

An important property of the theta functions is that they factorize the exponentials of a quadratic and linear form of u. This fact is a consequence of the modular invariance of the theta function and was used extensively in the study of holomorphic blocks [7].

Lemma 13 *For integers r and s we have:*

$$\mathrm{e}^{\pi\mathrm{i}(u+rc_{\mathrm{b}})^2} = \mathrm{e}^{-\frac{\pi\mathrm{i}}{12}(\tau+\tau^{-1})}\theta((-q^{\frac{1}{2}})^r x;q)\theta((-\tilde{q}^{-\frac{1}{2}})^r\tilde{x};\tilde{q}^{-1}) \tag{111a}$$

$$\mathrm{e}^{\pi\mathrm{i}ru^2+2\pi\mathrm{i}sc_{\mathrm{b}}u} = \mathrm{i}^s\mathrm{e}^{\frac{\pi\mathrm{i}}{12}(3s-r)(\tau+\tau^{-1})}\theta(x;q)^{r-s}\theta(-q^{\frac{1}{2}}x;q)^s \times \theta(\tilde{x};\tilde{q}^{-1})^{r-s}\theta(-\tilde{q}^{-\frac{1}{2}}\tilde{x};\tilde{q}^{-1})^s \tag{111b}$$

for integers r and s.

Note that we the above factorization formulas are by no means unique, and this is a reflection of the dependence of the above formulas on a theta divisor.

Proof When $x = \mathrm{e}^{2\pi \mathsf{b} u}$, $q = e(\tau)$, $\tilde{q} = e(-1/\tau)$ and $|q| < 1$, then we claim

$$\mathrm{e}^{-\frac{1}{4\pi \mathrm{i}\tau}(\log x)^2} = \mathrm{e}^{\pi \mathrm{i} u^2} = \Phi_{\mathsf{b}}(0)^{-2}\Phi_{\mathsf{b}}(u)\Phi_{\mathsf{b}}(-u) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{12}(\tau+\tau^{-1})}\theta(x;q)\theta(\tilde{x};\tilde{q}^{-1}). \quad (112)$$

The first equality is easy, the second one follows from the inversion formula of Faddeev's quantum dilogarithm, and the third one follows from the product expansion of Faddeev's quantum dilogarithm or from the modular invariance of the theta function. Note also that $\Phi_{\mathsf{b}}(0)^2 = (q/\tilde{q})^{\frac{1}{24}} = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}(\tau+\tau^{-1})}$. Equation (111a) follows easily from the above, and Equation (111b) follows from the above using, for example, $\mathrm{e}^{\pi \mathrm{i} r u^2 + 2\pi \mathrm{i} s c_{\mathsf{b}} u} = \mathrm{e}^{(r-s)\pi \mathrm{i} u^2}\mathrm{e}^{s\pi i(u+c_{\mathsf{b}})^2}\mathrm{e}^{-s\pi \mathrm{i} c_{\mathsf{b}}^2}$. □

5 The 4_1 knot

5.1 Asymptotic series

Our starting point will be the state-integral for the $\mathbf{4}_1$ knot [3, Eqn.38] (after removing a prefactor that depends on u alone)

$$Z_{\mathbf{4}_1}(u;\tau) = \mathrm{e}^{-2\pi \mathrm{i} u^2}\int_{\mathbb{R}+\mathrm{i}0} \Phi_{\mathsf{b}}(v)\,\Phi_{\mathsf{b}}(u+v)\,\mathrm{e}^{-\pi \mathrm{i}(v^2+4uv)}\mathrm{d}v. \quad (113)$$

The above state-integral (and all the subsequent ones) is a holomorphic function of $\tau \in \mathbb{C}'$ and u when $|\mathrm{Im}(u)| < |\mathsf{b}+\mathsf{b}^{-1}|/2$ and extends to an entire function of u (see Theorem 14).

After a change of variables $u \mapsto u/(2\pi\mathsf{b})$ (see Equation (2)) and $v \mapsto v/(2\pi\mathsf{b})$, the asymptotic expansion of the quantum dilogarithm (see, for instance, [3, Prop.6]) implies that the integrand of $Z_{\mathbf{4}_1}(u_{\mathsf{b}};\tau)$ has a leading term given by $\mathrm{e}^{V(u,v)/(2\pi \mathrm{i}\tau)}$ where

$$V(u,v) = \mathrm{Li}_2(-\mathrm{e}^v) + \mathrm{Li}_2(-\mathrm{e}^{u+v}) + \frac{1}{2}(v)^2 + 2uv. \quad (114)$$

Taking derivative with respect to v gives the equation for the critical point

$$2u + v - \log(1+\mathrm{e}^v) - \log(1+\mathrm{e}^{u+v}) = 0 \quad (115)$$

which implies that $(x, y) = (e^u, -e^v)$ is a complex point points of the affine curve S given by

$$S : -x^2 y = (1-y)(1-xy) \quad (116)$$

and (u, v) is a point of the exponentiated curve S^* given by the above equation where $(x, y) = (\mathrm{e}^u, -\mathrm{e}^v)$. Moreover, we have

$$V(u,v) = \mathrm{Li}_2(y) + \mathrm{Li}_2(xy) + \frac{1}{2}(\log(-y))^2 + 2\log x \log(-y). \quad (117)$$

Note that (115) has more information than (116) since it chooses the logarithms of $1+\mathrm{e}^v$ and $1+\mathrm{e}^{u+v}$ such that (115) holds. This ultimately implies that V is a holomorphic $\mathbb{C}/2\pi^2\mathbb{Z}$-valued function on the exponentiated curve S^*. Note that when $u = 0$, Equation (116) becomes (38).

The constant term of the asymptotic expansion is given by the Hessian of $V(u,v)$ at a critical point (u,v), and it is a rational function of x and y is given by

$$\delta(x,y) = -\frac{1-xy^2}{x^2y}. \tag{118}$$

Note that $\delta(x,y) = 0$ on S if and only if x is a root of the discriminant of S with respect to y, i.e.,

$$(1-3x+x^2)(1+x+x^2) = 0. \tag{119}$$

In other words, δ vanishes precisely when two branches of $y = y(x)$ coincide.

Beyond the leading asymptotic expansion and its constant term, the asymptotic series has the form $\Phi(x,y;\tau)$ where

$$\Phi(x,y;\tau) = \exp\left(\frac{V(u,v)}{2\pi\mathrm{i}\tau}\right)\varphi(x,y;\tau), \qquad \varphi(x,y;\tau) \in \frac{1}{\sqrt{\mathrm{i}\delta}}\mathbb{Q}[x^{\pm},y^{\pm},\delta^{-1}][[2\pi\mathrm{i}\tau]] \tag{120}$$

where δ is given in (118) and $\sqrt{\mathrm{i}\delta}\,\varphi(x,y;0) = 1$. In other words, the coefficient of every power of $2\pi\mathrm{i}\tau$ in $\sqrt{\delta}\,\varphi(x,y;\tau)$ is a rational function on S. There is a natural projection $S \to \mathbb{C}^*$ given by $(x,y) \to x$, and we denote by $y_\sigma(x)$ the choice of a local section (an algebraic function of x), for $\sigma \in \mathcal{P} = \{\sigma_1,\sigma_2\}$. We denote the corresponding series $\Phi(x,y_\sigma(x);\tau)$ simply by $\Phi_\sigma(x;\tau)$. Note that

$$\delta(x,y_{1,2}(x)) = \pm\frac{\sqrt{(1-x-x^{-1})^2-4}}{x} \tag{121}$$

and that the two series are related by

$$\Phi_2(x;\tau) = \mathrm{i}\Phi_1(x;-\tau). \tag{122}$$

The power series $\sqrt{\mathrm{i}\delta}\varphi_\sigma(x;\tau)$ can be computed by applying Gaussian expansion to the state-integral (113). One can compute up to 20 terms in a few minutes, and the first few terms agree with an independent computation using the WKB method (see [19, Eqn.(4.39)] as well as [28]), and given by $\sqrt{\mathrm{i}\delta}\varphi_\sigma(x;\tau)$

$$\begin{aligned}\sqrt{\mathrm{i}\delta}\varphi_{1,2}\left(x;\frac{\tau}{2\pi\mathrm{i}}\right) = 1 &- \frac{1}{24\gamma_{1,2}^3(x)}\left(x^{-3}-x^{-2}-2x^{-1}+15-2x-x^2+x^3\right)\tau\\ &+ \frac{1}{1152\gamma_{1,2}^6(x)}\left(x^{-6}-2x^{-5}-3x^{-4}+610x^{-3}\right.\\ &-606x^{-2}-1210x^{-1}+3117\\ &\left.-1210x-606x^2+610x^3-3x^4-2x^5+x^6\right)\tau^2+O(\tau^3),\end{aligned} \tag{123}$$

where

$$\gamma_{1,2}(x) = x\delta(x,y_{1,2}(x)) = \pm\sqrt{x^{-2}-2x^{-1}-1-2x+x^2}. \tag{124}$$

sets x to numerical values, one can compute 300 terms of this power series.

5.2 Holomorphic blocks

In this section, we give the definition of the holomorphic blocks (and their descendants) which factorize the state-integral (and its descendants) and discuss their analytic properties, and their linear q-difference equations. Note that in this section, as well as in Sect. 5.3, all the statements are theorems, whose proofs we provide.

Motivated by the state-integral $Z_{\mathbf{4}_1}(u;\tau)$ of the $\mathbf{4}_1$ knot given in (113), and by the descendant state-integral $Z_{\mathbf{4}_1,m,\mu}(0;\tau)$ given in (67), we introduce the descendant state-integral of the $\mathbf{4}_1$ knot

$$Z_{\mathbf{4}_1,m,\mu}(u;\tau) = \mathrm{e}^{-2\pi \mathrm{i} u^2} \int_{\mathcal{D}} \Phi_{\mathsf{b}}(\nu)\, \Phi_{\mathsf{b}}(u+\nu)\, \mathrm{e}^{-\pi \mathrm{i}(\nu^2+4u\nu)+2\pi(m\mathsf{b}-\mu\mathsf{b}^{-1})\nu}\, \mathrm{d}\nu \tag{125}$$

for integers m and μ, which agrees with the Andersen-Kashaev invariant of the $\mathbf{4}_1$ knot when $m=\mu=0$. Here, the contour $\mathcal{D}$ was introduced in (67). It is expressed in terms of two descendant holomorphic blocks, which we denote by A_m and B_m instead of $B_m^{\sigma_1}$ and $B_m^{\sigma_2}$, in order to simplify the notation. For $|q|\neq 1$, $A_m(x;q)$ and $B_m(x;q)$ are given by

$$A_m(x;q) = \theta(-q^{\frac{1}{2}}x;q)^{-2} x^{2m} J(q^m x^2, x; q)\,, \tag{126a}$$

$$B_m(x;q) = \theta(-q^{-\frac{1}{2}}x;q) x^m J(q^m x, x^{-1}; q)\,, \tag{126b}$$

where $J(x,y;q^\varepsilon) := J^\varepsilon(x,y;q)$ for $|q|<1$ and $\varepsilon=\pm$ is the q-Hahn Bessel function

$$J^+(x,y;q) = (qy;q)_\infty \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^n}{(q;q)_n (qy;q)_n}\,, \tag{127a}$$

$$J^-(x,y;q) = \frac{1}{(y;q)_\infty} \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^n y^{-n}}{(q;q)_n (qy^{-1};q)_n}\,. \tag{127b}$$

The next theorem expresses the descendant state-integrals bilinearly in terms of descendant holomorphic blocks.

Theorem 14 (a) *The descendant state-integral can be expressed in terms of the descendant holomorphic blocks by*

$$Z_{\mathbf{4}_1,m,\mu}(u_{\mathsf{b}};\tau) = (-1)^{m+\mu} q^{m/2} \tilde{q}^{\mu/2} \Big(\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4} - \frac{\pi\mathrm{i}}{6}(\tau+\tau^{-1})} A_m(x;q) A_{-\mu}(\tilde{x};\tilde{q}^{-1}) \tag{128}$$

$$+ \mathrm{e}^{\frac{3\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{3}(\tau+\tau^{-1})} B_m(x;q) B_{-\mu}(\tilde{x};\tilde{q}^{-1}) \Big)\,. \tag{129}$$

(b) *The functions $A_m(x;q)$, $B_m(x;q)$ are holomorphic functions of $|q|\neq 1$ and meromorphic functions of $x\in\mathbb{C}^*$ with poles in $x\in q^{\mathbb{Z}}$ of order at most* 1.
(c) *Let*

$$W_m(x;q) = \begin{pmatrix} A_m(x;q) & B_m(x;q) \\ A_{m+1}(x;q) & B_{m+1}(x;q) \end{pmatrix} \qquad (|q|\neq 1)\,. \tag{130}$$

For all integers m and μ, the state-integral $Z_{\mathbf{4}_1,m,\mu}(u;\tau)$ and the matrix-valued function

$$W_{m,\mu}(u;\tau) = W_{-\mu}(\tilde{x};\tilde{q}^{-1}) \Delta(\tau) W_m(x;q)^T\,, \tag{131}$$

where

$$\Delta(\tau) = \begin{pmatrix} \mathrm{e}^{-\frac{3\pi\mathrm{i}}{4} - \frac{\pi\mathrm{i}}{6}(\tau+\tau^{-1})} & 0 \\ 0 & \mathrm{e}^{\frac{3\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{3}(\tau+\tau^{-1})} \end{pmatrix}, \tag{132}$$

are holomorphic functions of $\tau \in \mathbb{C}'$ *and entire functions of* $u \in \mathbb{C}$.

Proof Part (a) follows by applying the residue theorem to the state-integral (125), along the lines of the proof of Theorem 1.1 in [33]. A similar result was stated in [20].

Part (b) follows from the fact that when $|q| < 1$, the ratio test implies that $J^+(x, y; q)$ is an entire function of $(x, y) \in \mathbb{C}^2$ and $J^-(x, y; q)$ is a meromorphic function of (x, y) with poles in $y \in q^{\mathbb{Z}}$.

For part (c), one uses parts (a) and (b) to deduce that $W_{m,\mu}(u; \tau)$ is holomorphic of $\tau \in \mathbb{C}'$ and meromorphic in u with potential simple poles at $\mathsf{i}\mathsf{b}\mathbb{Z} + \mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$. An expansion at these points, done by the method of Sect. 5.3, demonstrates that the function is analytic at the points $\mathsf{i}\mathsf{b}\mathbb{Z} + \mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$. □

Note that the summand of J^+ (a proper q-hypergeometric function) is equal to that of J^- after replacing q by q^{-1}. This implies that $J^{\pm}$ have a common annihilating ideal $\mathcal{I}_J$ with respect to x, y which can be computed (rigorously, along with a provided certificate) using the creative telescoping method of Zeilberger [63] implemented in the `HolonomicFunctions` package of Koutschan [51,52]. Below, we will abbreviate this package by `HF`.

Lemma 15 *The annihilating ideal of* $\mathcal{I}_J$ *of* $J^{\pm}$ *is given by*

$$\mathcal{I}_J = \langle (-x + y) + xS_y - yS_x,\ 1 + (-1 - x + qy)S_y + xS_y^2 \rangle \tag{133}$$

where S_x *and* S_y *are the shifts* x *to* qx *and* y *to* qy.

The next theorem concerns the properties of the linear q-difference equations satisfied by the descendant holomorphic blocks.

Theorem 16 (a) *The pair* $A_m(x; q)$ *and* $B_m(x; q)$ *are* q*-holonomic functions in the variables* (m, x) *with a common annihilating ideal*

$$\mathcal{I}_{\mathbf{4}_1} = \langle q^m x^2 + (-q^m + q^{1+2m}x^2)S_m + x^3 S_x,\ (1 - x^{-1}S_m)(1 - x^{-2}S_m) + q^{1+m}S_m \rangle \tag{134}$$

where S_m *is the shift of* m *to* $m + 1$ *and* S_x *is the shift of* x *to* qx. $\mathcal{I}_{\mathbf{4}_1}$ *has rank 2 and the two functions form a basis of solutions of the corresponding system of linear equations.*
(b) *As functions of an integer* m, $A_m(x; q)$ *and* $B_m(x; q)$ *form a basis of solutions of the linear* q*-difference equation* $\widehat{B}_{\mathbf{4}_1}(S_m, x, q^m, q) f_m(x; q) = 0$ *for* $|q| \neq 1$ *where*

$$\widehat{B}_{\mathbf{4}_1}(S_m, x, q^m, q) = (1 - x^{-1}S_m)(1 - x^{-2}S_m) + q^{1+m}S_m\,. \tag{135}$$

(c) *The Wronskian* $W_m(x; q)$ *of* (135), *defined in* (130), *satisfies*

$$\det(W_m(x; q)) = x^{3m+3} \qquad (m \in \mathbb{Z})\,. \tag{136}$$

(d) *The Wronskian satisfies the orthogonality relation*

$$W_{-1}(x;q)\begin{pmatrix}1&0\\0&-1\end{pmatrix}W_{-1}(x;q^{-1})^T=\begin{pmatrix}x^{-2}+x^{-1}-1&1\\1&0\end{pmatrix}. \tag{137}$$

It follows that for all integers m and ℓ

$$W_m(x;q)\begin{pmatrix}1&0\\0&-1\end{pmatrix}W_\ell(x;q^{-1})^T\in \mathrm{GL}(2,\mathbb{Z}[q^{\pm},x^{\pm}]). \tag{138}$$

(e) *As functions of x, $A_m(x;q)$ and $B_m(x;q)$ form a basis of a linear q-difference equation $\hat{A}_{\mathbf{4}_1}(S_x,x,q^m,q)f_m(x;q)=0$ where*

$$\hat{A}_{\mathbf{4}_1}(S_x,x,q^m,q)=\sum_{j=0}^{2}C_j(x,q^m,q)S_x^j, \tag{139}$$

S_x is the operator that shifts x to qx and

$$C_0=q^{2+3m}x^2(-1+q^{3+m}x^2) \tag{140a}$$

$$C_1=-q^m(-1+q^{2+m}x^2)(1-qx-q^{1+m}x^2 -q^{3+m}x^2-q^{3+m}x^3+q^{4+2m}x^4) \tag{140b}$$

$$C_2=q^2x^2(-1+q^{1+m}x^2). \tag{140c}$$

(f) *The Wronskian of* (139)

$$\mathcal{W}_m(x;q)=\begin{pmatrix}A_m(x;q)&B_m(x;q)\\A_m(qx;q)&B_m(qx;q)\end{pmatrix},\qquad (|q|\neq 1) \tag{141}$$

satisfies

$$\det(\mathcal{W}_m(x;q))=q^mx^{3m}(1-q^{m+1}x^2)\qquad (m\in\mathbb{Z}). \tag{142}$$

Proof Since $A_m(x;q)$ and $B_m(x;q)$ are given in terms of q-proper hypergeometric multisums, it follows from the fundamental theorem of Zeilberger [63,71,72] (see also [35]) that they are q-holonomic functions in both variables m and x. Part (a) follows from an application of the HF package of Koutschan [51,52].

Part (b) follows from the HF package. The fact that they are a basis follows from (c).

For part (c), Equation (135) implies that the determinant of the Wronskian satisfies the first-order equation $\det(W_{m+1}(x;q))=x^3\det(W_m(x;q))$ (see [32, Lem.4.7]). It follows that $\det(W_m(x;q))=x^{3m}\det(W_0(x;q))$ with initial condition a function of x given by Swarttouw [64]

$$\det(W_0(x;q))=x^3\qquad (|q|\neq 1). \tag{143}$$

We recall the details of the proof which will be useful in the case of the $\mathbf{5}_2$ knot. When $|q|<1$, the q-Hahn Bessel function $J(x,y;q)$ satisfies the recursion relation

$$yJ(qx,y;q)-(1+y-x)J(x,y;q)+J(q^{-1}x,y;q)=0. \tag{144}$$

This follows from [64], and can also be proved using the `HF` package. It then follows that

$$\mathcal{J}_{\nu,1}(z;q) := J(z,q^\nu;q), \quad \mathcal{J}_{\nu,2}(z;q) := z^{-\nu} J(q^{-\nu}z, q^{-\nu};q) \tag{145}$$

are two independent solutions to

$$q^\nu \mathcal{J}(qz;q) - (1+q^\nu - x)\mathcal{J}(z;q) + \mathcal{J}(q^{-1}z;q) = 0. \tag{146}$$

The corresponding Wronskian

$$\mathcal{W}_\nu(z;q) = \begin{pmatrix} \mathcal{J}_{\nu,1}(z;q) & \mathcal{J}_{\nu,2}(z;q) \\ \mathcal{J}_{\nu,1}(qz;q) & \mathcal{J}_{\nu,2}(qz;q) \end{pmatrix} \tag{147}$$

satisfies the recursion relation (see [32, Lem.4.7])

$$\det \mathcal{W}_\nu(z;q) = q^{-\nu} \det \mathcal{W}_\nu(q^{-1}z;q) \tag{148}$$

which implies that the determinant of $U(z;q) = z^\nu \mathcal{W}_\nu(z;q)$ is an elliptic function

$$\det U(qz;q) = \det U(z;q). \tag{149}$$

It can be computed by the following limit

$$\begin{aligned} \det U(z;q) &= \lim_{k\to\infty} \det U(q^k z;q) = \lim_{z\to 0} \det U(z;q) \\ &= \lim_{z\to 0} \left(q^{-\nu} J(z,q^\nu;q) J(q^{1-\nu}z, q^{-\nu};q) - J(q^{-\nu}z, q^{-\nu};q) J(qz,q^\nu;q) \right) \\ &= (q^{-\nu}-1)(q^{1+\nu};q)_\infty (q^{1-\nu};q)_\infty, \end{aligned} \tag{150}$$

where in the last step we just used the q-expansion definition of the q-Hahn Bessel function. We thus have

$$z^\nu \det \mathcal{W}_\nu(z;q) = -(qq^\nu;q)_\infty (q^{-\nu};q)_\infty. \tag{151}$$

Using the substitution

$$z \mapsto x^2, \quad z^\nu \mapsto x \tag{152}$$

in the above equation and cancelling with the θ-prefactors of $A_m(x;q)$ and $B_m(x;q)$, we obtain Equation (143) for $|q| < 1$. The case of $|q| > 1$ can be obtained by analytic continuation on both sides of (143).

For part (d), Equation (135) implies that

$$W_{m+1}(x;q) = \begin{pmatrix} 0 & 1 \\ -x^3 & x^2 + x - q^{1+m}x^3 \end{pmatrix} W_m(x;q). \tag{153}$$

Hence, Equation (138) follows from (137). The latter is a direct consequence of the analytic continuation formula

$$J(x,y;q) = \theta(-q^{1/2}y;q) J(y^{-1}x, y^{-1};q^{-1}) \tag{154}$$

which one easily sees by comparing (127a) and (127b).

Part (e) follows from the HF package. The fact that they are a basis follows from (f).

For part (f), since $A_m(x;q)$ (as well as $B_m(x;q)$) are annihilated by the first generator of (134), it follows that

$$q^m x^2 A_m(x;q) - q^m(1-q^{1+m}x^2)A_{m+1}(x;q) + x^3 A_m(qx;q) = 0. \tag{155}$$

After solving for the above for $A_m(qx;q)$ (and same for $B_m(qx;q)$) and substituting into the Wronskian (141), it follows that the two Wronskians are related by

$$\mathcal{W}_m(x;q) = \begin{pmatrix} 1 & 0 \\ -q^m x^{-1} & q^m x^{-3}(1-q^{1+m}x^2) \end{pmatrix} W_m(x;q). \tag{156}$$

After taking determinants, it follows that

$$\det(\mathcal{W}_m(x;q)) = q^m x^{-3}(1-q^{1+m}x^2)\det(W_m(x;q)). \tag{157}$$

This, together with (136) concludes the proof of (142). □

We now come to Conjecture 5 concerning a refinement of the $\widehat{A}$-polynomial. Combining Theorems 14 and 16, we obtain explicit linear q-difference equations for the descendant integrals with respect to the u and the m variables. To simplify our presentation keeping an eye on Equation (129), let us define a normalized version of the descendant state-integral by

$$z_{\mathbf{4}_1,m,\mu}(u;\tau) = (-1)^{m+\mu} q^{-m/2} \tilde{q}^{-\mu/2} Z_{\mathbf{4}_1,m,\mu}(u;\tau). \tag{158}$$

Theorem 17 *$z_{\mathbf{4}_1,m,k}(u_{\mathsf{b}};\tau)$ is a q-holonomic function of (m,u) with annihilator ideal $\mathcal{I}_{\mathbf{4}_1}$ given in (134). As a function of u (resp., m) it is annihilated by the operators $\widehat{A}_{\mathbf{4}_1}(S_x,x,q^m,q)$ and $\widehat{B}_{\mathbf{4}_1}(S_m,x,q^m,q)$ (given, respectively, by (139) and (135)), whose classical limit is*

$$\begin{aligned} &\widehat{A}_{\mathbf{4}_1}(S_x,x,q^m,1) \\ &= (-1+q^m x^2)(x^2 S_x^2 - q^m(1-x-2q^m x^2 - q^m x^3 + q^{2m}x^4)S_x + q^{3m}x^2) \end{aligned} \tag{159}$$

and

$$\widehat{B}_{\mathbf{4}_1}(S_m,x,q^m,1) = (1-x^{-1}S_m)(1-x^{-2}S_m) + q^m S_m. \tag{160}$$

$\widehat{A}_{\mathbf{4}_1}(S_x,x,1,1)$ is the A-polynomial of the knot, $\widehat{A}_{\mathbf{4}_1}(S_x,x,1,q)$ is the (homogeneous part) of the $\widehat{A}$-polynomial of the knot and $\widehat{B}_{\mathbf{4}_1}(x^2y,x,1,1)$ is the defining equation of the curve (116).

Note that although the two equations (135) and (139) look quite different, they come from the common annihilating ideal (134) of rank 2. This explains their common order, assuming that the ideal is generic. The annihilating ideal is easier to describe than the S_m-free element (139) of it. In fact, the first generator of $\mathcal{I}_{\mathbf{4}_1}$ expresses S_x as a polynomial in S_m, and eliminating S_m, one obtains equation (139) from (135). The characteristic variety of $\mathcal{I}_{\mathbf{4}_1}$ is a complex is a two-dimensional complex surface in $(\mathbb{C}^*)^4$ and its intersection with a complex 3-torus contains two special curves, namely the A-polynomial and the B-polynomial of the $\mathbf{4}_1$ knot.

5.3 Taylor series expansion at $u = 0$

The descendant state-integral is a meromorphic function of u which is analytic at $u = 0$ and factorizes in terms of descendant holomorphic blocks (129). In this section, we compute the Taylor series of the holomorphic blocks and of the state-integral at $u = 0$ and show how the factorization of the descendant state-integral (129) reproduces (69).

We begin with some general comments valid for descendant holomorphic blocks and state-integrals. Since the descendant holomorphic blocks are products of theta functions times q-hypergeometric sums, we need to compute the Taylor expansion of each piece. For Taylor expansion of the q-hypergeometric sums, we use

$$\phi_n(u) := \frac{(q^{1+n}\mathrm{e}^u;q)_\infty}{(q^{1+n};q)_\infty} = \exp\left(-\sum_{l=1}^{\infty}\frac{1}{l!}E_l^{(n)}(q)u^l\right) \tag{161a}$$

$$\widetilde{\phi}_n(u) := \frac{(\tilde{q};\tilde{q})_\infty}{(\tilde{q}\mathrm{e}^u;\tilde{q})_\infty}\frac{(\tilde{q}^{-1};\tilde{q}^{-1})_n}{(\tilde{q}^{-1}\mathrm{e}^u;\tilde{q}^{-1})_n} = \exp\left(\sum_{l=1}^{\infty}\frac{1}{l!}\widetilde{E}_l^{(n)}(\tilde{q})u^l\right) \tag{161b}$$

from [33, Prop.2.2], where

$$E_l^{(n)}(q) = \sum_{s=1}^{\infty}\frac{s^{l-1}q^{s(n+1)}}{1-q^s} \tag{162}$$

and

$$\widetilde{E}_l^{(n)}(\tilde{q}) = \begin{cases} -n + E_l^{(n)}(\tilde{q}) & l = 1 \\ E_l^{(n)}(\tilde{q}) & l > 1 \text{ odd} \\ 2E_l^{(0)}(\tilde{q}) - E_l^{(n)}(\tilde{q}) & l > 1 \text{ even}. \end{cases} \tag{163}$$

For the Taylor series of the theta functions, we use the well-known identity that expresses them in terms of quasi-modular forms (see, e.g., [13, Sec.8, Eqn(76)]), or alternatively observe that the theta functions that appear in the bilinear expressions of the holomorphic blocks are exponentials of quadratic and linear forms in u; see, for instance, (111a)–(111b). Yet alternatively (and this is the method that we will use below), when r and s are nonzero integers with r odd and positive, we can use the identity

$$\theta(-q^{\frac{r}{2}}x^s;q) = (-1)^{\frac{r+1}{2}}q^{\frac{r^2-1}{8}}x^{-\frac{s(r+1)}{2}}(1-x^s)\phi_0(su)\phi_0(-su) \tag{164}$$

, whereas when r is odd and negative, we can use the q-difference equation (109) to bring it to the case of r odd and positive.

One last comment is that the descendant holomorphic blocks are in general meromorphic functions of u. However, their bilinear combination that appears in the descendant state-integrals is regular at $u = 0$.

We now give the details of the Taylor series expansion of $Z_{\mathbf{4}_1,m,\mu}(u;\tau)$ and of the descendant holomorphic blocks for the $\mathbf{4}_1$ knot. Using the definition of $J(x,y;q)$ and (161a)–(161b), we find

$$\begin{aligned} A_m(\mathrm{e}^u;q) &= (1-\mathrm{e}^{-u})^{-2}(q;q)_\infty^{-3}\phi_0(u)^{-2}\phi_0(-u)^{-2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{q^{nm}}{(q;q)_n(q^{-1};q^{-1})_n}\phi_n(u)\mathrm{e}^{(2n+2m)u} \\ &= u^{-2}(q;q)_\infty^{-3}\left(\alpha_0^{(m)}(q)+u\alpha_1^{(m)}(q)+O(u^2)\right) \end{aligned} \tag{165}$$

where

$$\alpha_0^{(m)}(q) = G_m^0(q), \tag{166a}$$

$$\alpha_1^{(m)}(q) = \sum_{n=0}^{\infty} \frac{q^{nm}}{(q;q)_n(q^{-1};q^{-1})_n}(1+2n+2m-E_1^{(n)}(q)), \tag{166b}$$

and

$$\begin{aligned} B_m(\mathrm{e}^u;q) &= (1-\mathrm{e}^u)(q;q)_\infty^3\phi_0(u)\phi_0(-u)\sum_{n=0}^{\infty}\frac{q^{nm}}{(q;q)_n(q^{-1};q^{-1})_n}\phi_n(-u)\mathrm{e}^{(n+m)u} \\ &= -u(q;q)_\infty^3\left(\beta_0^{(m)}(q)+u\beta_1^{(m)}(q)+O(u^2)\right) \end{aligned} \tag{167}$$

where

$$\beta_0^{(m)}(q) = G_m^0(q), \tag{168a}$$

$$\beta_1^{(m)}(q) = \sum_{n=0}^{\infty}\frac{q^{nm}}{(q;q)_n(q^{-1};q^{-1})_n}\left(\frac{1}{2}+n+m+E_1^{(n)}(q)\right). \tag{168b}$$

We notice that

$$\beta_1^{(m)}(q)-\alpha_1^{(m)}(q) = -\frac{1}{2}G_m^1(q). \tag{169}$$

Similarly, using the definition of $J(x,y;q^{-1})$, we find

$$\begin{aligned} A_m(\mathrm{e}^u;q^{-1}) &= (1-\mathrm{e}^u)(q;q)_\infty^3\phi_0(u)^2\phi_0(-u)^2\sum_{n=0}^{\infty}\frac{q^{-nm}}{(q;q)_n(q^{-1};q^{-1})_n}\widetilde{\phi}_n(u)\mathrm{e}^{(2n+2m)u} \\ &= -u(q;q)_\infty^3\left(\widetilde{\alpha}_0^{(m)}(q)+u\widetilde{\alpha}_1^{(m)}(q)+O(u^2)\right) \end{aligned} \tag{170}$$

where

$$\widetilde{\alpha}_0^{(m)}(q) = G_{-m}^0(q), \tag{171a}$$

$$\widetilde{\alpha}_1^{(m)}(q) = \sum_{n=0}^{\infty}\frac{q^{-nm}}{(q;q)_n(q^{-1};q^{-1})_n}\left(\frac{1}{2}+n+2m+E_1^{(n)}(q)\right), \tag{171b}$$

and

$$B_m(\mathrm{e}^u;q^{-1}) = \frac{\phi_0(u)^{-1}\phi_0(-u)^{-1}}{(1-\mathrm{e}^{-u})^2(q;q)_\infty^3}\sum_{n=0}^{\infty}\frac{q^{-nm}}{(q;q)_n(q^{-1};q^{-1})_n}\widetilde{\phi}_n(-u)\mathrm{e}^{(n+m)u}$$

$$= u^{-2}(q;q)_\infty^{-3}\left(\widetilde{\beta}_0^{(m)}(q) + u\widetilde{\beta}_1^{(m)}(q) + O(u^2)\right) \tag{172}$$

where

$$\widetilde{\beta}_0^{(m)}(q) = G^0_{-m}(q), \tag{173a}$$

$$\widetilde{\beta}_1^{(m)}(q) = \sum_{n=0}^{\infty} \frac{q^{-nm}}{(q;q)_n(q^{-1};q^{-1})_n}(1+2n+m-E_1^{(n)}(q)). \tag{173b}$$

We also notice

$$\widetilde{\beta}_1^{(m)}(q) - \widetilde{\alpha}_1^{(m)}(q) = \frac{1}{2}G^1_{-m}(q). \tag{174}$$

Applying these results to the right-hand side of (129), we find the $O(1/u)$ contributions from (170) and (172) cancel, and the $O(u^0)$ contributions reproduce exactly (69). Notice that the $\sqrt{\tau}$ terms that appear in (69) come from expanding in terms of $2\pi\mathsf{b}\,u$ and $2\pi\mathsf{b}^{-1}\,u$ in (129).

As an application of the above computations, we obtain proof of a simplified formula for the q-series $G^1(q)$ from (43b) which was found experimentally in [44].

Proposition 18 *For $|q|<1$, we have:*

$$G^1(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}(6n+1)\,. \tag{175}$$

Proof We first show that the definition (43b) can equally be written as

$$G^1(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}\left(1+2n-4E_1^{(n)}(q)\right). \tag{176}$$

By definition

$$E_1^{(n)}(q) = \sum_{s=1}^{\infty}\frac{q^{s(n+1)}}{1-q^s}, \tag{177}$$

they satisfy the recursion relation

$$E_1^{(n)}(q) - E_1^{(n-1)}(q) = -\frac{q^n}{1-q^n}, \tag{178}$$

and therefore,

$$E_1^{(n)}(q) = E_1^{(0)}(q) - \sum_{j=1}^{n}\frac{q^j}{1-q^j}. \tag{179}$$

Using the identification $E_1(q) = 1 - 4E_1^{(0)}(q)$, one can then easily show that (176) is the same as (43b).

(175) follows from (176) thanks to the non-trivial identity

$$\sum_{n=0}^{\infty}\frac{n+E_1^{(n)}(q)}{(q;q)_n(q^{-1};q^{-1})_n} = 0, \quad |q|<1, \tag{180}$$

which now we prove. The crucial fact we use is that when $|q| < 1$, $J(x, y; q)$ is symmetric between x and y (see a proof in [7]). Let us consider the following expansion in small u

$$\begin{aligned} J(\mathrm{e}^{-u}, \mathrm{e}^{u}; q) =& \sum_{n=0}^{\infty} (q^{1+n}\mathrm{e}^{u}; q)_\infty \frac{\mathrm{e}^{-nu}}{(q^{-1}; q^{-1})_n} = (q;q)_\infty \sum_{n=0}^{\infty} \frac{\phi_n(u)\mathrm{e}^{-nu}}{(q;q)_n (q^{-1};q^{-1})_n} \\ =& (q;q)_\infty \sum_{n=0}^{\infty} \frac{1 - (n + E_1^{(n)}(q))u}{(q;q)_n (q^{-1};q^{-1})_n} + O(u^2). \end{aligned} \tag{181}$$

Since $J(\mathrm{e}^{-u}, \mathrm{e}^{u}; q) = J(\mathrm{e}^{u}, \mathrm{e}^{-u}; q)$, the coefficient of u (and in fact, of any odd power of u) in the expansion above vanishes, which leads to (180). □

As a second application, we demonstrate that Theorem 9, especially the identities (52), (55), as well as the recursion relation (49), can be proved by taking the $u = 0$ limit of the analogue identities in Theorem 16.

Using the expansion formulas of holomorphic blocks (165), (167), (170), (172), the Wronskians can be expanded as

$$\begin{aligned} W_m(\mathrm{e}^{u}; q) =& \begin{pmatrix} G^0_m(q) + u\alpha_1^{(m)}(q) & G^0_m(q) + u\beta_1^{(m)}(q) \\ G^0_{m+1}(q) + u\alpha_1^{(m+1)}(q) & G^0_{m+1}(q) + u\beta_1^{(m+1)}(q) \end{pmatrix} \\ & \times \begin{pmatrix} u^{-2}(q;q)_\infty^{-3} & 0 \\ 0 & -u(q;q)_\infty^{3} \end{pmatrix}, \end{aligned} \tag{182}$$

$$\begin{aligned} W_m(\mathrm{e}^{u}; q^{-1}) =& \begin{pmatrix} G^0_{-m}(q) + u\widetilde{\alpha}_1^{(m)}(q) & G^0_{-m}(q) + u\widetilde{\beta}_1^{(m)}(q) \\ G^0_{-m-1}(q) + u\widetilde{\alpha}_1^{(m+1)}(q) & G^0_{-m-1}(q) + u\widetilde{\beta}_1^{(m+1)}(q) \end{pmatrix} \\ & \times \begin{pmatrix} -u(q;q)_\infty^{3} & 0 \\ 0 & u^{-2}(q;q)_\infty^{-3} \end{pmatrix}. \end{aligned} \tag{183}$$

Taking the determinant of (182), we find

$$\det W_m(\mathrm{e}^{u}; q) = \frac{1}{2} \det W_m(q) + O(u) \tag{184}$$

which together with the u-expansion of the right-hand side of (136) leads to the determinant identity (52). Furthermore, by substituting (182), (183) into the Wronskian relation (137), the latter also reduces in the leading order to the determinant identity (52). On the other hand, the Wronskian relation (54) is equivalent to

$$W_m(q) = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(W_m(q)^{-1}\right)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{185}$$

which can be proved directly by expressing the inverse matrix on the right-hand side by minors and determinant, using the explicit value of the determinant given by the identity (52).

Finally, from the expression (166a),(168a) of the leading-order coefficients $\alpha_0^{(m)}(q)$, $\beta_0^{(m)}(q)$ of $A_m(\mathrm{e}^{u}; q)$, $B_m(\mathrm{e}^{u}; q)$ in the expansion of u, one concludes that the recursion relation (49) should be the $u = 0$ limit of the recursion relation (135) in m, and one can easily check it is indeed the case.

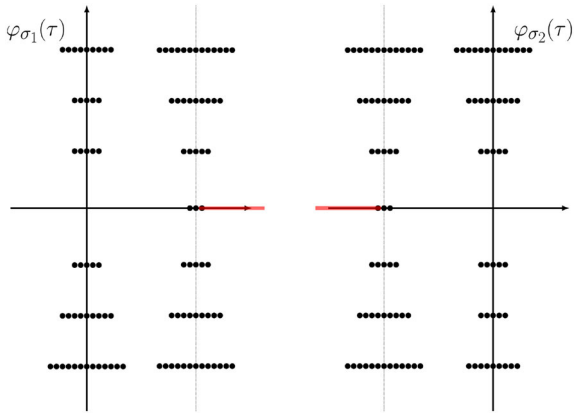

Fig. 6 The singularities in the Borel plane for the series $\varphi_{\sigma_j}(x;\tau)$ with $j = 1, 2$ of knot $\mathbf{4}_1$ where x is close to 1. The shortest vertical spacing between singularities is $2\pi\mathrm{i}$, and the horizontal spacing between neighboring singularities in each cluster is $\log x$

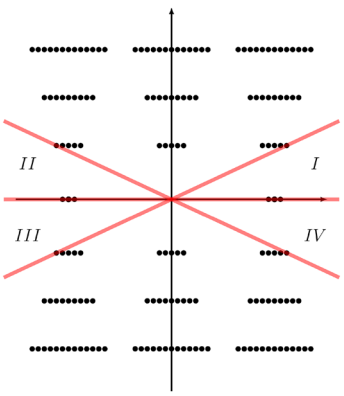

Fig. 7 Four different sectors in the τ-plane for $\Phi(u;\tau)$ of knot $\mathbf{4}_1$ with u close to zero

5.4 Stokes matrices near $u = 0$

In this section, we conjecture a formula for the Stokes matrices of the asymptotic series $\varphi(x;\tau)$. When we turn on the non-vanishing deformation parameter u, the resurgent structure discussed in Sect. 3.1 undergoes significant changes. Compared to Fig. 2, there are many more singular points in the Borel plane whose positions depend on u in addition to τ, and the Stokes matrices also become u-dependent. However, if we focus on the case when u is not far away from zero, equivalent to x not far away from 1, the resurgent data are holomorphic in u and reduce to those in Sect. 3.1 in the $u = 0$ limit. For instance, each singular point in Fig. 2 splits to a cluster of neighboring singular points separated with distance $\log x$ as shown in Fig. 6. In particular, each singular point $\iota_{i,j}$ on the real axis splits to a cluster of three, in accord with the off-diagonal entries ± 3 in (60), and if we choose real x, the split singularities still lie on the real axis. As in Sect. 3.1, we label the four regions separating singularities on real axis and all the others by *I, II, III, IV* (see Fig. 7). In each of the four regions, we have the following the results.

Conjecture 19 The asymptotic series and the holomorphic blocks are related by (15) with the diagonal matrix $\Delta(\tau)$ as in (132), where matrices $M_R(x;q)$ are given in terms of the matrices $W_{-1}(x;q)$ as follows

$$M_I(x;q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_{-1}(x;q)^T \begin{pmatrix} 0 & 1 \\ -x^{-1} & 1 \end{pmatrix}, \qquad |q|<1, \tag{186a}$$

$$M_{II}(x;q) = W_{-1}(x^{-1};q)^T \begin{pmatrix} -x^{-1} & 0 \\ -x^{-1} & 1 \end{pmatrix}, \qquad |q|<1, \tag{186b}$$

$$M_{III}(x;q) = W_{-1}(x^{-1};q)^T \begin{pmatrix} -x^{-1} & 0 \\ 1+x & 1 \end{pmatrix}, \qquad |q|>1, \tag{186c}$$

$$M_{IV}(x;q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_{-1}(x;q)^T \begin{pmatrix} 0 & 1 \\ -x^{-1} & -x^{-1}-x^{-2} \end{pmatrix}, \qquad |q|>1. \tag{186d}$$

The above conjecture completely determines the resurgent structure of $\Phi(\tau)$. Indeed, it implies that the Stokes matrices, defined in Equations (34) and (33), are explicitly given by:

$$\mathsf{S}^+(x;q) = \begin{pmatrix} 0 & -1 \\ x^{-1} & -1-x \end{pmatrix} \cdot W_{-1}(x^{-1};q^{-1}) \cdot W_{-1}(x;q)^T \cdot \begin{pmatrix} 0 & x \\ -1 & -1-x^{-1} \end{pmatrix}, \quad |q|<1, \tag{187a}$$

$$\mathsf{S}^-(x;q) = \begin{pmatrix} x & x \\ 0 & -1 \end{pmatrix} \cdot W_{-1}(x;q) \cdot W_{-1}(x^{-1};q^{-1})^T \cdot \begin{pmatrix} x^{-1} & 0 \\ x^{-1} & -1 \end{pmatrix}, \quad |q|<1. \tag{187b}$$

We remark that since $s(\Phi_\sigma)(x;\tau)$ for $\sigma = 1,2$ transform under the reflection $\pi : x \mapsto x^{-1}$ uniformly by (210) (see the comment below), the Stokes matrices should be invariant under π, and we have checked that (187a),(187b) indeed satisfy this consistency condition.

In the $q \mapsto 0$ limit,

$$\mathsf{S}^+(x;0) = \begin{pmatrix} 1 & x^{-1}+1+x \\ 0 & 1 \end{pmatrix}, \quad \mathsf{S}^-(x;0) = \begin{pmatrix} 1 & 0 \\ -x^{-1}-1-x & 1 \end{pmatrix} \tag{188}$$

A curious corollary of our computation is that the matrices of integers (60) from [30,45] which relates the asymptotics of the coefficients of $\varphi(\tau)$ to the coefficients themselves spread out to the matrices (188) with entries in $\mathbb{Z}[x^{\pm 1}]$.

Using the unique factorization Lemma 8 and the Stokes matrix S from above, we can compute the Stokes constants and the corresponding matrix $\mathcal{S}$ of Equation (36) to arbitrary order in q, and we find that

$$\begin{aligned} \mathcal{S}^+_{\sigma_1,\sigma_1}(x;q) =& \mathsf{S}^+(x;q)_{1,1} - 1 \\ =& (-2-x^{-2}-2x^{-1}-2x-x^2)q \\ &+ (-3-x^{-2}-2x^{-1}-2x-x^2)q^2 + O(q^3), \end{aligned} \tag{189}$$

$$\begin{aligned} \mathcal{S}^+_{\sigma_1,\sigma_2}(x;q) =& \mathsf{S}^+(x;q)_{1,2}/\mathsf{S}^+(x;q)_{1,1} - (\mathcal{S}^{(1,0)}_{\sigma_1,\sigma_2}x + \mathcal{S}^{(0,0)}_{\sigma_1,\sigma_2} + \mathcal{S}^{(-1,0)}_{\sigma_1,\sigma_2}x^{-1}) \\ =& (3+x^{-2}+2x^{-1}+2x+x^2)q \\ &+ (17+x^{-4}+4x^{-3}+9x^{-2} \\ &+ 15x^{-1}+15x+9x^2+4x^3+x^4)q^2 + O(q^3), \end{aligned} \tag{190}$$

$$\begin{aligned} \mathcal{S}^+_{\sigma_2,\sigma_1}(x;q) =& \mathsf{S}^+(x;q)_{2,1}/\mathsf{S}^+(x;q)_{1,1} \\ =& (-3-x^{-2}-2x^{-1}-2x-x^2)q \end{aligned}$$

$$
\begin{aligned}
&+(-17-x^{-4}-4x^{-3}-9x^{-2}-15x^{-1}\\
&\quad -15x-9x^2-4x^3-x^4)q^2+O(q^3),
\end{aligned}
\tag{191}
$$

$$
\begin{aligned}
\mathcal{S}^{+}_{\sigma_2,\sigma_2}(x;q) =& \mathsf{S}^{+}(x;q)_{2,2}-1-\mathsf{S}^{+}(x;q)_{1,2}\mathsf{S}^{+}(x;q)_{2,1}/\mathsf{S}^{+}(x;q)_{1,1}\\
=&(2+x^{-2}+2x^{-1}+2x+x^2)q\\
&+(17+x^{-4}+4x^{-3}+9x^{-2}+14x^{-1}\\
&\quad +14x+9x^2+4x^3+x^4)q^2+O(q^3).
\end{aligned}
\tag{192}
$$

They enjoy the symmetry

$$
\mathcal{S}^{+}_{\sigma_1,\sigma_2}(x;q)=-\mathcal{S}^{+}_{\sigma_2,\sigma_1}(x;q)\,,
\tag{193}
$$

and experimentally, it appears that the entries of the matrix $\mathcal{S}^{+}(x;q)=(\mathcal{S}^{+}_{\sigma_i,\sigma_j}(x;q))$ (except the upper-left one) are (up to a sign) in $\mathbb{N}[x^{\pm 1}][[q]]$. Similarly, we can extract the Stokes constants $\mathcal{S}^{(\ell,-k)}_{\sigma_i,\sigma_j}$ associated with the singularities in the lower half plane, and assemble into q^{-1}-series $\mathcal{S}^{-}_{\sigma_i,\sigma_j}(x;q^{-1})$. We find they are related to $\mathcal{S}^{+}_{\sigma_i,\sigma_j}(x;q)$ by

$$
\begin{aligned}
&\mathcal{S}^{-}_{\sigma_i,\sigma_j}(x;q)=-\mathcal{S}^{+}_{\sigma_j,\sigma_i}(x;q),\quad i\neq j\\
&\mathcal{S}^{-}_{\sigma_1,\sigma_1}(x;q)=\mathcal{S}^{+}_{\sigma_2,\sigma_2}(x;q),\quad \mathcal{S}^{-}_{\sigma_2,\sigma_2}(x;q)=\mathcal{S}^{+}_{\sigma_1,\sigma_1}(x;q).
\end{aligned}
\tag{194}
$$

Let us now verify Conjecture 4. From (187a), we find that indeed

$$
\mathsf{S}^{+}(x;q)\doteq W_{-1}(x^{-1};q^{-1})\cdot W_{-1}(x;q)^{T}.
\tag{195}
$$

Using the recursion relation (153) and the relation between two Wronskians (156), we further find

$$
\mathsf{S}^{+}(x;q)\doteq\doteq \mathcal{W}_0(x^{-1};q^{-1})\cdot \mathcal{W}_0(x;q)^{T}.
\tag{196}
$$

If we use the uniform notation for all holomorphic blocks

$$
(B^{\alpha}_{K}(x;q))_{\alpha=1,2}=(A_0(x;q),B_0(x;q)),
\tag{197}
$$

the right-hand side of (196) reads

$$
\mathcal{W}_0(x^{-1};q^{-1})\cdot \mathcal{W}_0(x;q)^{T}=\left(\sum_{\alpha}B^{\alpha}_{\mathbf{4}_1}(q^{j}x;q)B^{\alpha}_{\mathbf{4}_1}(q^{-i}x^{-1};q^{-1})\right)_{i,j=0,1}.
\tag{198}
$$

which is precisely the right-hand side of (20) in Conjecture 4 following (19).[1] In addition, the forms of the accompanying matrices on the left and on the right are such that the

[1] Note that because the form of the state-integral in [7] is slightly different from that in [3], which we adopt, our convention for holomorphic blocks is also different from [7]. As a result, the entries of (198) equate the DGG indices computed in [7] up to a prefactor

$$
\left(\mathcal{W}_0(x^{-1};q^{-1})\cdot \mathcal{W}_0(x;q)^{T}\right)_{i+1,j+1}=(-q^{1/2})^{j-i}\mathrm{Ind}^{\mathrm{rot}}_{\mathbf{4}_1}(j-i,q^{\frac{j+i}{2}}x;q),\quad i,j=0,1.
\tag{199}
$$

If we take this into account, Conjecture 4 should be modified slightly by stating the accompanying matrices on the left and on the right are in $GL(2,\mathbb{Z}(x,q^{1/2}))$.

(1, 1) entry of $\mathsf{S}^+(x;q)$ equates exactly the DGG index with no magnetic flux. By explicit calculation,

$$\begin{aligned}\mathsf{S}^+(x;q)_{1,1} &= J(x,x^{-1};q)J(x,x^{-1};q^{-1}) + J(x^{-2},x^{-1};q)J(x^2,x;q^{-1})\\ &= 1-(2x^{-2}+x^{-1}+2+x+2x^2)q\\ &\quad -(x^{-2}+2x^{-1}+3+2x+x^2)q^2+O(q^3),\end{aligned} \tag{200}$$

which is the $\mathrm{Ind}^{\mathrm{rot}}_{\mathbf{4}_1}(0,x;q)$ given in [7].

5.5 The Borel resummations of the asymptotic series Φ

In this section, we explain how Conjecture 19 identifies the Borel resummations of the factorially divergent series $\Phi(x;\tau)$ with the descendant state-integrals, thus lifting the Borel resummation to holomorphic functions on the cut-plane $\mathbb{C}'$. This is interesting theoretically, but also practically in the numerical computation of Borel resummations.

After multiplying the inverse of $M_R(\tilde{x},\tilde{q})$ from the left on both sides of (15), we can also express the Borel sums $s_R(\Phi)(x;\tau)$ in each region in terms of holomorphic functions of $\tau\in\mathbb{C}\backslash\mathbb{R}$ as follows

Corollary 20 (of Conjecture 19) *We have*

$$s_I(\Phi)(x;\tau) = \begin{pmatrix}-\tilde{x} & 1+\tilde{x}^{-1}\\ 0 & 1\end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1})\Delta(\tau)B(x;q), \tag{201a}$$

$$s_{II}(\Phi)(x;\tau) = \begin{pmatrix}0 & -\tilde{x}\\ 1 & -\tilde{x}-\tilde{x}^2\end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1}) \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Delta(\tau)B(x;q), \tag{201b}$$

$$s_{III}(\Phi)(x;\tau) = \begin{pmatrix}0 & -\tilde{x}\\ 1 & 1\end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1}) \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Delta(\tau)B(x;q), \tag{201c}$$

$$s_{IV}(\Phi)(x;\tau) = \begin{pmatrix}-\tilde{x} & -\tilde{x}\\ 0 & 1\end{pmatrix} W_{-1}(\tilde{x},\tilde{q}^{-1})\Delta(\tau)B(x;q). \tag{201d}$$

where the right-hand side of (201a)–(201d) *are holomorphic functions of* $\tau\in\mathbb{C}'$, *as they are linear combinations of the descendants* (129).

The asymptotic series of the $\mathbf{4}_1$ knot have the symmetry (122) due to the fact that it is an amphichiral knot. This gives a symmetry of the state-integral.

Proposition 21 *(Assuming Conjecture 19) We have:*

$$Z_{\mathbf{4}_1}(u;\tau) = e^{2\pi(\mathsf{b}+\mathsf{b}^{-1})u} Z_{\mathbf{4}_1}(-u;\tau). \tag{202}$$

Proof The second line of (201a) indicates that in region I

$$Z_{\mathbf{4}_1}(u;\tau) = s_I(\Phi_2)(x;\tau). \tag{203}$$

Recall the structure of $\Phi_2(x;\tau)$ from (120)

$$\Phi_2(x;\tau) = \exp\left(\frac{V(x,y_2(x))}{2\pi\mathrm{i}\tau}\right)\frac{1}{\sqrt{\mathrm{i}\delta(x,y_2(x))}}\sqrt{\mathrm{i}\delta}\varphi(x,y_2(x);\tau). \tag{204}$$

Here, $\sqrt{\mathrm{i}\delta}\varphi_2(x;\tau) := \sqrt{\mathrm{i}\delta}\varphi(x, y_2(x);\tau)$ is an asymptotic series in τ with $\sqrt{\mathrm{i}\delta}\varphi_2(x;0) = 1$. The coefficients of the series $\sqrt{\mathrm{i}\delta}\varphi_2(x;\tau)$ are invariant under the transformation $x \mapsto 1/x$ (see, for instance, (123); this is also true for $\sqrt{\mathrm{i}\delta}\varphi_1(x;\tau) = \sqrt{\mathrm{i}\delta}\varphi(x, y_1(x);\tau)$), and thus,

$$s(\sqrt{\mathrm{i}\delta}\varphi_2)(1/x;\tau) = s(\sqrt{\mathrm{i}\delta}\varphi_2)(x;\tau). \tag{205}$$

On the other hand, from definition (121) of $\delta_2(x) := \delta(x, y_2(x))$, it is clear that

$$\delta_2(1/x) = x^2\delta_2(x). \tag{206}$$

Finally, to study the behavior of $V_2(x) := V(x, y_2(x))$ under the transformation $x \mapsto 1/x$, it is convenient to do the change of variables $y = x^{-1} + \tilde{y}$, so that Equation (116) satisfied by y becomes

$$1 - (1 - x - x^{-1})\tilde{y} + \tilde{y}^2 = 0 \tag{207}$$

which is manifestly invariant under this transformation, and thus, $\tilde{y}(1/x) = \tilde{y}(x)$. Expressed in terms of this variable

$$\begin{aligned} V_2(x) = &- \mathrm{Li}_2(-x^{-1}\tilde{y}_2) - \mathrm{Li}_2(-x\tilde{y}_2) - \frac{\pi}{3} - \frac{1}{2}\log^2(x\tilde{y}_2^{-1}) \\ &+ \log(1 + x\tilde{y}_2^{-1})\log(-x\tilde{y}_2^{-1}) - \log(1 + x\tilde{y}_2)\log(-x\tilde{y}_2) \\ &- \frac{1}{2}\log^2(-1 - x\tilde{y}_2^{-1}) + \frac{1}{2}\log^2(-x^{-1} - \tilde{y}_2) + 2\log x \log(-x^{-1} - \tilde{y}_2), \end{aligned} \tag{208}$$

and it has the property that

$$V_2(x^{-1}) = V_2(x) - 2\pi\mathrm{i}x. \tag{209}$$

This can be proven by differentiating both sides with respect to x, and reducing it to an identity of rational functions on the curve S. Combining (205),(206),(209), we have

$$s_I(\Phi_2)(x^{-1};\tau) = x^{-1}\tilde{x}^{-1}s_I(\Phi_2)(x;\tau) \tag{210}$$

which implies (202). We comment in the passing that the identity (210) is true for both $s(\Phi_{1,2})(x;\tau)$ for any $\tau \in \mathbb{C}$ whenever the asymptotic series is Borel summable.

Once we have established the identity (202) in region I, it can be extended to $\tau \in \mathbb{C}'$ by the holomorphicity of (129). □

5.6 Stokes matrices for real *u*

In Sect. 5.4, we only considered the resurgent structure for x near 1, or equivalently, $u = \log x$ near 0. When x is arbitrary, the resurgent structure of the vector $\Phi_\sigma(x;\tau)$ could be very different. According to the Picard–Lefschetz theory (for review, see, for instance, [70]), when a set of asymptotic series originates from a (path) integral, the Borel sum of each asymptotic series is the evaluation of the integral along a Lefschetz thimble anchored to a critical point. In the x-plane, there are walls of marginal stability which start from the roots to the discriminant (119) and which end at infinity. When we cross

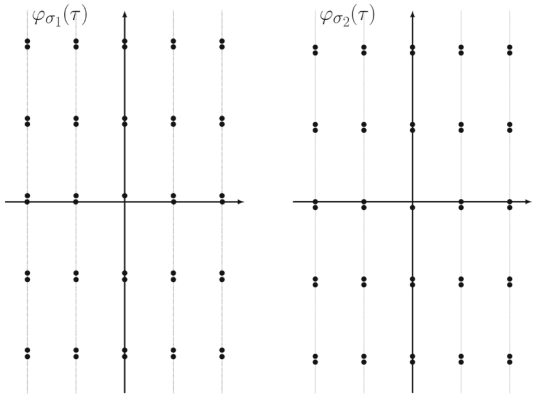

Fig. 8 The singularities in the Borel plane for the series $\varphi_{\sigma_j}(u;\tau)$ with $j=1,2$ of knot $\mathbf{4}_1$ for $x=\mathrm{e}^{2\pi \mathbf{b} u}$ in the first or the third interval of the positive axis

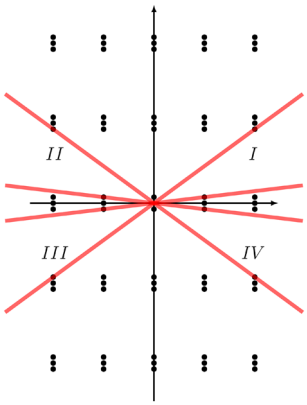

Fig. 9 Four different sectors in the τ-plane for $\Phi(u;\tau)$ of knot $\mathbf{4}_1$ with $x=\mathrm{e}^{2\pi \mathbf{b} u}$ in the first or the third interval of the positive axis

such a Stokes line, Lefschetz thimbles jump leading to linear transformations of the Borel summed asymptotic series. In this section we extend slightly the discussion of Sect. 5.4 by considering the resurgent structure of $\Phi_\sigma(x;\tau)$ for generic positive x (see [7] for a similar discussion in complex Chern–Simons theory). The positive real axis is divided by the two real solutions to (119)

$$x_\pm = \frac{1}{2}(3 \pm \sqrt{5}) \tag{211}$$

to three intervals

$$(0, x_-),\ (x_-, x_+),\ (x_+, \infty). \tag{212}$$

The middle interval is covered by Sect. 5.4, while the first (labeled by $<$) and third (labeled by $>$) intervals are discussed below.

First of all, we notice that the first and third intervals are related by the reflection $x \mapsto x^{-1}$. In fact, due to the property (205) of the asymptotic series, the Borel plane singularities for $\Phi_\sigma(x;\tau)$ and $\Phi_\sigma(x^{-1};\tau)$ are identical, and we illustrate them uniformly in

Fig. 8. The positions of singularities are still described by (14). However, the difference of action $(V(\sigma_1)-V(\sigma_2))/(2\pi\mathrm{i})$ is now imaginary and it describes the vertical spacing between neighboring singularities. The shortest horizontal spacing is $\log x$. Finally, all singularities are repeated vertically by the spacing $2\pi\mathrm{i}$. Similar to the discussion in Sect. 5.4, we label in the τ-plane by *I, II, III, IV* the four sectors which separate the 12 singularities close to the real axis and other singularities along the imaginary axis or away from the real axis, as in Fig. 9. In each of the four sectors, the Borel summed vector $s_R^{\gtrless}(\Phi)(x;\tau)$ is a linear transformation of the Borel summed vector $s_R(\Phi)(x;\tau)$ in the middle interval, as per the Picard–Lefschetz theory

$$s_R^{\gtrless}(\Phi)(x;\tau) = T_R^{\gtrless}(\tilde{x}) \cdot s_R(\Phi)(x;\tau). \tag{213}$$

It is most convenient to compute the transformation matrix $T_R^{\gtrless}$ by comparing the left-hand side with the holomorphic lifts of $s_R(\Phi)(x;\tau)$ summarized in Corollary 20. By doing so, we find in the first interval

$$T_I^{<}(\tilde{x}) = \begin{pmatrix} \tilde{x} & 0 \\ -\tilde{x} & 1 \end{pmatrix}, \qquad T_{II}^{<}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ -\tilde{x} & 1 \end{pmatrix} \tag{214}$$

$$T_{III}^{<}(\tilde{x}) = \begin{pmatrix} \tilde{x} & 1 \\ -\tilde{x} & 0 \end{pmatrix}, \qquad T_{IV}^{<}(\tilde{x}) = \begin{pmatrix} \tilde{x} & 1 \\ 0 & 1 \end{pmatrix}, \tag{215}$$

while in the third interval

$$T_I^{>}(\tilde{x}) = \begin{pmatrix} \tilde{x}^{-1} & 0 \\ -\tilde{x}^{-1} & 1 \end{pmatrix}, \qquad T_{II}^{>}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ -\tilde{x}^{-1} & 1 \end{pmatrix} \tag{216}$$

$$T_{III}^{>}(\tilde{x}) = \begin{pmatrix} \tilde{x}^{-1} & 1 \\ -\tilde{x}^{-1} & 0 \end{pmatrix}, \qquad T_{IV}^{>}(\tilde{x}) = \begin{pmatrix} \tilde{x}^{-1} & 1 \\ 0 & 1 \end{pmatrix}. \tag{217}$$

They are indeed related by

$$T_R^{<}(\tilde{x}) = T_R^{>}(\tilde{x}^{-1}). \tag{218}$$

Once the linear combinations are known, the Stokes matrices can be computed using the Stokes matrices in the middle interval given in Sect. 5.4

$$\mathfrak{S}_{R\to R'}^{\gtrless}(x;q) = T_{R'}^{\gtrless}(x) \cdot \mathfrak{S}_{R\to R'}(x;q) \cdot \left(T_R^{\gtrless}(x)\right)^{-1}. \tag{219}$$

We find

$$\mathfrak{S}_{I\to II}^{>}(x;q) = \begin{pmatrix} 0 & 1 \\ -x^{-1} & 1 \end{pmatrix} \cdot \mathfrak{S}_{I\to II}(x;q) \cdot \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix} \tag{220}$$

$$\mathfrak{S}_{II\to III}^{>}(x) = \begin{pmatrix} 1-x-x^2 & x+x^2 \\ -x-x^2 & 1+x+x^2 \end{pmatrix} \tag{221}$$

$$\mathfrak{S}_{III\to IV}^{>}(x;q) = \begin{pmatrix} x^{-1} & 1 \\ -x^{-1} & 0 \end{pmatrix} \cdot \mathfrak{S}_{III\to IV}(x;q) \cdot \begin{pmatrix} x & -x \\ 0 & 1 \end{pmatrix} \tag{222}$$

$$\mathfrak{S}^{>}_{IV\to I}(x) = \begin{pmatrix} 1+x^{-1}+x^{-2} & x^{-1}+x^{-2} \\ -x^{-1}-x^{-2} & 1-x^{-1}-x^{-2} \end{pmatrix}, \tag{223}$$

and thanks to (218),

$$\mathfrak{S}^{<}_{R\to R'}(x;q) = \mathfrak{S}^{>}_{R\to R'}(x^{-1};q). \tag{224}$$

Note that $\mathfrak{S}^{>}_{III\to I}(x)$ and $\mathfrak{S}^{>}_{II\to IV}(x)$ encode all the 12 singularities near the real axis illustrated in Fig. 9, as well as the Stokes constants associated with them. Furthermore, the Stokes matrices also have the property

$$\mathfrak{S}^{>}_{IV\to III}(x^{-1};q)^{-1} = \mathfrak{S}^{>}_{I\to II}(x;q)^{T}, \tag{225}$$

in accord with Conjecture 5.

5.7 Numerical verification

In this section, we explain the numerical verification of Conjecture 19. This involves, on the one hand, a numerical computation of the asymptotics of the holomorphic blocks and on the other hand, a numerical computation of the Borel resummation by the Laplace integral of a Padé approximation. Taking the two computations into account, we found out numerically, integers appearing at two exponentially small scales, namely $\tilde{q}$ and $\tilde{x}$, and guessing these integers eventually led to Conjecture 19.

We found ample numerical evidence for the resurgent data (186a)–(186d). First of all, due to the symmetry $\Phi_1(x;-\tau) = -\mathrm{i}\Phi_2(x;\tau)$, $\Phi_2(x;-\tau) = \mathrm{i}\Phi_1(x;\tau)$, the resurgent behavior of $s(\Phi)(x;\tau)$ for τ in the lower half plane can be deduced from that for τ in the upper half plane. We only have to numerically test the resurgent data for τ in regions *I, II* in the upper half plane.

The first piece of evidence comes from analyzing the radial asymptotics of the left-hand side of (15). Note that the matrix $\Delta(\tau) = \mathrm{diag}(\Delta_\sigma(\tau))$ is always diagonal, and each row of (15) is

$$\Delta_\sigma(\tau)B^\sigma(x;q) = \sum_{\sigma'} M_R(\tilde{x},\tilde{q})_{\sigma,\sigma'} s_R(\Phi_{\sigma'})(x;\tau). \tag{226}$$

If we take $\tau = \mathrm{e}^{\mathrm{i}\alpha}/k$ with the argument α depending on a ray in the region R and k a very large integer, the difference between $\exp\left(\frac{V(\sigma')}{2\pi\mathrm{i}\tau}\right)$ associated with different critical points is greatly magnified, and the right-hand side of (226) is dominated by a single series. Furthermore, when τ is in the upper (lower) half plane, $\tilde{q}$ ($1/\tilde{q}$) is exponentially suppressed and the correction $M_R(\tilde{x},\tilde{q})_{\sigma,\sigma'}$ as a series in $\tilde{q}$ ($1/\tilde{q}$) is dominated by the leading term. (226) thus becomes

$$\Delta_\sigma(\tau)B^\sigma(x;q) \sim \exp\left(\frac{V(\hat{\sigma})+\omega_{\hat{\sigma}}}{2\pi\,\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}}k - \log(\mathrm{i}\delta(x,y_{\hat{\sigma}})) + \sum_{n=1}^{\infty} S_n(x,y_{\hat{\sigma}})\mathrm{e}^{n\mathrm{i}\alpha}k^{-n}\right), \quad k\gg 1, \tag{227}$$

where $\omega_{\hat{\sigma}}$ is possible contribution from the leading term of $M_R(\tilde{x},\tilde{q})_{\sigma,\hat{s}}$, and the series in k^{-1} is $\log\varphi(x,y_{\hat{\sigma}};\tau)$. As pointed out in [44], this equation can be tested numerically with the help of Richardson transformations (see, for instance, [10]).

Table 1 Numerical tests of holomorphic lifts of Borel sums of asymptotic series for knot $\mathbf{4}_1$

(A) Region *I*: $\tau = \frac{1}{20}e^{\frac{\pi i}{5}}$				
	$\lvert\frac{s_I(\Phi)(x;\tau)}{P_I(x;\tau)} - 1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\lvert\tilde{x}(x,\tau)\rvert$
σ_1	3.2×10^{-66}	9.7×10^{-66}	8.3×10^{-33}	0.05
σ_2	1.9×10^{-94}	5.2×10^{-94}		
(B) Region *II*: $\tau = \frac{1}{20}e^{\frac{4\pi i}{5}}$				
	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{P_{II}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\lvert\tilde{x}(x,\tau)^{-1}\rvert$
σ_1	1.9×10^{-94}	5.2×10^{-94}	8.3×10^{-33}	0.05
σ_2	3.2×10^{-66}	9.7×10^{-66}		
(C) Region *III*: $\tau = \frac{1}{20}e^{-\frac{4\pi i}{5}}$				
	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{P_{IV}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\lvert\tilde{x}(x,\tau)\rvert$
σ_1	1.9×10^{-94}	5.2×10^{-94}	8.3×10^{-33}	0.05
σ_2	3.2×10^{-66}	9.7×10^{-66}		
(D) Region *IV*: $\tau = \frac{1}{20}e^{-\frac{\pi i}{5}}$				
	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{P_{III}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\lvert\tilde{x}(x,\tau)^{-1}\rvert$
σ_1	3.2×10^{-66}	9.7×10^{-66}	8.3×10^{-33}	0.05
σ_2	1.9×10^{-94}	5.2×10^{-94}		

We perform the Borel–Padé resummation on $\Phi(x;\tau)$ with 280 terms at $x = 6/5$ and τ in four different regions, and compute the relative difference between them and the right-hand side of (201a)–(201d), which we denote by $P_R(x;\tau)$. They are within the error margins of Borel–Padé resummation, which are estimated by redo the resummation with 276 terms, denoted by $s'_R(\bullet)$ in the tables. The relative errors are much smaller than $|\tilde{q}^{\pm 1}|, |\tilde{x}^{\pm 1}|$, possible sources of additional corrections

Next, we can test (226) directly. One way of doing this is to compute Borel–Padé resummation $s_R(\Phi_{\sigma'})(x;\tau)$ for various values of $x \in \mathbb{R}$ and τ in the same region R, and by comparing with the left-hand side extract terms of $M_R(\tilde{x},\tilde{q})_{\sigma,\sigma'}$ order by order. To facilitate this operation, instead of $M_R(\tilde{x};\tilde{q})$ we consider

$$\widetilde{M}_R(\tilde{x};\tilde{q}) = \begin{pmatrix} \theta(-\tilde{q}^{-1/2}\tilde{x};\tilde{q})^2 & 0 \\ 0 & \theta(-\tilde{q}^{1/2}\tilde{x};\tilde{q})^{-1} \end{pmatrix} M_R(\tilde{x};\tilde{q}) \tag{228}$$

whose entries are $\tilde{q}$-series with coefficients in $\mathbb{Z}[\tilde{x}^{\pm 1}]$ instead of in $\mathbb{Z}(\tilde{x})$. Using 280 terms of $\Phi_\sigma(x;\tau)$, we find the results for τ in the upper half plane

$$\widetilde{M}_I(x;q) = \begin{pmatrix} -x + (x^2+x^3)q + (x^2+x^3)q^2 & 1 - (x+x^2+x^3)q - (x+x^2)q^2 \\ -1 + (x^{-1}+x)q + (x^{-1}+x)q^2 & 1 - (x^{-1}+1+x)q - (x^{-1}+x)q^2 \end{pmatrix} + O(q^3) \tag{229a}$$

$$\widetilde{M}_{II}(x;q) = \begin{pmatrix} -x + (1+x^{-1}+x^{-2})q + (1+x^{-1})q^2 & 1 - (x^{-1}+x^{-2})q - (x^{-1}+x^{-2})q^2 \\ -1 + (x+1+x^{-1})q + (x+x^{-1})q^2 & 1 - (x+x^{-1})q - (x+x^{-1})q^2 \end{pmatrix} + O(q^3) \tag{229b}$$

and they agree with (186a), (186b). Another more decisive way is to compute both sides of (226) numerically assuming (186a), (186b) and compare them. Alternatively, we can compare two sides of the equations of holomorphic lift (201a), (201b). We find that the difference between the two sides is always within the error margin of Borel–Padé resummation, and much smaller than $\tilde{q}^{\pm 1}, \tilde{x}^{\pm 1}$, i.e., possible additional corrections. We illustrate this comparison by one example with $x = 6/5$ and $\tau = \frac{1}{20}e^{\pm\frac{\pi i}{5}}, \frac{1}{20}e^{\pm\frac{4\pi i}{5}}$ in four regions in Table 1.

A final way to test these results is to see that in the $x \mapsto 1$ limit, the resurgent data as well as the Stokes matrices (187a)–(187b) are compatible with the results in Sect. 3.1 where $x = 1$. This is a non-trivial test since the matrix $W_{-1}(x^{-1}; q^{-1})$ ($|q| < 1$) in (187a)–(187b) is divergent in the limit $x \mapsto 1$.

6 The 5_2 knot

6.1 Asymptotic series

Our second example that we discuss in detail will be the case of the $\mathbf{5}_2$ knot. The state-integral for the $\mathbf{5}_2$ knot [3, Eqn.(39)] (after removing a prefactor that depends on u alone)

$$Z_{\mathbf{5}_2}(u;\tau) = \int_{\mathbb{R}+\mathrm{i}0} \Phi_{\mathsf{b}}(v)\,\Phi_{\mathsf{b}}(v+u)\,\Phi_{\mathsf{b}}(v-u)\,\mathrm{e}^{-2\pi\mathrm{i}v^2}\,\mathrm{d}v\,. \tag{230}$$

After a change of variables $u \mapsto u/(2\pi\mathsf{b})$ (see Equation (2)) and $v \mapsto v/(2\pi\mathsf{b})$, it follows that the integrand of $Z_{\mathbf{5}_2}(u_{\mathsf{b}};\tau)$ has a leading term given by $\mathrm{e}^{V(u,v)/(2\pi\mathrm{i}\tau)}$ where

$$V(u,v) = \mathrm{Li}_2(-\mathrm{e}^{v}) + \mathrm{Li}_2(-\mathrm{e}^{u+v}) + \mathrm{Li}_2(-\mathrm{e}^{-u+v}) + (v)^2\,. \tag{231}$$

Taking derivative with respect to v gives the equation for the critical point

$$2v - \log(1+\mathrm{e}^{v}) - \log(1+\mathrm{e}^{u+v}) - \log(1+\mathrm{e}^{-u+v}) = 0 \tag{232}$$

which implies that $x = \mathrm{e}^{2\pi\mathsf{b}u}$ and $y = -\mathrm{e}^{2\pi\mathsf{b}v}$ are points of the affine curve S given by

$$S : y^2 = (1-y)(1-xy)(1-x^{-1}y) \tag{233}$$

and (u, v) are points of the exponentiated curve S^* given by the above equation with $(x, y) = (\mathrm{e}^{u}, -\mathrm{e}^{v})$. Moreover,

$$V(u,v) = \mathrm{Li}_2(y) + \mathrm{Li}_2(xy) + \mathrm{Li}_2(x^{-1}y) + (\log(-y))^2 \tag{234}$$

is a holomorphic $\mathbb{C}/2\pi^2\mathbb{Z}$-valued function on the exponentiated curve S^*. Note that when $u = 0$, Equation (233) becomes (73).

The constant term of the asymptotic expansion is given by the Hessian of $V(u, v)$ at a critical point (u, v), and it is a rational function of x and y is given by

$$\delta(x,y) = y - (1 + x + x^{-1})y^{-1} + 2y^{-2}\,. \tag{235}$$

Note that $\delta(x, y) = 0$ on S if and only if x is a root of the discriminant of S with respect to y, i.e.,

$$1 - 6x + 11x^2 - 12x^3 - 11x^4 - 12x^5 + 11x^6 - 6x^7 + x^8 = 0\,. \tag{236}$$

This happens at two points in the real line given approximately by $x \approx 0.235344$ and $x \approx 4.24909$. Moreover, when x is a root of (236), exactly two out of the three branches of $y = y(x)$ collide, and the corresponding branch point is simple.

Beyond the leading asymptotic expansion and its constant term, the asymptotic series has the form $\Phi(x,y;\tau)$ where

$$\Phi(x,y;\tau) = \exp\left(\frac{V(u,v)}{2\pi\mathrm{i}\tau}\right)\varphi(x,y;\tau), \qquad \varphi(x,y;\tau) \in \frac{1}{\sqrt{\mathrm{i}\delta}}\mathbb{Q}[x^{\pm},y^{\pm},\delta^{-1}][[2\pi\mathrm{i}\tau]] \tag{237}$$

where δ is given in (235) and $\sqrt{\mathrm{i}\delta}\,\varphi(x,y;0) = 1$. In other words, the coefficient of every power of $2\pi\mathrm{i}\tau$ in $\sqrt{\delta}\,\varphi(x,y;\tau)$ is a rational function on S. There is a natural projection $S \to \mathbb{C}^*$ given by $(x,y) \mapsto x$ and we denote by $y_\sigma(x)$ the choice of a local section (an algebraic function of x), for $\sigma \in \mathcal{P} = \{\sigma_1,\sigma_2,\sigma_3\}$. We denote the corresponding series $\varphi(x,y_\sigma(x);\tau)$ simply by $\varphi_\sigma(x;\tau)$. When x is close to 1, we order $\mathcal{P}$ so that $\sigma_1,\sigma_2,\sigma_3$ correspond to small deformations away from geometric, conjugate and real connections at $x=1$. Note for σ_3, we only keep the real part of V. The power series $\sqrt{\mathrm{i}\delta}\varphi_\sigma(x;\tau)$ can be computed by applying Gaussian expansion on the state-integral (230), and one can compute up to 15 terms in a few minutes. Let us write down the first few terms of $\varphi_\sigma(x;\tau)$

$$\begin{aligned}\varphi_\sigma(x;\frac{\tau}{2\pi\mathrm{i}}) = 1 + \frac{\tau}{24\delta_\sigma^3}\Big(&81 + 112s - 78s^2 - 70s^3 + 94s^4 - 38s^5 + 5s^6\\ &+ \big(138 - 254s + 127s^2 + 44s^3 - 89s^4 + 38s^5 - 5s^6\big)y_\sigma\\ &+ \big(135 - 101s - 11s^2 + 61s^3 - 33s^4 + 5s^5\big)y_\sigma^2\Big) + O(\tau^2),\end{aligned} \tag{238}$$

where

$$s = s(x) = x^{-1} + 1 + x\,. \tag{239}$$

On the other hand, if one sets x to numerical values, the power series can be computed to 200 terms.

6.2 Holomorphic blocks

Motivated by the case of the $\mathbf{4}_1$ knot, we define the descendant state-integral of the $\mathbf{5}_2$ knot by

$$Z_{\mathbf{5}_2,m,\mu}(u;\tau) = \int_{\mathcal{D}} \Phi_{\mathsf{b}}(v)\,\Phi_{\mathsf{b}}(v+u)\,\Phi_{\mathsf{b}}(v-u)\,\mathrm{e}^{-2\pi\mathrm{i}v^2+2\pi(m\mathsf{b}-\mu\mathsf{b}^{-1})v}\mathrm{d}v \tag{240}$$

for integers m and μ, which agrees with the Andersen-Kashaev invariant of the $\mathbf{5}_2$ knot when $m=\mu=0$. Here the contour $\mathcal{D}$ was introduced in (67). It is expressed in terms of three descendant holomorphic blocks, which we denote by A_m, B_m and C_m instead of $B_m^{\sigma_j}$ for $j=1,2,3$. For $|q|\neq 1$, $A_m(x;q)$, $B_m(x;q)$ and $C_m(x;q)$ are given by

$$A_m(x;q) = H(x,x^{-1},q^m;q) \tag{241a}$$

$$B_m(x;q) = \theta(-q^{1/2}x;q)^{-2}x^m H(x,x^2,q^m x^2;q) \tag{241b}$$

$$C_m(x;q) = \theta(-q^{-1/2}x;q)^{-2}x^{-m} H(x^{-1},x^{-2},q^m x^{-2};q) \tag{241c}$$

where $H(x,y,z;q^\varepsilon) := H^\varepsilon(x,y,z;q)$ for $|q|<1$ and $\varepsilon = \pm$ and

$$H^+(x,y,z;q) = (qx;q)_\infty(qy;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}z^n}{(q;q)_n(qx;q)_n(qy;q)_n} \tag{242a}$$

$$H^-(x,y,z;q) = \frac{1}{(x;q)_\infty (y;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^{-n} y^{-n} z^n}{(q;q)_n (qx^{-1};q)_n (qy^{-1};q)_n}. \tag{242b}$$

Note that the summand of H^+ (a proper q-hypergeometric function) is equal to that of H^- after replacing q by q^{-1}. This implies that $H^\pm$ have a common annihilating ideal $\mathcal{I}_H$ with respect to x, y, z which can be computed as in the case of Lemma 15.

The next theorem expresses the descendant state-integrals bilinearly in terms of descendant holomorphic blocks.

Theorem 22 (a) *The descendant state-integral can be expressed in terms of the descendant holomorphic blocks by*

$$\begin{aligned} Z_{\mathfrak{5}_2,m,\mu}(u_{\mathsf{b}};\tau) =& (-1)^{m+\mu} q^{m/2} \tilde{q}^{\mu/2} \Big(\mathrm{e}^{\frac{3\pi\mathrm{i}}{4} + \frac{5\pi\mathrm{i}}{12}(\tau+\tau^{-1})} A_m(x;q) A_{-\mu}(\tilde{x};\tilde{q}^{-1}) \\ &+ \mathrm{e}^{-\frac{\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{12}(\tau+\tau^{-1})} B_m(x;q) B_{-\mu}(\tilde{x};\tilde{q}^{-1}) \\ &+ \mathrm{e}^{-\frac{\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{12}(\tau+\tau^{-1})} C_m(x;q) C_{-\mu}(\tilde{x};\tilde{q}^{-1}) \Big). \end{aligned} \tag{243}$$

(b) *The functions $A_m(x;q)$, $B_m(x;q)$ and $C_m(x;q)$ are holomorphic functions of $|q| \neq 1$ and meromorphic functions of $x \in \mathbb{C}^*$ with poles in $x \in q^{\mathbb{Z}}$ of order at most* 2.
(c) *Let*

$$W_m(x;q) = \begin{pmatrix} A_m(x;q) & B_m(x;q) & C_m(x;q) \\ A_{m+1}(x;q) & B_{m+1}(x;q) & C_{m+1}(x;q) \\ A_{m+2}(x;q) & B_{m+2}(x;q) & C_{m+2}(x;q) \end{pmatrix} \qquad (|q| \neq 1). \tag{244}$$

For all integers m and μ, state-integral $Z_{\mathfrak{5}_2,m,\mu}(u;\tau)$ and the matrix-valued function

$$W_{m,\mu}(u;\tau) = W_{-\mu}(\tilde{x};\tilde{q}^{-1}) \Delta(\tau) W_m(x;q)^T, \tag{245}$$

where

$$\Delta(\tau) = \begin{pmatrix} \mathrm{e}^{\frac{3\pi\mathrm{i}}{4} + \frac{5\pi\mathrm{i}}{12}(\tau+\tau^{-1})} & 0 & 0 \\ 0 & \mathrm{e}^{-\frac{\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{12}(\tau+\tau^{-1})} & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{\pi\mathrm{i}}{4} + \frac{\pi\mathrm{i}}{12}(\tau+\tau^{-1})} \end{pmatrix}, \tag{246}$$

are holomorphic functions of $\tau \in \mathbb{C}'$ and entire functions of $u \in \mathbb{C}$.

Proof Part (a) follows by applying the residue theorem, just as in the proof of part (a) of Theorem 14. A similar result was stated in [20].

Part (b) follows from the fact that when $|q| < 1$, the ratio test implies that $H^+(x,y,z;q)$ is an entire function of $(x,y,z) \in \mathbb{C}^3$ and $J^-(x,y,z;q)$ is a meromorphic function of $(x,y,z) \in \mathbb{C}^2 \times \mathbb{C}$ with poles at $x = q^{\mathbb{Z}}$ and $y \in q^{\mathbb{Z}}$.

For part (c), one uses parts (a) and (b) to deduce that $W_{m,\mu}(u;\tau)$ is holomorphic of $\tau \in \mathbb{C}'$ and meromorphic in u with possible poles of second order at $\mathsf{i}\mathsf{b}\mathbb{Z} + \mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$. An expansion at these points, done by the method of Sect. 6.3, demonstrates that the function is analytic at the points $\mathsf{i}\mathsf{b}\mathbb{Z} + \mathsf{i}\mathsf{b}^{-1}\mathbb{Z}$. □

Note that the holomorphic blocks have the symmetry

$$A_m(x^{-1};q) = A_m(x;q), \quad B_m(x^{-1};q) = C_m(x;q), \tag{247}$$

which implies the symmetry of the matrix $W_m(x;q)$

$$W_m(x^{-1};q) = W_m(x;q) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{248}$$

Consequently, $W_{m,\mu}(u;\tau)$ is invariant under the reflection $u \mapsto -u$.

Lemma 23 (a) *The annihilating ideal of $\mathcal{I}_H$ of $H^{\pm}$ is given by*

$$\begin{aligned} \mathcal{I}_J = \langle & yzS_x - xzS_y + (x^2y - xy^2)S_z + (-x^2y + xy^2 + xz - yz), \\ & -xy^2S_z^2 + zS_y + (y^2 + xy^2 - qyz)S_z + (-y^2 - z), -qyS_yS_z + S_y - 1, \\ & zS_y^2 + (-x + qy + qxy - q^2y^2 - z - qz)S_y + qxyS_z + (x - qy - qxy + qz)\rangle \end{aligned} \tag{249}$$

where S_x, S_y and S_z are the shifts x to qx, y to qy and z to qz, respectively.
(b) *When $|q| < 1$, we have*

$$H(x,y,z;q^{-1}) = \frac{\det \begin{pmatrix} H(x^{-1},x^{-1}y,x^{-2}z;q) & H(y^{-1},y^{-1}x,y^{-2}z;q) \\ xH(x^{-1},x^{-1}y,q^{-1}x^{-2}z;q) & yH(y^{-1},y^{-1}x,q^{-1}y^{-2}z;q) \end{pmatrix}}{y\theta(-q^{-\frac{1}{2}}x;q)\theta(-q^{-\frac{1}{2}}y;q)\theta(-q^{-\frac{1}{2}}xy^{-1};q)}. \tag{250}$$

Proof Part (a) follows as in the proof of Lemma 15.

For part (b), observe that both sides of the equation are power series in z and q-holonomic functions of z. Using the `HF` package, we find that the (i,j)-entry of the determinant is annihilated by the operator r_{ij} given by

$$\begin{aligned} r_{11} &= -yS_z^3 + (x + y + xy - q^2z)S_z^2 + (-x - x^2 - xy)S_z + x^2 \\ r_{12} &= -xS_z^3 + (x + y + xy - q^2z)S_z^2 + (-y - xy - y^2)S_z + y^2 \\ r_{21} &= -yS_z^3 + (x + y + xy - qz)S_z^2 + (-x - x^2 - xy)S_z + x^2 \\ r_{22} &= -xS_z^3 + (x + y + xy - qz)S_z^2 + (-y - xy - y^2)S_z + y^2, \end{aligned}$$

whereas the left-hand side of (250), after being multiplied by the denominator of the right-hand side, is annihilated by the operator

$$r = S_z^3 + (-1 - x - y)S_z^2 + (x + y + xy - qz)S_z - xy. \tag{251}$$

Using the commands `DFiniteTimes` and `DFiniteTimes` from the `HF` package, we computed a 9th-order operator R (which is too long to type here) that annihilates the determinant, and using the command `OreReduce`, we proved that it is a left multiple of r. It follows that both sides of (250) satisfy the same 9th-order recursion with respect to z, with non-vanishing leading term. Thus, the identity follows once we prove that the coefficient

of z^k in both sides agree, for $k = 0, \dots, 8$. When $|q| < 1$, the coefficient of z^k in $H(x, y, z; q)$ (resp., $H(x, y, z; q^{-1})$) is in $(qx; q)_\infty (qy; q)_\infty \mathbb{Q}(x, y, q)$ (resp., $(x; q)_\infty^{-1} (y; q)_\infty^{-1} \mathbb{Q}(x, y, q)$), and this implies that the equality of the coefficient of z^k in the above identity reduces to an equality on the field $\mathbb{Q}(x, y, q)$ of rational functions in three variables. The latter is easy to check for $k = 0, \dots, 8$. This completes the proof of (250). □

The next theorem concerns the properties of the linear q-difference equations satisfied by the descendant holomorphic blocks.

Theorem 24 (a) *They are q-holonomic functions in the variables* (m, x) *with a common annihilating ideal*

$$\mathcal{I}_{\mathbf{5}_2} = \langle P_1, P_2, P_3 \rangle \tag{252}$$

where

$$\begin{aligned} P_1 =& x(1 - q^3x^2)(1 - qx^2 - q^2x^2 - q^{3+m}x^3 + q^3x^4) \\ & - (1 - qx)(1 + qx)(1 - qx^2)(1 - q^3x^2)S_m \\ & - x(1 - qx)(1 + qx)(1 - qx - qx^2 \\ & - q^3x^2 + q^2x^3 + q^4x^3 - q^{3+m}x^3 - q^{4+m}x^3 \\ & + q^4x^4 - q^5x^5)S_x - q^{4+m}x^4(1 - qx^2)S_x^2 \end{aligned} \tag{253a}$$

$$P_2 = x - S_m - xS_x + qx^2 S_x S_m \tag{253b}$$

$$\begin{aligned} P_3 =& x(1 - q^{1+m} - qx^2) - (1 - q^{1+m} + x - qx^2 + q^{2+m}x^2 - qx^3)S_m \\ & + (1 - qx^2)S_m^2 + q^{1+m}xS_x\,. \end{aligned} \tag{253c}$$

$\mathcal{I}_{\mathbf{5}_2}$ has rank 3 and the three functions form a basis of solutions of the corresponding system of linear equations.

(b) *As functions of an integer m, $A_m(x; q)$, $B_m(x; q)$ and $C_m(x; q)$ form a basis of solutions of the linear q-difference equation $\widehat{B}_{\mathbf{5}_2}(S_m, x, q^m, q) f_m(x; q) = 0$ for $|q| \neq 1$ where*

$$\widehat{B}_{\mathbf{5}_2}(S_m, x, q^m, q) = (1 - S_m)(1 - xS_m)(1 - x^{-1}S_m) - q^{2+m}S_m^2\,. \tag{254}$$

(c) *The Wronskian $W_m(x; q)$ of (254), defined in (244), satisfies*

$$\det(W_m(x; q)) = -\theta(-q^{-\frac{1}{2}}x; q)^{-2}\theta(-q^{-\frac{1}{2}}x^2; q) \qquad (|q| \neq 1)\,. \tag{255}$$

(d) *The Wronskian satisfies the orthogonality relation*

$$W_{-1}(x; q)\, W_{-1}(x; q^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x + x^{-1} \end{pmatrix}. \tag{256}$$

It follows that for all integers m, ℓ

$$W_m(x; q)\, W_\ell(x; q^{-1})^T \in \mathrm{PSL}(3, \mathbb{Z}[q^{\pm}, x^{\pm}]). \tag{257}$$

(e) *As functions of x, they form a basis of a linear q-difference equation* $\hat{A}_{\mathbf{5}_2}(S_x, x, q^m, q) f_m(x;q) = 0$ *where*

$$\hat{A}_{\mathbf{5}_2}(S_x, x, q^m, q) = \sum_{j=0}^{3} C_j(x, q^m, q) S_x^j, \tag{258}$$

S_x *is the operator that shifts x to qx and*

$$\begin{aligned}
C_0 =& -q^{2+m}x^2(1-q^2x)(1+q^2x)(1-q^5x^2) && (259a)\\
C_1 =&(1-qx)(1+qx)(1-q^5x^2)\times \\
&(1-qx-qx^2-q^4x^2+q^{2+m}x^2+q^{3+m}x^2 \\
&+q^2x^3+q^5x^3+q^5x^4+q^{5+m}x^4-q^6x^5) && (259b)\\
C_2 =&qx(1-q^2x)(1+q^2x)(1-qx^2)\times \\
&(1-q^2x-q^{2+m}x-q^2x^2-q^5x^2+q^4x^3+q^7x^3 \\
&-q^{5+m}x^3-q^{6+m}x^3+q^7x^4-q^9x^5) && (259c)\\
C_3 =&q^{8+m}x^4(1-qx)(1+qx)(1-qx^2) && (259d)
\end{aligned}$$

(f) *The Wronskian of Equation* (258)

$$\mathcal{W}_m(x;q) = \begin{pmatrix} A_m(x;q) & B_m(x;q) & C_m(x;q) \\ A_m(qx;q) & B_m(qx;q) & C_m(qx;q) \\ A_m(q^2x;q) & B_m(q^2x;q) & C_m(q^2x;q) \end{pmatrix}, \qquad (|q| \neq 1) \tag{260}$$

satisfies

$$\det \mathcal{W}_m(x;q) = q^{-5-2m}x^{-5}(1-q^2x^2)(1-qx^2)(1-q^3x^2)\theta(-q^{-\frac{1}{2}}x;q)^{-2}\theta(-q^{-\frac{1}{2}}x^2;q). \tag{261}$$

Proof Since $A_m(x;q)$, $B_m(x;q)$ and $C_m(x;q)$ are given in terms of q-proper hypergeometric multisums, it follows from the fundamental theorem of Zeilberger [63,71,72] (see also [35]) that they are q-holonomic functions in both variables m and x. Part (a) follows from an application of the `HF` package of Koutschan [51,52].

Part (b) follows from the `HF` package. The fact that they are a basis follows from (c).

For part (c), Equation (254) implies that the determinant of the Wronskian satisfies the first-order equation $\det(W_{m+1}(x;q)) = \det(W_m(x;q))$ (see [32, Lem.4.7]). It follows that $\det(W_m(x;q)) = \det(W_0(x;q))$ with initial condition a function of x given by

$$\det(W_0(x;q)) = -\theta(-q^{-\frac{1}{2}}x;q)^{-2}\theta(-q^{-\frac{1}{2}}x^2;q) \qquad (|q| \neq 1), \tag{262}$$

which can be proved in a manner similar to Sect. 5.2. Using Lemma 23 and the `HF` package, we find the following recursion relation for the q-function $H(x, y, z; q)$ when $|q| < 1$

$$\begin{aligned}
&xyH(x, y, qz; q) - (x + y + xy - z)H(x, y, z; q) \\
&+(1 + x + y)H(x, y, q^{-1}z; q) + H(x, y, q^{-2}z; q) = 0.
\end{aligned} \tag{263}$$

It then follows that

$$\mathcal{H}_{\mu,\nu,1}(z;q) = H(q^\mu, q^\nu, z; q), \tag{264}$$

$$\mathcal{H}_{\mu,\nu,2}(z;q) = z^{-\mu} H(q^{-\mu}, q^{\nu-\mu}, q^{-2\mu} z; q), \tag{265}$$

$$\mathcal{H}_{\mu,\nu,3}(z;q) = z^{-\nu} H(q^{-\nu}, q^{\mu-\nu}, q^{-2\nu} z; q), \tag{266}$$

are three independent solutions to

$$q^{\mu+\nu} H(qz;q) - (q^\mu + q^\nu + q^{\mu+\nu} - z) H(z;q) + (1 + q^\mu + q^\nu) H(q^{-1}z;q) + H(q^{-2}z;q) = 0. \tag{267}$$

The corresponding Wronskian

$$\mathcal{W}_{\mu,\nu}(z;q) = \begin{pmatrix} \mathcal{H}_{\mu,\nu,1}(q^{-1}z;q) & \mathcal{H}_{\mu,\nu,2}(q^{-1}z;q) & \mathcal{H}_{\mu,\nu,3}(q^{-1}z;q) \\ \mathcal{H}_{\mu,\nu,1}(z;q) & \mathcal{H}_{\mu,\nu,2}(z;q) & \mathcal{H}_{\mu,\nu,3}(z;q) \\ \mathcal{H}_{\mu,\nu,1}(qz;q) & \mathcal{H}_{\mu,\nu,2}(qz;q) & \mathcal{H}_{\mu,\nu,3}(qz;q) \end{pmatrix} \tag{268}$$

satisfies the recursion relation (see [32, Lem.4.7])

$$\det \mathcal{W}_{\mu,\nu}(z;q) = q^{-\mu-\nu} \det \mathcal{W}_{\mu,\nu}(q^{-1}z;q) \tag{269}$$

which implies that the determinant of $U(z;q) = z^{\mu+\nu} \mathcal{W}_{\mu,\nu}(z;q)$ is invariant under the shift $z \mapsto qz$. We can thus identify it with the limit $z \mapsto 0$, which is easy to compute. Since

$$\lim_{z\mapsto 0} H(x,y,z;q) = (qx;q)_\infty (qy;q)_\infty, \tag{270}$$

we have

$$\begin{aligned} &\lim_{z\mapsto 0} \det U(z;q) \\ &= \lim_{z\mapsto 0} \det \begin{pmatrix} H(q^\mu, q^\nu, q^{-1}z;q) & q^\mu H(q^{-\mu}, q^{\nu-\mu}, q^{-2\mu}q^{-1}z;q) & q^\nu H(q^{-\nu}, q^{\mu-\nu}, q^{-2\nu}qz^{-1};q) \\ H(q^\mu, q^\nu, z;q) & H(q^{-\mu}, q^{\nu-\mu}, q^{-2\mu}z;q) & H(q^{-\nu}, q^{\mu-\nu}, q^{-2\nu}z;q) \\ H(q^\mu, \mathrm{e}^\nu, qz;q) & q^{-\mu} H(q^{-\mu}, q^{\nu-\mu}, q^{-2\mu}qz;q) & q^{-\nu} H(q^{-\nu}, q^{\mu-\nu}, q^{-2\nu}qz;q) \end{pmatrix} \\ &= (qq^\mu;q)_\infty (qq^\nu;q)_\infty (qq^{-\mu};q)_\infty (qq^{\nu-\mu};q)_\infty (qq^{-\nu};q)_\infty (qq^{\mu-\nu};q)_\infty \det \begin{pmatrix} 1 & q^\mu & q^\nu \\ 1 & 1 & 1 \\ 1 & q^{-\mu} & q^{-\nu} \end{pmatrix} \\ &= q^{-\mu-\nu}(1-q^\mu)(1-q^\nu)(q^\mu - q^\nu) \\ &\quad \times (qq^\mu;q)_\infty (qq^\nu;q)_\infty (qq^{-\mu};q)_\infty (qq^{\nu-\mu};q)_\infty (qq^{-\nu};q)_\infty (qq^{\mu-\nu};q)_\infty. \end{aligned} \tag{271}$$

We thus have

$$z^{\mu+\nu} \det \mathcal{W}_{\mu,\nu}(z;q) = -(q^\mu;q)_\infty (qq^{-\mu};q)_\infty (qq^\nu;q)_\infty (q^{-\nu};q)_\infty (qq^{\mu-\nu};q)_\infty (q^{\nu-\mu};q)_\infty. \tag{272}$$

Using the substitution

$$q^\mu = x^{-1}, \quad q^\nu = x, \quad z = 1 \tag{273}$$

in the above equation and cancelling with the θ-prefactors of $B_m(x;q)$ and $C_m(x;q)$, we obtain Equation (262) for $|q| < 1$. The case for $|q| > 1$ can be obtained by analytic continuation on both sides of (262).

For part (d), Equation (256) follows from (250). To see this, let us introduce

$$\widetilde{W}_m(x;q) = \begin{pmatrix} H(x,x^{-1},q^m;q) & x^m H(x,x^2,q^m x^2;q) & x^{-m} H(x^{-1},x^{-2},q^m x^{-2};q) \\ H(x,x^{-1},q^{m+1};q) & x^{m+1} H(x,x^2,q^m x^2;q) & x^{-m-1} H(x^{-1},x^{-2},q^{m+1} x^{-2};q) \\ H(x,x^{-1},q^{m+2};q) & x^{m+2} H(x,x^2,q^{m+2} x^2;q) & x^{-m-2} H(x^{-1},x^{-2},q^{m+2} x^{-2};q) \end{pmatrix}. \tag{274}$$

Then, Equation (256) can equally be written as

$$\widetilde{W}_{-1}(x;q^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x+x^{-1} \end{pmatrix} \left(\widetilde{W}_{-1}(x;q)^{-1}\right)^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-2} & 0 \\ 0 & 0 & x^2 \end{pmatrix}, \tag{275}$$

consisting of 9 scalar equations. Each of these equations is a specialization of (250), sometimes after applying the recursion relation (254).

Observe that Equation (254) written in matrix form implies that

$$W_{m+1}(x;q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -s(x) & s(x) - q^{2+m} \end{pmatrix} W_m(x;q) \tag{276}$$

where

$$s(x) = x + x^{-1} + 1\,. \tag{277}$$

Equation (257) follows from (256) together with (276).

Part (e) follows from the `HF` package. The fact that they are a basis follows from (f).

For part (f), the first and third generators of the annihilating ideal (252), which annihilate $A_m(x;q)$, $B_m(x;q)$, $C_m(x;q)$, allow expressing $A_m(qx;q)$, $A_m(q^2x;q)$ in terms of $A_m(x;q), A_{m+1}(x;q), A_{m+2}(x;q)$, and similarly for $B_m(q^j x;q)$, $C_m(q^j x;q)$ $(j = 1, 2)$. It follows that the Wronskian (260) and the Wronskian (244) are related

$$\mathcal{W}_m(x;q) = M_m(x;q) W_m(x;q) \tag{278}$$

where $M_m(x;q)$ is a 3×3 matrix with entries

$$\begin{aligned}
(M_m(x;q))_{1,1} =& 1, \quad (M_m(x;q))_{1,2} = 0, \quad (M_m(x;q))_{1,3} = 0, \\
(M_m(x;q))_{2,1} =& -q^{-1-m}(1 - q^{1+m} - qx^2) \\
(M_m(x;q))_{2,2} =& q^{-1-m}x^{-1}(1 - q^{1+m} + x + (-q + q^{2+m})x^2 - qx^3) \\
(M_m(x;q))_{2,3} =& -q^{-1-m}x^{-1}(1 - qx^2) \\
(M_m(x;q))_{3,1} =& q^{-5-2m}x^{-3}\big(1 + (-q + q^{2+m})x - (q + q^2 + q^3)x^2 \\
& + (q^2 + q^3 + q^4 - q^{3+m} - 2q^{4+m} - q^{5+m} + q^{5+2m})x^3 \\
& + (q^3 + q^4 + q^5)x^4 + (-q^4 - q^5 - q^6 + q^{5+m} \\
& + q^{6+m} + q^{7+m})x^5 - q^6 x^6 + q^7 x^7\big) \\
(M_m(x;q))_{3,2} =& -q^{-5-2m}x^{-4}(1 - qx)(1 + qx)\big(1 + (1 - q + q^{2+m})x \\
& + (-2q - q^3 + q^{2+m})x^2
\end{aligned}$$

$$
\begin{aligned}
&+(-q+q^2-q^3+q^4-2q^{3+m}-q^{4+m}-q^{5+m}+q^{4+2m}+q^{5+2m})x^3\\
&+(q^2+2q^4-q^{3+m}-q^{4+m}-q^{5+m})x^4+(q^4-q^5+q^{6+m})x^5-q^5x^6\big)\\
(M_m(x;q))_{3,3} =&q^{-5-2m}x^{-4}(1-qx)(1+qx)\big(1-qx-(q+q^3)x^2\\
&+(q^2+q^4-q^{3+m}-q^{4+m})x^3+q^4x^4-q^5x^5\big)
\end{aligned}
\tag{279}
$$

After taking determinants on both sides, one finds that

$$
\det(\mathcal{W}_m(x;q)) = -q^{-5-2m}x^{-5}(1-qx)(1+qx)(1-qx^2)(1-q^3x^2)\det(W_m(x;q)) \tag{280}
$$

This, together with (255) concludes the proof of (261). □

We now come to Conjecture 5 concerning a refinement of the $\widehat{A}$-polynomial. As in Sect. 5.2, we can use Theorems 22 and 24 to obtain explicit linear q-difference equations for the descendant integrals with respect to the u and the m variables, and in doing so, we will obtain a refinement of the $\widehat{A}$-polynomial. To simplify Equation (129), let us define a normalized version of the descendant state-integral

$$
z_{\mathbf{5}_2,m,\mu}(u;\tau) = (-1)^{m+\mu}q^{-m/2}\tilde{q}^{-\mu/2}Z_{\mathbf{5}_2,m,\mu}(u;\tau)\,. \tag{281}
$$

Our next theorem confirms Conjecture 5 for the $\mathbf{5}_2$ knot.

Theorem 25 *$z_{\mathbf{5}_2,m,k}(u_{\mathsf{b}};\tau)$ is a q-holonomic function of (m,u) with annihilator ideal $\mathcal{I}_{\mathbf{5}_2}$ given in (252). As a function of u (resp., m), it is annihilated by the operators $\widehat{A}_{\mathbf{5}_2}(S_x,x,q^m,q)$ (resp., $\widehat{B}_{\mathbf{5}_2}(S_m,x,q^m,q)$) given by (254) and (258), whose classical limit is*

$$
\begin{aligned}
&\widehat{A}_{\mathbf{5}_2}(S_x,x,q^m,1) =\\
&-(1-x)^2(1+x)^2\Big(q^mx^2-(1-x-2(1-q^m)x^2+2x^3+(1+q^m)x^4-x^5)S_x\\
&-x(1-(1+q^m)x-2x^2+2(1-q^m)x^3+x^4-x^5)S_x^2-q^mx^4S_x^3\Big)\,,
\end{aligned}
\tag{282}
$$

and

$$
\widehat{B}_{\mathbf{5}_2}(S_m,x,q^m,1) = (1-S_m)(1-xS_m)(1-x^{-1}S_m)-q^mS_m^2\,. \tag{283}
$$

$\widehat{A}_{\mathbf{5}_2}(S_x,x,1,1)$ is the A-polynomial of the knot, $\widehat{A}_{\mathbf{5}_2}(S_x,x,1,q)$ is the (homogeneous part) of the $\widehat{A}$-polynomial of the knot, and $\widehat{B}_{\mathbf{5}_2}(y,x,1,1)$ is the defining equation of the curve (233).

6.3 Taylor series expansion at $u=0$

In this section, we discuss the Taylor expansion of the descendant holomorphic blocks and descendant state-integral of the $\mathbf{5}_2$ knots at $u=0$. Using the definition of $H(x,y,z;q)$ and (161a)–(161b), we find

$$
\begin{aligned}
A_m(\mathrm{e}^u;q) &= (q;q)_\infty^2\sum_{n=0}^{\infty}\frac{q^{n(n+1)+nm}}{(q;q)_n^3}\phi_n(u)\phi_n(-u)\\
&= (q;q)_\infty^2\left(\alpha_0^{(m)}(q)+u^2\alpha_2^{(m)}(q)+O(u^3)\right)
\end{aligned}
\tag{284}
$$

where

$$\alpha_0^{(m)}(q) = H^+_{0,m}(q), \tag{285}$$

$$\alpha_2^{(m)}(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)+nm}}{(q;q)_n^3}(-E_2^{(n)}(q)), \tag{286}$$

and

$$\begin{aligned} C_m(\mathrm{e}^{-u};q) &= B_m(\mathrm{e}^u;q) \\ &= (1-\mathrm{e}^{-u})^{-2}(q;q)_\infty^{-2}\phi_0(u)^{-2}\phi_0(-u)^{-2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{q^{n(n+1)+nm}}{(q;q)_n^3}\phi_n(u)\phi_n(2u)\mathrm{e}^{(m+2n)u} \\ &= u^{-2}(q;q)_\infty^{-2}\left(\beta_0^{(m)}(q) + u\beta_1^{(m)}(q) + u^2\beta_2^{(m)}(q) + O(u^3)\right) \end{aligned} \tag{287}$$

where the coefficients satisfy

$$\beta_0^{(m)}(q) = H^+_{0,m}(q), \tag{288}$$

$$\beta_1^{(m)}(q) = H^+_{1,m}(q), \tag{289}$$

$$\beta_2^{(m)}(q) = \frac{1}{2}H^+_{2,m}(q) + \alpha_2^{(m)}(q). \tag{290}$$

Similarly, using the definition of $H(x,y,z;q^{-1})$, we find

$$\begin{aligned} A_m(\mathrm{e}^u;q^{-1}) &= \frac{1}{(1-\mathrm{e}^u)(1-\mathrm{e}^{-u})(q;q)_\infty^2}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n+1)-nm}}{(q;q)_n^3}\widetilde{\phi}_n(u)\widetilde{\phi}_n(-u) \\ &= -u^{-2}(q;q)_\infty^{-2}\left(\widetilde{\alpha}_0^{(m)}(q) + u^2\widetilde{\alpha}_2^{(m)}(q) + O(u^3)\right) \end{aligned} \tag{291}$$

where

$$\widetilde{\alpha}_0^{(m)}(q) = H^-_{0,-m}(q), \tag{292}$$

$$\widetilde{\alpha}_2^{(m)}(q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n+1)-nm}}{(q;q)_n^3}\left(-\frac{1}{12} + 2E_2^{(0)}(q) - E_2^{(n)}(q)\right), \tag{293}$$

and

$$\begin{aligned} C_m(\mathrm{e}^{-u};q^{-1}) &= B_m(\mathrm{e}^u;q^{-1}) \\ &= (1-\mathrm{e}^u)(1-\mathrm{e}^{2u})^{-1}(q;q)_\infty^2\phi_0(u)^2\phi_0(-u)^2 \\ &\quad \times \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(n+1)-nm}}{(q;q)_n^3}\widetilde{\phi}_n(u)\widetilde{\phi}_n(2u)\mathrm{e}^{(2n+m)u} \\ &= \frac{1}{2}(q;q)_\infty^2\left(\widetilde{\beta}_0^{(m)}(q) + u\widetilde{\beta}_1^{(m)}(q) + u^2\widetilde{\beta}_2^{(m)}(q) + O(u^3)\right) \end{aligned} \tag{294}$$

where the coefficients satisfy

$$\widetilde{\beta}_0^{(m)}(q) = H^-_{0,-m}(q), \tag{295}$$

$$\widetilde{\beta}_1^{(m)}(q) = -H^-_{1,-m}(q), \tag{296}$$

$$\widetilde{\beta}_2^{(m)}(q) = \frac{1}{2}H^-_{2,-m}(q) + \widetilde{\alpha}_2^{(m)}(q). \tag{297}$$

Applying these results to the right-hand side of (243), as well as using the trick

$$\frac{1}{2\pi \mathrm{i}} = -\frac{1}{12}(\tau E_2(q) - \tau^{-1}E_2(\tilde{q})), \tag{298}$$

we find the $O(1/u^2)$ and $O(1/u)$ contributions from (291) and (294) cancel, and the $O(u^0)$ contributions reproduce (103).

As an application of the above computations, we demonstrate that Theorem 11, especially (91), (93), as well as the recursion relation (86), can be proved by taking the $u = 0$ limit of the analogue identities in Theorem 24.

Using the expansion formulas of holomorphic blocks (284), (287), (291), (294), the Wronskians can be expanded as

$$W_m(\mathrm{e}^u; q) = \left(\sum_{j=0} W^+_{m,j}(q)u^j\right) \begin{pmatrix} (q;q)_\infty^2 & 0 & 0 \\ 0 & u^{-2}(q;q)_\infty^{-2} & 0 \\ 0 & 0 & u^{-2}(q;q)_\infty^{-2} \end{pmatrix}, \tag{299}$$

$$W_m(\mathrm{e}^u; q^{-1}) = \left(\sum_{j=0} W^-_{m,j}(q)u^j\right) \begin{pmatrix} -u^{-2}(q;q)_\infty^{-2} & 0 & 0 \\ 0 & \frac{1}{2}(q;q)_\infty^2 & 0 \\ 0 & 0 & \frac{1}{2}(q;q)_\infty^2 \end{pmatrix} \tag{300}$$

where

$$W^+_{m,j}(q) = \begin{pmatrix} \alpha_j^{(m)} & \beta_j^{(m)} & (-1)^j\beta_j^{(m)} \\ \alpha_j^{(m+1)} & \beta_j^{(m+1)} & (-1)^j\beta_j^{(m+1)} \\ \alpha_j^{(m+2)} & \beta_j^{(m+2)} & (-1)^j\beta_j^{(m+2)} \end{pmatrix}(q),$$

$$W^-_{m,j}(q) = \begin{pmatrix} \widetilde{\alpha}_j^{(m)} & \widetilde{\beta}_j^{(m)} & (-1)^j\widetilde{\beta}_j^{(m)} \\ \widetilde{\alpha}_j^{(m+1)} & \widetilde{\beta}_j^{(m+1)} & (-1)^j\widetilde{\beta}_j^{(m+1)} \\ \widetilde{\alpha}_j^{(m+2)} & \widetilde{\beta}_j^{(m+2)} & (-1)^j\widetilde{\beta}_j^{(m+2)} \end{pmatrix}(q). \tag{301}$$

Note that $\alpha_j^{(m)}(q) = \widetilde{\alpha}_j^{(m)}(q) = 0$ if j is odd. Taking the determinant of (299), we find

$$\det W_m(\mathrm{e}^u; q) = u^{-1}(q;q)_\infty^{-2} \det W_m(q) + O(u^0) \tag{302}$$

which together with the u-expansion of the right-hand side of (255) leads to the determinant identity (91). Furthermore, by substituting (299), (300) into the Wronskian relation (256), we find the left-hand side reduces to

$$\frac{1}{2}W_{-1}(q)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} W_{-1}(q^{-1})\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + O(u^1), \tag{303}$$

which together with the u-expansion of the right-hand side leads in the leading order to (92) for $m = 1$. The more general case follows from the identity of $m = 1$ by applying the recursion relations (86) on the Wronskians.

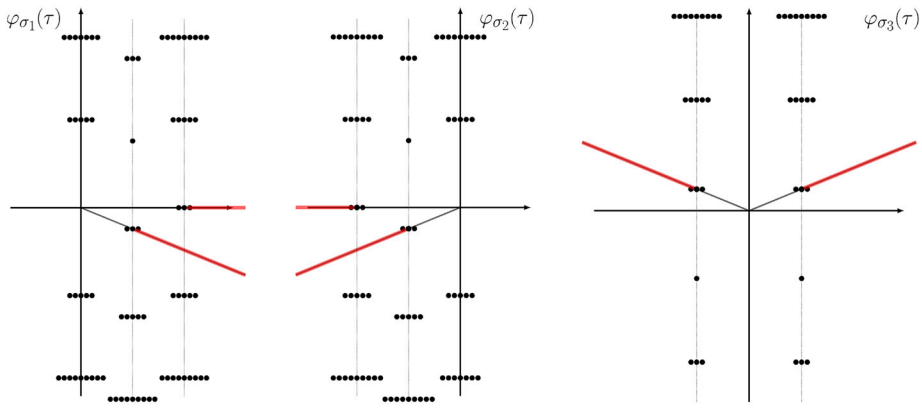

Fig. 10 The singularities in the Borel plane for the series $\varphi_{\sigma_j}(u;\tau)$ with $j = 1, 2, 3$ of knot $\mathbf{5}_2$ where u is close to zero

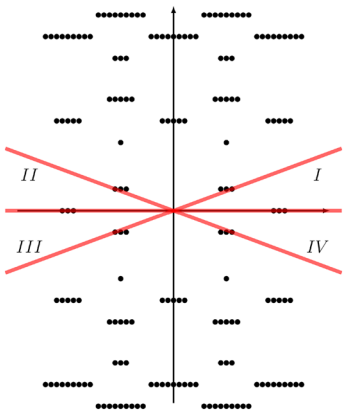

Fig. 11 Four different sectors (and additional two auxiliary regions) in the τ-plane for $\Phi(u;\tau)$ of knot $\mathbf{5}_2$ with u close to zero

Finally, from the expression (285),(288) of the leading-order coefficients $\alpha_0^{(m)}(q)$, $\beta_0^{(m)}(q)$, of $A_m(e^u;q)$, $B_m(e^u;q)$, $C_m(e^u;q)$ in the expansion of u, one concludes that the recursion relation (86) should be the $u = 0$ limit of the recursion relation (254) in m, and one can easily check it is indeed the case.

6.4 Stokes matrices near $u = 0$

In this section, we give a conjecture for the Stokes matrices of the asymptotic series $\varphi(x;\tau)$.

We only consider the case when u is not far away from zero, or equivalently x not far away from 1. To be more precise, we focus on real x and constrain x to be in the interval containing 1 between the two real solutions to the discriminant (236). In this case, there are mild changes to the resurgent structure discussed in Sect. 3.2. Each singular point in Fig. 4 splits to a cluster of neighboring singular points as shown in Fig. 10. In particular, each of the six singular points $\iota_{i,j}$ $(i \neq j)$ splits to a cluster of neighboring three separated from each other by $\log x$. We label the four regions separating singularities on positive and negative real axis and the other singularities by *I, II, III, IV* (see Fig. 11). In each of the four regions, we have the following the results.

Conjecture 26 The asymptotic series and the holomorphic blocks are related by (15) with the diagonal matrix $\Delta(u)$ as in (246) where the matrices $M_R(x;q)$ are given in terms of the $W_{-1}(x;q)$ as follows

$$M_I(x;q) = W_{-1}(x;q)^T \begin{pmatrix} 0 & 0 & -1 \\ 1 & -s(x) & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad |q| < 1, \tag{304a}$$

$$M_{II}(x;q) = W_{-1}(x^{-1};q)^T \begin{pmatrix} 0 & 0 & -1 \\ -s(x) & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad |q| < 1, \tag{304b}$$

$$M_{III}(x;q) = W_{-1}(x^{-1};q)^T \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad |q| > 1, \tag{304c}$$

$$M_{IV}(x;q) = W_{-1}(x;q)^T \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad |q| > 1, \tag{304d}$$

and where $s(x)$ is given in (277).

Just as in the case of the $\mathbf{4}_1$ knot, the above conjecture completely determines the resurgent structure of $\Phi(\tau)$. Indeed, it implies that the Stokes matrices given by Equations (34) and (33) are explicitly given by:

$$\mathsf{S}^+(x;q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(x^{-1};q^{-1}) W_{-1}(x;q)^T \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{305a}$$

$$\mathsf{S}^-(x;q) = \begin{pmatrix} 0 & -s(x) & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(x;q) W_{-1}(x^{-1};q^{-1})^T \begin{pmatrix} 0 & 0 & -1 \\ -s(x) & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{305b}$$

for $|q| < 1$. Note that Equations (233), (234), (235), (238) imply that (one can also see this from (230))

$$s(\Phi_\sigma)(x^{-1};\tau) = s(\Phi_\sigma)(x;\tau) \tag{306}$$

for any $\tau \in \mathbb{C}$ whenever the asymptotic series is Borel summable. It follows that the Stokes matrices must be invariant under the reflection $\pi : x \mapsto x^{-1}$. Using the property (248) of $W_m(x;q)$, it is easy to show that the Stokes matrices (305a),(305b) indeed satisfy this consistency condition.

The $q \mapsto 0$ limit of the Stokes matrices factorizes

$$\begin{aligned} \mathsf{S}^+(x;0) &= \mathfrak{S}_{\sigma_3,\sigma_1}(x)\mathfrak{S}_{\sigma_3,\sigma_2}(x)\mathfrak{S}_{\sigma_1,\sigma_2}(x) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s(x) & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & s(x)+1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathsf{S}^-(x;0) &= \mathfrak{S}_{\sigma_1,\sigma_3}(x)\mathfrak{S}_{\sigma_2,\sigma_3}(x)\mathfrak{S}_{\sigma_2,\sigma_1}(x) \end{aligned} \tag{307}$$

$$= \begin{pmatrix} 1 & 0 & s(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -s(x) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -s(x) - 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{308}$$

where the non-vanishing off-diagonal entries of $\mathfrak{S}_{\sigma_i,\sigma_j}(x)$ encode the Stokes constants associated with the Borel singularities split from $\iota_{i,j}$. Using the unique factorization Lemma 8 and the Stokes matrix S from above, we can compute the Stokes constants and the corresponding matrix $\mathcal{S}$ of Equation (36) to arbitrary order in q, and we find that

$$\begin{aligned}
\mathcal{S}^+_{\sigma_1,\sigma_1} =& \mathsf{S}^+(q)_{1,1} - 1 \\
=& (-4 - x^{-2} - 3x^{-1} - 3x - x^2)q + (-1 + x^{-3} + x^{-2} + x^2 + x^3)q^2 + O(q^3), && \text{(309a)} \\
\mathcal{S}^+_{\sigma_1,\sigma_2} =& (4 + x^{-2} + 3x^{-1} + 3x + x^2)q \\
& + (37 + x^{-4} + 5x^{-3} + 16x^{-2} + 30x^{-1} + 30x + 16x^2 + 5x^3 + x^4)q^2 + O(q^3), && \text{(309b)} \\
\mathcal{S}^+_{\sigma_1,\sigma_3} =& q + (1 + x^{-1} + x)q^2 + O(q^3), && \text{(309c)} \\
\mathcal{S}^+_{\sigma_2,\sigma_1} =& -(4 + x^{-2} + 3x^{-1} + 3x + x^2)q \\
& - (37 + x^{-4} + 5x^{-3} + 16x^{-2} + 30x^{-1} + 30x + 16x^2 + 5x^3 + x^4)q^2 + O(q^3), && \text{(309d)} \\
\mathcal{S}^+_{\sigma_2,\sigma_2} =& (4 + x^{-2} + 3x^{-1} + 3x + x^2)q \\
& + (37 + x^{-4} + 5x^{-3} + 16x^{-2} + 30x + 30x + 16x^2 + 5x^3 + x^4)q^2 + O(q^3), && \text{(309e)} \\
\mathcal{S}^+_{\sigma_2,\sigma_3} =& -(7 + 2x^{-2} + 5x^{-1} + 5x + 2x^2)q \\
& - (59 + 2x^{-4} + 10x^{-3} + 27x^{-2} \\
& \quad + 49x^{-1} + 49x + 27x^2 + 10x^3 + 2x^4)q^2 + O(q^3), && \text{(309f)} \\
\mathcal{S}^+_{\sigma_3,\sigma_1} =& -q - (1 + x^{-1} + x)q^2 + O(q^3) && \text{(309g)} \\
\mathcal{S}^+_{\sigma_3,\sigma_2} =& (7 + 2x^{-2} + 5x^{-1} + 5x + 2x^2)q \\
& + (59 + 2x^{-4} + 10x^{-3} + 27x^{-2} \\
& \quad + 49x^{-1} + 49x + 27x^2 + 10x^3 + 2x^4)q^2 + O(q^3), && \text{(309h)} \\
\mathcal{S}^+_{\sigma_3,\sigma_3} =& 0. && \text{(309i)}
\end{aligned}$$

Note that the series $\mathcal{S}^+_{\sigma_1,\sigma_2}$ and $\mathcal{S}^+_{\sigma_2,\sigma_2}$ are different, even though their first few terms are coincidental. They differ in higher orders, as one can already see in the $u = 0$ limit in Sect. 3.2. The matrix $\mathcal{S}$ satisfies the symmetry

$$\mathcal{S}^+_{\sigma_i,\sigma_j}(x;q) = -\mathcal{S}^+_{\sigma_{\varphi(i)},\sigma_{\varphi(j)}}(x;q), \quad i \neq j, \tag{310}$$

with $\varphi(1) = 2, \varphi(2) = 1, \varphi(3) = 3$, and they display the familiar feature that the entries of the matrix$\mathcal{S}^+(x;q) = (\mathcal{S}^+_{\sigma_i,\sigma_j}(x;q))$ (except the upper-left one) are (up to a sign) in $\mathbb{N}[x^{\pm 1}][[q]]$. Similarly, we can extract the Stokes constants $\mathcal{S}^{(\ell,k)}$ $(k < 0)$ associated with the singularities below $\iota_{i,j}$ in the lower half plane and assemble into q^{-1}-series $\mathcal{S}^-_{\sigma_i,\sigma_j}(x;q^{-1})$. We find the relation

$$\mathcal{S}^-_{\sigma_i,\sigma_j}(x;q) = -\mathcal{S}^+_{\sigma_j,\sigma_i}(x;q), \quad i \neq j \quad \text{and} \quad \mathcal{S}^-_{\sigma_i,\sigma_i}(x;q) = +\mathcal{S}^+_{\sigma_{\varphi(i)},\sigma_{\varphi(i)}}(x;q). \tag{311}$$

Let us now verify Conjecture 4 for the $\mathbf{5}_2$ knot. The same logic presented at the end of Sect. 5.4 also holds here. From the form of the Stokes matrix (305a) as well as (276),(278), we immediately conclude that

$$\mathsf{S}^+(x;q) \doteq \mathcal{W}_0(x^{-1};q^{-1}) \cdot \mathcal{W}_0(x;q)^T. \tag{312}$$

Using the uniform notation for all holomorphic blocks

$$(B^{\alpha}_{\mathbf{5}_2}(x;q))_{\alpha=1,2,3} = (A_0(x;q), B_0(x;q), C_0(x;q)), \tag{313}$$

the right-hand side of (312) reads[2]

$$\begin{aligned}\mathcal{W}_0(x^{-1};q^{-1}) \cdot \mathcal{W}_0(x;q)^T &= \left(\sum_{\alpha} B^{\alpha}_{\mathbf{5}_2}(q^j x;q) B^{\alpha}_{\mathbf{5}_2}(q^{-i}x^{-1};q^{-1})\right)_{i,j=0,1,2} \\ &= \left(\mathrm{Ind}^{\mathrm{rot}}_{\mathbf{5}_2}(j-i, q^{\frac{j+i}{2}}x;q)\right)_{i,j=0,1,2},\end{aligned} \tag{315}$$

reproducing the right-hand side of (20) in Conjecture 4. Furthermore, the forms of the accompanying matrices on the left and on the right are such that the (1, 1) entry of $\mathsf{S}^+(x;q)$ equates exactly the DGG index with no magnetic flux. By explicit calculation,

$$\begin{aligned}\mathsf{S}^+(x;q)_{1,1} =& H(x,x^{-1},1;q)H(x^{-1},x,1;q^{-1}) + H(x,x^2,x^2;q)H(x^{-1},x^{-2},x^{-2};q^{-1}) \\ &+ H(x^{-1},x^{-2},x^{-2};q)H(x,x^2,x^2;q^{-1}) \\ =& 1-(x^{-2}+3x^{-1}+4+3x+x^2)q \\ &+ (x^{-3}+x^{-2}-1+x^2+x^3)q^2 + O(q^3).\end{aligned} \tag{316}$$

The right-hand side of the first two lines in (316) is the formula for the DGG index given in [7].

6.5 The Borel resummations of the asymptotic series Φ

As in the case of the $\mathbf{4}_1$ knot, Conjecture 26 identifies the Borel resummations of the factorially divergent series $\Phi(x;\tau)$ with the descendant state-integrals, thus lifting the Borel resummation to holomorphic functions on the cut-plane $\mathbb{C}'$.

Corollary 27 (of Conjecture 26) *We have*

$$s_I(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1})\Delta(\tau)B(x;q), \tag{317a}$$

$$s_{II}(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})\Delta(\tau)B(x;q), \tag{317b}$$

[2] Note that if we compare with the DGG indices computed in [7], we have to modify the last line in (315) slightly due to different conventions for holomorphic blocks

$$\left(\mathcal{W}_0(x^{-1};q^{-1}) \cdot \mathcal{W}_0(x;q)^T\right)_{i+1,j+1} = q^{\frac{3}{2}(i^2-j^2)}x^{-3(j-i)}\mathrm{Ind}^{\mathrm{rot}}_{\mathbf{5}_2}(j-i, q^{-\frac{j+i}{2}}x^{-1};q), \quad i,j=0,1,2. \tag{314}$$

The right-hand side of (20) should be modified accordingly. This, however, does not affect (21).

$$s_{III}(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -s(\tilde{x}) & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})\Delta(\tau)B(x;q), \tag{317c}$$

$$s_{IV}(\Phi)(x;\tau) = \begin{pmatrix} 0 & -s(\tilde{x}) & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x},\tilde{q}^{-1})\Delta(\tau)B(x;q). \tag{317d}$$

where the right-hand side of (317a)–(317d) *are holomorphic functions of* $\tau \in \mathbb{C}'$*, as they are linear combinations of the descendants* (243)*.*

6.6 Numerical verification

In this section, we explain the numerical verification of Conjecture 26, which involves a richer resurgent structure than that of the $\mathbf{4}_1$ knot. We found ample numerical evidence for the resurgent data (304a)–(304d). These numerical tests are parallel to those performed for knot $\mathbf{4}_1$, so we will be sketchy here. Besides, we will mostly focus on τ in the upper half plane, while the lower half plane is similar.

The first test is the analysis of radial asymptotics of the left-hand side of (226), which can be easily done. The second test is to compute the Borel resummation $s_R(\Phi'_\sigma)(x;\tau)$ and by comparing with the left-hand side extract terms of $M_R(\tilde{x},\tilde{q})_{\sigma,\sigma'}$ order by order. To expediate the operation of extraction, instead of $M_R(\tilde{x};\tilde{q})$ we consider

$$\widetilde{M}_R(\tilde{x};\tilde{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta(-\tilde{q}^{-1/2}\tilde{x};\tilde{q})^2 & 0 \\ 0 & 0 & \theta(-\tilde{q}^{1/2}\tilde{x};\tilde{q})^2 \end{pmatrix} M_R(\tilde{x};\tilde{q}) \tag{318}$$

whose entries are $\tilde{q}$-series with coefficients in $\mathbb{Z}[\tilde{x}^{\pm 1}]$ instead of in $\mathbb{Z}(\tilde{x})$. Using 180 terms of $\Phi_\sigma(x;\tau)$ at various values of x and τ, we find entries of $\widetilde{M}_{I,II}(x;q)$ up to $O(q^2)$ following results

$$\begin{aligned}
\widetilde{M}_I(x;q)_{1,1} =& 1-(x^{-1}+x)q-(x^{-1}-2+x)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{1,2} =& -x^{-1}-x+(x^{-2}+2+x^2)q+(x^{-2}-2x^{-1}+1-2x+x^2)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{1,3} =& -1+(x^{-1}-1+x)q+(x^{-1}-2+x)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{2,1} =& x^2-(x^3+x^4)q-(x^3-x^5)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{2,2} =& -x-x^2+(x^2+2x^3+x^4)q+(x^2+x^3-x^4-2x^5)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{2,3} =& -x+x^2q+(x^2-x^4)q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{3,1} =& x^{-2}-(x^{-4}+x^{-3})q+(x^{-5}-x^{-3})q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{3,2} =& -x^{-2}-x^{-1}+(x^{-4}+2x^{-3}+x^{-2})q-(2x^{-5}+x^{-4}-x^{-3}-x^{-2})q^2+O(q^3),\\
\widetilde{M}_I(x;q)_{3,3} =& -x^{-1}+x^{-2}q-(x^{-4}-x^{-2})q^2+O(q^3),
\end{aligned} \tag{319}$$

and

$$\begin{aligned}
\widetilde{M}_{II}(x;q)_{1,1} =& -x-x^{-1}+(x^2+2+x^{-2})q+(x^2-2x+1-2x^{-1}+x^{-2})q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{1,2} =& 1-(x+x^{-1})q-(x-2+x^{-1})q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{1,3} =& -1+(x-1+x^{-1})q+(x-2+x^{-1})q^2+O(q^3),
\end{aligned}$$

Table 2 Numerical tests of holomorphic lifts of Borel sums of asymptotic series for knot $\mathbf{5}_2$

(A) Region *I*: $\tau = \frac{1}{18}e^{\frac{\pi i}{12}}$				
	$\lvert\frac{s_I(\Phi)(x;\tau)}{P_I(x;\tau)} - 1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\lvert\tilde{x}(x,\tau)^{-1}\rvert$
σ_1	1.1×10^{-38}	1.1×10^{-38}	1.9×10^{-13}	0.042
σ_2	2.2×10^{-63}	1.7×10^{-62}		
σ_3	2.7×10^{-48}	2.5×10^{-48}		
(B) Region *II*: $\tau = \frac{1}{18}e^{\frac{11\pi i}{12}}$				
	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{P_{II}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\lvert\tilde{x}(x,\tau)\rvert$
σ_1	2.2×10^{-63}	1.7×10^{-62}	1.9×10^{-13}	0.042
σ_2	1.1×10^{-38}	1.1×10^{-38}		
σ_3	2.7×10^{-48}	2.5×10^{-48}		

We perform the Borel–Padé resummation on $\Phi(x;\tau)$ with 180 terms at $x = 6/5$ and τ in regions *I*, *II* and compute the relative difference between them and the right-hand side of (317a)–(317b), which we denote by $P_R(x;\tau)$. They are within the error margins of Borel–Padé resummation, which are estimated by redo the resummation with 276 terms, denoted by $s'_R(\bullet)$ in the tables. The relative errors are much smaller than $|\tilde{q}^{\pm 1}|, |\tilde{x}^{\pm 1}|$, possible sources of additional corrections

$$\begin{aligned}
\widetilde{M}_{II}(x;q)_{2,1} =& -x-1+(1+2x^{-1}+x^{-2})q+(1+x^{-1}-x^{-2}-2x^{-3})q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{2,2} =& 1-(x^{-1}+x^{-2})q-(x^{-1}-x^{-3})q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{2,3} =& -x+q+(1-x^{-2})q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{3,1} =& -1-x^{-1}+(x^2+2x+1)q-(2x^3+x^2-x-1)q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{3,2} =& 1-(x^2+x)q+(x^3-x)q^2+O(q^3),\\
\widetilde{M}_{II}(x;q)_{3,3} =& -x^{-1}+q-(x^2-1)q^2+O(q^3).
\end{aligned} \tag{320}$$

They are in agreement with (304a),(304b). More decisively, we can compare the numerical evaluation of both sides of the equations of holomorphic lifts (317a),(317b). We find the relative difference between the two sides is always within the error margin of Borel–Padé resummation, and much smaller than $\tilde{q}^{\pm 1}, \tilde{x}^{\pm 1}$, possible sources of additional corrections. We illustrate this by one example with $x = 6/5$ and $\tau = \frac{1}{18}e^{\frac{\pi i}{12}}, \frac{1}{18}e^{\frac{11\pi i}{12}}$ in regions *I*, *II* in Table 2. Finally, we can test the resurgent data by checking that in the $x \mapsto 1$ limit the Stokes matrices (305a), (305b) reduce properly to (95a), (95b). This is a non-trivial test as $W_{-1}(x;q^{-1})$ ($|q| < 1$) in (305a), (305b) itself diverges in the limit $x \mapsto 1$.

7 One-dimensional state-integrals and their descendants

In a sense, the results of our paper are not about the asymptotics and resurgence of complex Chern–Simons theory, but rather involve power series and q-difference equations that arise from K_2-Lagrangians (clearly advocated in Kontsevich's talks [50]), or from symplectic matrices (advocated in [45, Sec.7]). The connection with complex Chern–Simons theory comes via ideal triangulations of a 3-manifold with torus boundary components, a concept introduced by Thurston for the study of complete hyperbolic structures and their deformations [65]. The gluing equations of such triangulations are encoded by matrices which are the upper half of a symplectic matrix (see Neumann-Zagier [61]). The upper half of these symplectic matrices define state-integrals, as well as the asymptotic series $\Phi(x;\tau)$ (this was the approach taken in [16]) and the 3D-index (see [18]).

In this section, we discuss briefly general one-dimensional state-integrals and their descendants, and their asymptotic series. We will not aim for maximum generality, but

instead consider the one-dimensional state-integral

$$Z_{A,r}(u,t;\tau) = \int_{\mathbb{R}+\mathrm{i}0} \left(\prod_{j=1}^{r} \Phi_{\mathsf{b}}(v+u_j) \right) \mathrm{e}^{-A\pi \mathrm{i} v^2 + 2\pi \mathrm{i} v t} \mathrm{d}v, \tag{321}$$

where $u = (u_1, \dots, u_r) \in \mathbb{C}^r$ with $|\mathrm{Im} u_j| < |\mathrm{Im} c_{\mathsf{b}}|$ (this ensures that all poles of the integrand are above the real axis), $t \in \mathbb{C}$ and A and r integers with $r > A > 0$. (This ensures that the integrand decays exponentially at infinity, and hence, the integral is absolutely convergent.) We have already encountered two special cases in Equations (125) and (240):

$$\mathbf{4}_1 : (A,r) = (1,2), \ (u_1,u_2) = (u,0), \ w = -2u, \tag{322}$$

$$\mathbf{5}_2 : (A,r) = (2,3), \ (u_1,u_2,u_3) = (0,u,-u), \ w = 0. \tag{323}$$

We are interested in the descendants of Z defined by

$$z_{A,r,m,\mu}(u,t;\tau) = (-1)^{m+\mu} q^{-\frac{1}{2}m} \tilde{q}^{-\frac{1}{2}\mu} Z_{A,r}(u, t - m\mathsf{i}\mathsf{b} + \mu \mathsf{i}\mathsf{b}^{-1}; \tau) \tag{324}$$

for integers m and μ, where the extra factor was inserted to simplify the formulas below.

We will express the factorization of the state-integral (324) in terms of the auxiliary function

$$G_{A,r}(y,z;q) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^{An} q^{\frac{A}{2}n(n+1)} z^n \prod_{j=1}^{r} (q^{1+n} y_j; q)_\infty \tag{325}$$

for $y = (y_1, \dots, y_r)$ and its specialization

$$b_m^{(k)}(x,w;q) = \frac{1}{x_k^m} G_{A,r}\left(\frac{1}{x_k} x, x_k^{-A} w^{-1} q^m; q \right) q \tag{326}$$

for $x = (x_1, \dots, x_r)$ and its renormalization

$$B_m^{(k)}(x,w;q) = \theta(-q^{-1/2}x_k;q)^{-A+1} \theta(-q^{-1/2}x_k w;q)^{-1} \theta(w;q) b_m^{(k)}(x,w;q) \tag{327}$$

for $k = 1, \dots, r$.

Theorem 28 (a) *The descendant state-integral can be expressed in terms of the descendant holomorphic blocks by*

$$z_{A,r,m,\mu}(u,t;\tau) = B_{-\mu}(\tilde{x}, \tilde{w}; \tilde{q}^{-1})^T \Delta(\tau) B_m(x,w;q), \qquad (m, \mu \in \mathbb{Z}) \tag{328}$$

where

$$x_j = \mathrm{e}^{2\pi \mathsf{b} u_j}, \quad \tilde{x}_j = \mathrm{e}^{2\pi \mathsf{b}^{-1} u_j}, \quad w = \mathrm{e}^{2\pi \mathsf{b} t}, \quad \tilde{w} = \mathrm{e}^{2\pi \mathsf{b}^{-1} t}. \tag{329}$$

and

$$\Delta(\tau) = \mathrm{e}^{-\frac{\mathrm{i}\pi}{4} + \frac{(A-2)\mathrm{i}\pi}{12}(\tau + \tau^{-1})} \mathbf{1} \tag{330}$$

and $B_m(x,w;q) = (B_m^{(1)}(x,w;q),\dots,B_m^{(r)}(x,w;q))^T$. *Consider the matrix* $W_m(x,w;q)$ *defined by*

$$W_m(x,w;q) = \begin{pmatrix} B_m^{(1)}(x,w;q) & \dots & B_m^{(r)}(x,w;q) \\ \vdots & & \vdots \\ B_{m+r-1}^{(1)}(x,w;q) & \dots & B_{m+r-1}^{(r)}(x,w;q) \end{pmatrix} \tag{331}$$

(b) *The entries of* $W_m(x,w;q)$ *are holomorphic functions of* $|q| \neq 1$ *and meromorphic functions of* $(x,w) \in (\mathbb{C}^*)^r \times \mathbb{C}^*$ *with poles in* $x_j \in q^{\mathbb{Z}}$ *(for* $j = 1,\dots,r$*) and* $w \in q^{\mathbb{Z}}$ *of order at most* r.

(c) *The columns of* $W_m(x,w;q)$ *are a basis of solutions of the linear* q*-difference equation* $\widehat{B}(S_m, q^m, x, w, q) f_m(x;q) = 0$ *for* $|q| \neq 1$ *and* $m \in \mathbb{Z}$ *where*

$$\widehat{B}(S_m, q^m, x, w, q) = \prod_{k=1}^{r}(1 - x_i S_m) - (-q)^A w^{-1} q^m S_m^A. \tag{332}$$

In particular, $m \mapsto z_{A,r,m,\mu}(u,t;\tau)$ *is annihilated by the operator* $\widehat{B}(S_m, q^m, x, w, q)$.

Proof For part (a), summing up all the residues of the integrand of (321) in the upper half plane as in [33], we find that

$$z_{A,r,m,\mu}(u,t;\tau) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{12} + \frac{1}{3}\pi \mathrm{i} c_{\mathrm{b}}^2 + 2\pi \mathrm{i} c_{\mathrm{b}} t} \sum_{k=1}^{r} \mathrm{e}^{-A\pi \mathrm{i}(u_k - c_{\mathrm{b}})^2 - 2\pi \mathrm{i} u_k t} b_m^{(k)}(x,w;q) b_\mu^{(k)}(\tilde{x},\tilde{w};\tilde{q}^{-1}) \tag{333}$$

$$= \mathrm{e}^{-\frac{\pi \mathrm{i}}{12} + \frac{1}{3}\pi \mathrm{i} c_{\mathrm{b}}^2 + \frac{\pi \mathrm{i}}{12}(A-1)(\tau + \tau^{-1})} \sum_{k=1}^{r} B_m^{(k)}(x,w;q) B_\mu^{(k)}(\tilde{x},\tilde{w};\tilde{q}^{-1}) \tag{334}$$

The last equation follows from (111a) (which takes care of $\mathrm{e}^{-A\pi \mathrm{i}(u_k - c_{\mathrm{b}})^2}$) and (111b) (which takes care of the t-terms under the assumption that $u_k t = \rho_k u$ and $t = \rho u$ for integers ρ and ρ_k)

For part (b), note that $G_{A,r}(y,z;q)$ is symmetric with respect to permutation of the coordinates of y and that the specialization to $y_r = 1$ is given by

$$G_{A,r}(y,z;q)|_{y_r=1} = \sum_{n=0}^{\infty} (-1)^{An} \frac{q^{\frac{A}{2}n(n+1)}}{(q;q)_n} z^n \prod_{j=1}^{r-1} (q^{1+n} y_j; q)_\infty, \qquad (|q| \neq 1). \tag{335}$$

It follows that $G_{A,r}(y,z;q)|_{y_r=1}$ is holomorphic for $(y,z) \in \mathbb{C}^{r-1} \times \{1\} \times \mathbb{C}$ when $|q| < 1$ and meromorphic in $(y,z) \in (\mathbb{C}^*)^{r-1} \times \{1\} \times \mathbb{C}^*$ with poles in $y_j \in q^{\mathbb{N}}$ for $j = 1,\dots,r$. Since $b_m^{(k)}(x,w;q)$ are expressed in terms of a specialization of $G_{A,r}(y,z;q)|_{y_r=1}$, part (b) follows.

Part (c) follows from Equation (326) and the fact (proven by a standard creative telescoping argument) that the function $z \mapsto G_{A,r}(y,z;q)$ is annihilated by the operator

$$\prod_{k=1}^{r}(1 - y_k L_z) - (-q)^A z L_z^A \tag{336}$$

where L_z shifts z to qz. □

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Data availability
The authors declare that most of the data supporting the findings of this study are available within the paper, with the exception of the asymptotic power series, which are available from the corresponding author on reasonable request.

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