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On finite type 3-manifold invariants II

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1. Introduction

1.1. History

The present paper is a continuation of [Oh] and [Ga], devoted to the study of finite type invariants of *oriented integral homology 3-spheres* ($\mathbb{Z}HS$ for short). Our purpose is, among other things, to relate the seemingly unrelated notions of surgical equivalence of links in S^3 ([Le1]) and the notion of finite type invariants of oriented integral homology 3-spheres, due to T. Ohtsuki [Oh]. Finite type invariants of $\mathbb{Z}HS$ were originally introduced by Ohtsuki [Oh] in his seminal paper (for a precise definition, see Definition 3.2). There are at least two sources of motivation/analogy/inspiration: the finite type knot invariants (see [B-N1], [BL], [Va]) and Chern-Simons theory in 3 dimensions (see [Wi2], [Rz1], [Rz2]).

Recall that a type m (otherwise called Vassiliev) invariant of knots in S^3 (for a precise definition see the references above) satisfies a difference formula with respect to cutting along $m+1$ spheres, twisting and gluing back. Vassiliev invariants of knots in S^3 form a unifying way of thinking about the Alexander, Jones, HOMFLY, Kauffman (and others) polynomials of knots.

Similarly, a type m invariant of $\mathbb{Z}HS$ satisfies a "difference formula" with respect to cutting $m+1$ solid torii from a $\mathbb{Z}HS$, twisting them, and gluing them back. For a precise statement, see Definition 3.2. It is hoped that finite type invariants of $\mathbb{Z}HS$ will provide a unifying way of dealing with 3-manifold invariants and with Chern-Simons theory in a way analogous to the case of Vassiliev invariants.

In either of the above sources of inspiration, we can think of type m invariants of $\mathbb{Z}HS$ as "polynomials of degree m" on the infinite dimensional vector space $\mathcal{M}$. Here and in the rest of this paper $\mathcal{M}$ is the vector space (over $\mathbb{Q}$) with basis the set of $\mathbb{Z}HS$. It is an important question to ask whether finite type invariants of $\mathbb{Z}HS$ separate points in $\mathcal{M}$.

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1.2. Statement of results; plan of the proof

This paper consists of two parts. In the first part, which consists of Sect. 2 we classify pure braids and string links modulo the relation of surgical equivalence. In Sect. 2.1 we recall some definitions of string links and pure braids. In Sect. 2.2 we study a subgroup $A(F^4(n))$ of the automorphism group of the nilpotent quotient $F^4(n)$ of a free group and show in Theorem 1 how to express $A(F^4(n))$ as a semidirect product of $A(F^4(n-1))$ together with a nilpotent quotient of a free group. In Sect. 2.3 we prove that the group of surgical equivalence classes of pure braids is isomorphic to the corresponding group of string links (Theorem 2). We also provide two descriptions of the above mentioned group $P^{\mathbb{SE}}(n)$ of surgical equivalence classes of n strand pure braids: one as a semidirect product of $P^{\mathbb{SE}}(n-1)$ together with an explicit quotient of the free group and another description (Theorem 3) as a group of automorphisms of a nilpotent quotient of a free group.

In the second part which consists of Sect. 3 we apply our results on surgery equivalence to study the finite type invariants of $\mathbb{Z}HS$, originally introduced by Ohtsuki [Oh] and partially answer questions 1 and 2 from [Ga]. In Sect. 3.2 we reprove Ohtsuki's fundamental result which states that the space of type m invariants of $\mathbb{Z}HS$ is finite dimensional for every m. Our proof allows us to show a vanishing theorem (Corollary 3.8), namely that the graded space of degree m invariants of $\mathbb{Z}HS$ is zero dimensional unless m is divisible by 3. This partially answers question 1 of [Ga]. In Sect. 3.4 we study the map from knots (in S^3) to $\mathbb{Z}HS$ defined by mapping a knot K in S^3 to the $\mathbb{Z}HS$ $S^3_{K,+1}$ obtained by +1 surgery on K (see [Ro], and Sect. 3.1). We show (proposition 3.12) that type $5m+1$ invariants of $\mathbb{Z}HS$ map to type $4m$ invariants of knots, thus making progress towards answering question 2 of [Ga]. We conclude in Sect. 4 with a philosophical comment about the appearance of trivalent graphs in this paper. It turns out that both the notion of surgical equivalence of links in S^3 and the notion of finite type invariants of integral homology 3- spheres are ultimately related to invariants of vertex oriented trivalent graphs.

After the present work was completed we received a copy of [GrLi] in which they prove a very special case of Corollary 3.8 as well as a weak form of proposition 3.12.

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2. Surgical equivalence of string links and pure braids

In this section we give two different classification theorems of the group of surgical equivalence classes of string links and pure braids. As in the case of string link homotopy (see [HL]) these groups are isomorphic. These classification theorems are entirely analogous to the classical theorems of Artin [Ar] on isotopy classification of braids and the more recent results of Habegger-Lin [HL] on homotopy classification of string links. The first theorem is a recursive description of the n strand pure braid group in terms of the $n-1$ strand group and a certain nilpotent quotient of a free group. The second theorem gives an isomorphism with a certain group of automorphisms of another nilpotent quotient of a free group.

The proof of the first theorem is similar to that of [HL] while the second is rather different (*warning*: Lemma 1.9 of [HL] is not correct as stated and the proof of Theorem 1.7 of that paper seems to require some modification).

2.1. Preliminaries on surgical equivalence

We begin by recalling some definitions.

Definition 2.1. *(1) Let n be a positive integer, I be the unit interval* $[0,1]$ *and D^2 be the unit disc in the plane. An n-***component string link** *is a disjoint union of n (smooth) arcs $\sigma_1, \cdots, \sigma_n$ in the solid cylinder $I \times D^2$ so that the boundary of the i-th arc is $\partial I \times p_i$, where $\{p_i\}$ are n distinct points in D^2.*

*(2) An n-***strand pure braid** *is an n-component string link with the extra property that the tangent vector at any point of any σ_i is never horizontal (see Fig. 1).*

(3) A surgery on a string link σ produces another string link σ' as follows. Let γ be an unknotted closed curve in $I \times D^2 - \sigma$ whose linking number with each component of σ is zero. If we remove a tubular neighborhood of γ and sew it back in so that the new meridian is identified with a former longitude which links γ once, then $I \times D^2$ is converted into a new manifold Δ which is diffeomorphic to $I \times D^2$ again. We define $(I \times D^2, \sigma') = (\Delta, \sigma)$. A more concrete description of σ' is given by removing a tubular neighborhood of a disk bounded by γ and then reinserting it with a single clockwise or counterclockwise twist. We say two string links or pure braids are **surgically equivalent** *if one can be obtained from the other by a sequence of surgeries. It is clear that homotopic string links or braids (i.e., string links for which there is a homotopy h_t such that distinct components remain disjoint for all $t \in$ [0, 1] (see also [Mi1], [Mi2], [HL] and Fig. 4) are surgically equivalent. If σ, σ' are two string links, then the product string link $\sigma\sigma'$ is obtained by stacking σ on top of σ'. If $\overline{\sigma}$ is the reflection of σ about $\frac{1}{2} \times D^2$, then $\sigma\overline{\sigma}$ and $\overline{\sigma}\sigma$ are homotopic and, therefore, surgically equivalent to the trivial string link (see [HL]). Thus we obtain groups $P^{\mathbb{SE}}(n)$ (respectively, $SL^{\mathbb{SE}}(n)$) of surgical equivalence classes of n-strand pure braids (repectively, n-component string links). There is an obvious homomorphism $P^{\mathbb{SE}}(n) \to SL^{\mathbb{SE}}(n)$.*

Let $F(m)$ denote the free group on generators $\{x_1, \cdots, x_m\}$. Recall the homomorphism $\tau : F(n-1) \to P^{\mathbb{I}}(n)$ defined by inserting the n^{th} strand into a trivial $(n-1)$-strand braid. Here $P^{\mathbb{I}}(n)$ stands for the group of isotopy classes of pure braids in n strands. For any group G let G_q denote the subgroup generated

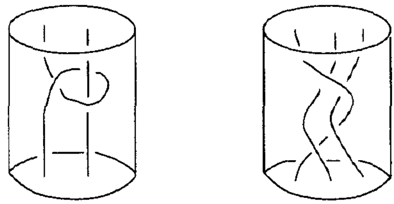

Fig. 1. A string link of 2 components (on the left) and a braid of 3 strands on the right

by all commutators of order q, with the convention that G_1 is equal to G and $G_2 = [G, G]$. Let G^q denote the quotient group G/G_q.

Claim 2.2. *For any $w \in F(n-1)$ the surgical equivalence class of $\tau(w)$ depends only on the image of w in $F^3(n-1)$.*

This follows directly from [Le1].

Thus we have a homomorphism $\tau : F^3(n-1) \longrightarrow P^{\mathbb{SE}}(n)$. We also recall the obvious split epimorphisms $P^{\mathbb{SE}}(n) \longrightarrow P^{\mathbb{SE}}(n-1)$ and $SL^{\mathbb{SE}}(n) \longrightarrow SL^{\mathbb{SE}}(n-1)$ defined by deleting the n^{th} strand. Let $\tau' : F^3(n-1) \longrightarrow SL^{\mathbb{SE}}(n)$ be the composition $F^3(n-1) \xrightarrow{\ \tau\ } P^{\mathbb{SE}}(n) \longrightarrow SL^{\mathbb{SE}}(n)$.

2.2. *The study of $A(F^4(n))$*

In this section we study a subgroup $A(F^4(n))$ of the group of automorphisms of $F^4(n)$ and prove Theorem 1, the key ingredient in determining the structure of the groups $P^{\mathbb{SE}}(n)$ and $SL^{\mathbb{SE}}(n)$ in the next section.

First consider all automorphisms α of $F^q(n)$ (for $q \geq 2$) which send each x_i to a conjugate of itself. In fact, for any sequence of elements $g_1, \cdots, g_n$ of $F(n)$, there is an automorphism α of $F^q(n)$ such that $\alpha(x_i) = g_i x_i g_i^{-1}$. These equations only define endomorphisms of $F(n)$, in general, but always define automorphisms of $F^q(n)$ for any $q \geq 2$. It is easy to see that α depends only on the class of the $\{g_i\}$ in $F^{q-1}(n)$. If we demand that g_i is *i-reduced*, i.e. the exponent sum of x_i in g_i is zero, then the $g_i \in F^{q-1}(n)$ are determined by α. We define $A(F^q(n))$ to be the subgroup of all such α satisfying the additional property:

$$\alpha(x_1 \ldots x_n) = x_1 \ldots x_n \tag{1}$$

The following lemma will be useful in our study of the group $A(F^4(n))$.

Lemma 2.3. *(a) If $\lambda \in F_2^4(n) \cap \langle x_n \rangle$, then there exist unique elements $\lambda_i \in F^3(n) \cap \langle x_n \rangle$ such that*

$$[x_1, \lambda_1] \cdots [x_n, \lambda_n] = \lambda \tag{2}$$

where $\langle x_n \rangle$ denotes the normal closure of x_n in $F_2^4(n) = F_2(n)/F_4(n)$.

(b) Suppose $\lambda_1, \cdots, \lambda_n \in F^3(n-1)$ *satisfy*

(3) $$\lambda_1 x_1 \lambda_1^{-1} \cdots \lambda_{n-1} x_{n-1} \lambda_{n-1}^{-1} = x_1 \cdots x_{n-1} \quad \text{in } F^4(n-1)$$

Then there exists unique $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n \in F^3(n)$ *such that* $\tilde{\lambda}_i \equiv \lambda_i \bmod \langle x_n \rangle$, $\tilde{\lambda}_n$ *is n-reduced and*

(4) $$\tilde{\lambda}_1 x_1 \tilde{\lambda}_1^{-1} \cdots \tilde{\lambda}_n x_n \tilde{\lambda}_n^{-1} = x_1 \cdots x_n \quad \text{in } F^4(n)$$

Remark 2.4. This lemma is *false* if we replace $F^4(n)$ and $F^3(n)$ by $F^q(n)$ and $F^{q-1}(n)$ if $q > 4$. Also, note that, in (b), $\tilde{\lambda}_i$ is i-reduced if and only if λ_i is i-reduced.

Proof. To prove part (a) recall the notion of a Hall basis (see [MKS]). In particular, any element of $F^4(n)$ can be uniquely written in the following form:

(5) $$x_1^{e_1} \cdots x_n^{e_n} \prod_{i<j} [x_i, x_j]^{e_{ij}} \prod_{\substack{i<j \\ k \leq j}} [x_k, [x_i, x_j]]^{e_{ijk}}$$

Thus we may uniquely write:

(6) $$\lambda = \prod_{i,n} [x_i, x_n]^{e_i} \prod_{\substack{i<n \\ j \leq n}} [x_j, [x_i, x_n]]^{e_{ij}}$$

Therefore, equation 2 has a solution as follows:

(7) $$\lambda_j = x_n^{e_j} \prod_{i<n} [x_i, x_n]^{e_{ij}}$$

We have set $e_n = 0$. Since any allowable solution λ_j has such a representation, uniqueness follows also.

In order to show part (b), we observe that any acceptable solution $\{\tilde{\lambda}_i\}$ of equation 4 can be written in the form $\tilde{\lambda}_i = \lambda_i \alpha_i x_n^{e_i}$ for some $\alpha_i \in F_2^3(n)$, with $e_n = 0$. Substituting this into equation 4 with a little bit of commutator manipulation and using the fact that $F_3^4(n)$ is the center of $F^4(n)$, we obtain the following formula in $F^4(n)$:

(8) $$\Big(\prod_{i=1}^{n-1} [\lambda_i, [x_n^{e_i}, x_i]]\Big)\Big(\prod_{i=1}^{n} [\alpha_i, x_i]\Big) \prod_{i=1}^{n} ([x_n^{e_i}, x_i] \lambda_i x_i \lambda_i^{-1}) = \prod_{i=1}^{n} x_i$$

Reducing to $F^3(n)$ we obtain the much simpler formula:

(9) $$\Big(\prod_{i=1}^{n-1} [x_n, x_i]^{e_i}\Big) \prod_{i=1}^{n} \lambda_i x_i \lambda_i^{-1} = \prod_{i=1}^{n} x_i$$

Substituting from equation 3 this becomes:

$$\prod_{i=1}^{n-1}[x_n, x_i]^{e_i} = [x_n, \lambda_n] \tag{10}$$

and so λ_n determines the $\{e_i\}$. Now that the $\{e_i\}$ are known, equation 8 can be written as $\prod_{i=1}^{n}[\alpha_i, x_i] = \tau$, where τ is a specified element of $F^4(n) \cap \langle x_n \rangle$. It follows from (a) that there is a unique solution for α_i in $F^3(n) \cap \langle x_n \rangle$. The uniqueness implies that $\alpha_i \in F_2^3(n)$ since, as can be easily checked, $\tau \in F_3^4(n)$. □

We can now use part (b) of Lemma 2.3 to define a homomorphism $\overline{\tau} : F^3(n-1) \to A(F^4(n))$ as follows. $\overline{\tau}(w)$ is the unique element $\alpha \in A(F^4(n))$ satisfying $\alpha(x_i) \equiv x_i \bmod \langle x_n \rangle$ for $i < n$ and $\alpha(x_n) = \tilde{w} x_n \tilde{w}^{-1}$ for some $\tilde{w} \equiv w \bmod \langle x_n \rangle$.

Combining this with the restriction $A(F^4(n)) \to A(F^4(n-1))$ we have the following result

Theorem 1. *The following sequence is split exact:*

$$1 \longrightarrow F^3(n-1) \xrightarrow{\overline{\tau}} A(F^4(n)) \longrightarrow A(F^4(n-1)) \longrightarrow 1$$

Remark 2.5. This theorem is *false* if we replace $F^4(n), F^4(n-1)$ and $F^3(n-1)$ by $F^q(n), F^q(n-1)$ and $F^{q-1}(n-1)$, respectively, if $q > 4$. For example consider the automorphism α of $F^5(2)$ defined by:

$$\alpha(x_1) = [x_2, [x_1, x_2]]x_1[x_2, [x_1, x_2]]^{-1} \tag{11}$$

$$\alpha(x_2) = [x_1, [x_2, x_1]]x_2[x_1, [x_2, x_1]]^{-1} \tag{12}$$

Proof of Theorem 1. This is an immediate consequence of part (b) of Lemma 2.3. □

Our next goal is to define a homomorphism $SL^{\mathbb{SE}}(n) \to A(F^4(n))$. First recall the definition of the $\overline{\mu}$-invariants of string links as formulated, for example, in [Le2]. Let σ be an n-string link; we define certain canonical *meridian* elements of $\pi(\sigma) = \pi_1(I \times D^2 - \sigma)$. Let m_i be a small circle in $0 \times D^2$ with $0 \times p_i$ as center. Let s_i be a straight line from a base point p in $0 \times (D^2 - \{p_i\})$ to m_i. Now let μ_i be the element of $\pi(\sigma)$ represented by $s_i m_i s_i^{-1}$. Let $\mu : F(n) \to \pi(\sigma)$ be defined by $\mu(x_i) = \mu_i$. It follows from a theorem of Stallings [Sta] that μ induces an isomorphism $F^q(n) \to \pi^q(\sigma)$ for each $q \geq 3$. We can also define canonical *longitude* elements of $\pi(\sigma)$. Let l_i be a curve on the boundary of a tubular neighborhood of σ_i which runs parallel from a point on m_i to the corresponding point on m_i', where m_i' is the projection of m_i into $1 \times D^2$. Let s_i' be the projection of s_i into $1 \times D^2$ and let $u = I \times p$ oriented from 1 to 0. Define $\tilde{\lambda}_i \in \pi$ to be represented by $s_i l_i (s_i')^{-1} u$ (see Fig. 2). Requiring that $\tilde{\lambda}_i$ have linking number 0 with σ_i determines l_i and, hence, $\tilde{\lambda}_i$. We now define $\lambda_i(\sigma) \in F^q(\sigma)$ to be the element (which is i-reduced) corresponding to $\tilde{\lambda}_i$ under the isomorphism induced by μ. These elements are invariants of the concordance class of σ. For a reference on concordance class of string links see [Li] and references therein.

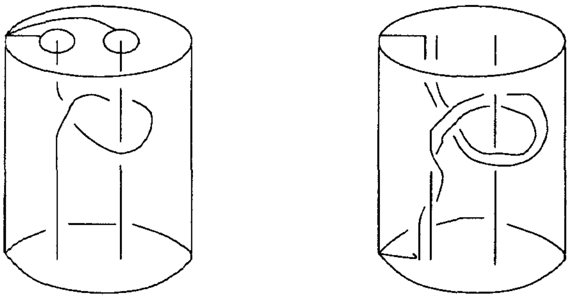

Fig. 2. On the left are shown the elements m_i, s_i (for $i = 1, 2$) and on the right is shown the element $\tilde{\lambda}_i$ of a string link of two components

Proposition 2.6. *The surgical equivalence class of σ determines the set $\{\lambda_i(\sigma)\} \subset F^3(n)$.*

Proof. This is essentially proved in [Le1] but for completeness we give a more direct argument here. Let σ be an n-strand string link and γ a simple closed curve in the complement of σ. We assume that the linking numbers of any two components of σ and any component of σ with γ is zero. Let $(I \times D^2, \sigma') = (\Delta, \sigma)$ denote the surgically equivalent link obtained by Dehn surgery along γ as in Definition 2.1. Let $X = (I \times D^2) - \sigma - \gamma$ and let

$$\pi = \pi_1(X), \quad \pi(\sigma) = \pi_1((I \times D^2) - \sigma), \quad \pi(\sigma') = \pi_1(\Delta - \sigma) = \pi_1((I \times D^2) - \sigma')$$

If m is a meridian element and l a longitude element for γ in π, then we have:

$$\pi(\sigma) \cong \pi/\langle m \rangle \quad \text{and} \quad \pi(\sigma') \cong \pi/\langle ml^{\pm 1} \rangle$$

We define homomorphisms:

$$F(n+1) \xrightarrow{\hat{\mu}_1} \pi, \quad F(n+1) \xrightarrow{\hat{\mu}_2} \pi, \quad F(n) \xrightarrow{\mu_1} \pi(\sigma),$$
$$F(n) \xrightarrow{\mu_2} \pi(\sigma')$$

all of which send x_i to a meridian of σ, σ', for $i \leq n$, and, in addition, $\hat{\mu}_1(x_{n+1}) = m$, $\hat{\mu}_2(x_{n+1}) = ml^{\pm 1}$. We have the following commutative diagram:

$$\begin{tikzcd}
\pi \arrow[d] & F(n+1) \arrow[l, "\hat{\mu}_2"'] \arrow[r, "\hat{\mu}_1"] \arrow[d] & \pi \arrow[d] \\
\pi(\sigma') & F(n) \arrow[l, "\mu_2"'] \arrow[r, "\mu_1"] & \pi(\sigma)
\end{tikzcd}$$

Here $F(n+1) \to F(n)$ is the obvious reduction defined by sending x_{n+1} to 1. Using results of Milnor [Mi1], [Mi2] and Stallings [Sta] we see that μ_1 and μ_2 induce isomorphisms $\mu_1^3 : F^3(n) \xrightarrow{\cong} \pi^3(\sigma)$ and $\mu_2^3 : F^3(n) \xrightarrow{\cong} \pi^3(\sigma')$, respectively, while $\hat{\mu}_1$ and $\hat{\mu}_2$ are epimorphisms. Furthermore, there are elements

λ and λ' in $F^3(n+1)$ such that the kernels of $\hat{\mu}_1$ and $\hat{\mu}_2$ (in $F^3(n+1)$) are normally generated, respectively, by $[x_{n+1}, \lambda]$ and $[x_{n+1}, \lambda']$. In particular, since $\lambda, \lambda' \in F_2(n+1)$, μ_1 and μ_2 induce isomorphisms $F^3(n) \xrightarrow{\cong} \pi^3$. If $\lambda_i, \lambda'_i \in F^3(n)$ are chosen to map to the longitude of σ_i and σ'_i in $\pi^3(\sigma)$ and $\pi^3(\sigma')$, respectively, then they reduce in $F^3(n)$ to $\lambda_i(\sigma)$ and $\lambda_i(\sigma')$.

Since $\lambda_i, \lambda'_i \in F^3_2(n)$, we only need prove that $\hat{\mu}_1$ and $\hat{\mu}_2$ agree on $F^3_2(n+1)$. But this will follow if $\hat{\mu}_2 \equiv \hat{\mu}_1 \bmod \pi^3_2$, which is clear since $l \in \pi_2$.

Now, we can introduce the following definition:

Definition 2.7. *Let $\Phi_n : SL^{\mathbb{SE}}(n) \to A(F^4(n))$ be the homomorphism defined by*

$$\Phi_n(\sigma)(x_i) = \lambda_i(\sigma) x_i \lambda_i(\sigma)^{-1} \tag{13}$$

To see that this is a homomorphism we give another interpretation of Φ_n. Let $X = I \times \partial D^2 \cup 1 \times (D^2 - \{p_i\}) \subseteq I \times D^2$. Then we have the following homomorphisms induced by inclusion maps:

$$\pi_1(0 \times (D^2 - \{p_i\})) \xrightarrow{\mu} \pi_1((I \times D^2) - \sigma) \xleftarrow{\mu'} \pi_1(X)$$

We identify the first and last groups with $F(n)$ equating x_i with the homotopy class of $s_i m_i s_i^{-1}$ and $u^{-1} s'_i m'_i (s'_i)^{-1} u$, respectively. Now Stallings Theorem tells us that the maps in equation 2.7 become isomorphisms when we pass to any lower central series quotient. Thus the composition of the maps in 2.7 define an automorphism of $F^3(n)$; we leave it as an exercise for the reader to show that this is $\Phi_n(\sigma)$. From this formulation, it is clear that Φ_n is a homomorphism.

2.3. Determining the groups $P^{\mathbb{SE}}(n)$ and $SL^{\mathbb{SE}}(n)$

In this section we determine the groups $P^{\mathbb{SE}}(n)$ and $SL^{\mathbb{SE}}(n)$.

Theorem 2. *(a) $P^{\mathbb{SE}}(n) \to SL^{\mathbb{SE}}(n)$ is an isomorphism.*
(b) The diagram:

$$\begin{tikzcd}
1 \arrow[r] & F^3(n-1) \arrow[r, "\tau"] \arrow[d, equal] & P^{\mathbb{SE}}(n) \arrow[r] \arrow[d] & P^{\mathbb{SE}}(n-1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F^3(n-1) \arrow[r, "\tau'"] & SL^{\mathbb{SE}}(n) \arrow[r] & SL^{\mathbb{SE}}(n-1) \arrow[r] & 1
\end{tikzcd}$$

is commutative and the rows are split short exact.

The exactness and commutativity in (b) is clear except for the injectivity of τ and τ'. This will be proved in Theorem 3 below. Clearly (a) will follow from (b).

Theorem 3. *Φ_n is an isomorphism.*

Proof of Theorems 2 and 3. We can combine the statements of Theorems 1, 2 and 3 into a single commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & F^3(n-1) \arrow[r, "\tau"] \arrow[d, equal] & P^{\mathbb{SE}}(n) \arrow[r] \arrow[d] & P^{\mathbb{SE}}(n-1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F^3(n-1) \arrow[r, "\tau'"] \arrow[d, equal] & SL^{\mathbb{SE}}(n) \arrow[r] \arrow[d] & SL^{\mathbb{SE}}(n-1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F^3(n-1) \arrow[r, "\overline{\tau}"] & A(F^4(n)) \arrow[r] & A(F^4(n)) \arrow[r] & 1
\end{tikzcd}$$

The commutativity of this diagram is clear. To establish that the first two rows are exact we need only confirm the injectivity of τ and τ'. But this now follows from the injectivity of $\overline{\tau}$. This concludes the proofs of Theorems 2 and 3. □

We can now deduce the following corollary to Theorem 3:

Corollary 2.8. *$P^{\mathbb{SE}}(n)$ is a nilpotent group of class two, i.e., with the notation of Sect. 2.1 we have that $P_3^{\mathbb{SE}}(n) = 1$. Furthermore, $P^{\mathbb{SE}}(n)/P_2^{\mathbb{SE}}(n)$ is a free abelian group in $\binom{n}{2}$ generators and $P_2^{\mathbb{SE}}(n)$ is a free abelian group in $\binom{n}{3}$ generators. In fact, $P_2^{\mathbb{SE}}(n)$ is a free abelian group on generators α_I for $I \in I_n := \{(i,j,k) | 1 \leq i,j,k \leq n\}$ (where all i,j,k are distinct) with identifications $\alpha_I = sgn(\sigma)\alpha_{\sigma(I)}$, where σ is any permutation of I. The element α_I can be represented, if $i < j < k$, by the braid B_I in Fig. 3.*

Remark 2.9. The relation of surgical equivalence on string links and pure braids is generated by the local moves of Figs. 4 and 5. This is proved in [Le1] for closed links but the proof is the same for string links.

3. Relations with finite type invariants of $\mathbb{Z}HS$

3.1. Preliminaries on finite type 3-manifold invariants

As a motivation to the notion of finite type 3-manifold invariants, let us recall the definition of type m Vassiliev invariants of (oriented) knots in (oriented) S^3 (after [B-N1], [BL], [Va]): V is a *type m* invariant of knots if for every knot K in S^3 and every choice $B_1, \dots, B_{m+1}$ of embedded 3-balls that intersect the knot as in Fig. 6 (the balls appear in the form of solid cylinders) we have that

$$\sum_{\epsilon_i \in \{0,1\}} \prod_{i=1}^{m+1} (-1)^{\epsilon_i} V(K_{\epsilon_1,\dots,\epsilon_{m+1}}) = 0 \tag{14}$$

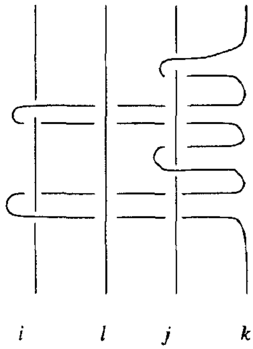

Fig. 3. A 4 strand braid $B_{i,j,k} = B_1 \cup B_2$ representing $\alpha_{i,j,k}$. Here B_1 consists of the 3 strands i, j, k and B_2 consists of all the rest strands (here only one). Note that the strands of B_2 are on top of the i, j, k strands, and that the closure of the 3 strand braid B_2 is a Borromean link of 3 components and that the link represented by the closure of the i, j strands is trivial. Furthermore, the longitude l_k of the k^{th} strand represents the element $[\mu_i, \mu_j]$ in the fundamental group of the complement of the link consisting of the i, j strands, where $\{\mu_j\}$ are the canonical free generators of the fundamental group of the complement of the i, j strands

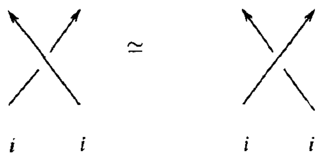

Fig. 4. A local move that generates the equivalence relation of string link homotopy. Here arcs labeled by the same letter (i in this figure and i, j in the next figure) belong to the same link component

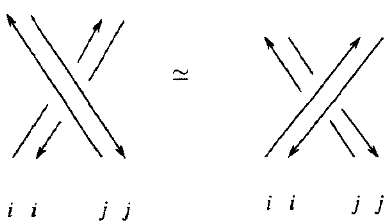

Fig. 5. A local move that is implied by the relation of surgical equivalence of string links

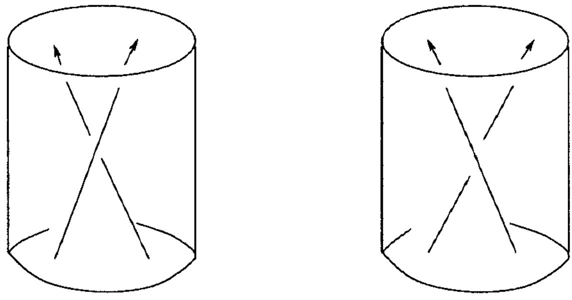

Fig. 6. An embedded 3-ball in S^3 intersecting a knot as in the left part of the figure. On the right is the result of reinserting the ball with a full counterclockwise twist about the vertical axis

where $K_{\epsilon_1,\dots,\epsilon_{m+1}}$ is the knot obtained by removing $B_1,\dots,B_{m+1}$ from S^3, twisting every B_i ϵ_i times as in Fig. 6 and gluing back.

Let $\mathscr{F}_m\mathscr{V}$ be the space of type m knot invariants, and let $\mathscr{F}_\star\mathscr{V} := \cup_{m\geq 0}\mathscr{F}_m\mathscr{V}$ be the space of finite type knot invariants. It is easy to show that $\mathscr{F}_\star\mathscr{V}$ is a filtered commutative algebra (with pointwise multiplication). Let $\mathscr{G}_\star\mathscr{V}$ denote the associated graded algebra (and more generally, let $\mathscr{G}_\star(obj)$ denote the associated graded object of the filtered object $\mathscr{F}_\star(obj)$) and let $\mathscr{W}_m$ denote the space of weight systems of degree m (i.e. linear functionals on the space of chord diagrams with m chords modulo $4T$ relations and framing independence relation, see references above). Then it is easy to show that there is a map $\mathscr{F}_m\mathscr{V} \to \mathscr{W}_m$ with kernel $\mathscr{F}_{m-1}\mathscr{V}$. Since $\mathscr{W}_m$ is *a priori* finite dimensional, so is $\mathscr{F}_m\mathscr{V}$.

Before we talk about 3-manifold invariants, let us establish some useful notation: Let $\mathscr{M}$ denote the vector space (over $\mathbb{Q}$) on the set of *oriented integral homology 3-spheres* ($\mathbb{Z}HS$ for short). A link $L \subseteq M$ in a $\mathbb{Z}HS$ is called *algebraically split* if the linking numbers between any two components vanish. A framing $\mathbf{f} = (f_1,\dots,f_n)$ for an n component link is a sequence of integers associated to each component.

Remark 3.1. A framing for a link L corresponds to a choice of *longitudes* for each component. If N_i is a tubular neighborhood of the component L_i, then the associated longitude γ_i on ∂N_i is required to be homologous to L_i in N_i and to have linking number f_i with L_i. To make sense of this we must impose an orientation on L_i, but then it is easy to see that γ_i is independent of this choice. Note that linking numbers make sense in any ZHS.

A framed link $(L,\mathbf{f})$ in a $\mathbb{Z}HS$ M is called *unimodular* if $f_i = \pm 1$ for all i. A framed link $(L,\mathbf{f})$ is called *AS-admissible* if it is algebraically split and unimodular. For every framed link $(L,\mathbf{f})$ in M we denote by $M_{L,f}$ the result of doing *Dehn surgery* on (L,f) in M [Ro], i.e. remove a tubular neighborhood N_i of each L_i and sew it back in so that γ_i now bounds a disk in N_i. For a framed link $(L,\mathbf{f})$ in M we denote

(15) $$[M, L, f] := \sum_{L' \subseteq L} (-1)^{|L'|} M_{L', f|_{L'}} \in \mathcal{M}$$

Recall that $\mathcal{M}$ is the $\mathbb{Q}$ vector space on the set of $\mathbb{Z}HS$, and that if (L, f) is a framed link in a $\mathbb{Z}HS$ M, then (L, f) is AS- admissible if and only if $M_{L', f|_{L'}}$ is a $\mathbb{Z}HS$ for every sublink L' of L.

In case L is an AS-admissible link in S^3 with framing +1 on each component, we will denote

(16) $$[L] = [S^3, L, \{+1, \cdots, +1\}]$$

Let us define a decreasing filtration $\mathcal{F}_\star^{Oh}$ on the vector space $\mathcal{M}$ as follows: $\mathcal{F}_m^{Oh}\mathcal{M}$ is the subspace spanned by $[M, L, \mathbf{f}]$ for all AS-admissible m component links in any $\mathbb{Z}HS$ M.

It is an immediate consequence of equation 15 that $\mathcal{M}$ is the vector space generated by all triples $[M, L, f]$ subject to the *fundamental relation*:

(17) $$[M, L, f] = [M, L', f|L'] - [M_{(l, f|_l)}, L', f|L']$$

where L' is any sublink of L obtained by removing a component l. If l bounds a disk D in M, then $M_{(l, f|_l)} \cong M$ and we may construct the link in M corresponding to L in $M_{(l, f|_l)}$ from L by just giving the bundle of strands of L which pass through D a full clockwise twist if $f = +1$ or counterclockwise twist if $f = -1$.

We can now recall the following definition from [Oh]:

Definition 3.2. *λ is a* **type** *m invariant of $\mathbb{Z}HS$ (with values in $\mathbb{Q}$) if $\lambda(\mathcal{F}_{m+1}^{Oh}\mathcal{M}) = 0$ i.e., if for every AS-admissible link L of $m+1$ components in a $\mathbb{Z}HS$ M we have that*

(18) $$\sum_{L' \subseteq L} (-1)^{|L'|} \lambda(M_{L', f|_{L'}}) = 0$$

Let $\mathcal{F}_m\mathcal{O}$ denote the vector space of type m invariants of $\mathbb{Z}HS$, and let $\mathcal{F}_\star\mathcal{O}$ denote the union $\cup_{m \geq 0}\mathcal{F}_m\mathcal{O}$. It is easy to see that $\mathcal{F}_\star\mathcal{O}$ is a filtered commutative algebra with pointwise multiplication.

3.2. Surgical equivalence and 3-manifold invariants

In this section we link the results from Sect. 2 with the notion of finite type invariants of integral homology 3- spheres. In particular, we reprove Ohtsuki's fundamental result (Theorem 5) which states that the space of type m invariants of $\mathbb{Z}HS$ is finite dimensional for every m.

We begin by observing that every $\sigma \in P^{\mathbb{I}}(n)$ can be closed to a link $\hat{\sigma}$ of n components in S^3. Furthermore, with the notation of Corollary 2.8 we have that $\sigma \in P_2^{\mathbb{I}}(n)$ if and only if $\hat{\sigma}$ is an algebraically split link. Let us now consider the map $P_2^{\mathbb{I}}(n) \to \mathcal{M}$ defined as $\sigma \to [\hat{\sigma}]$ with the notation of equation 16. We claim that this map descends to a well defined map (*not* a group homomorphism)

$$P_2^{\mathbb{SE}}(n) \longrightarrow \mathcal{G}_n^{Oh}\mathcal{M} \tag{19}$$

Indeed, it follows from the definition of $\mathbb{SE}$ equivalence of pure braids. This extends to a linear map from the rational group ring $\mathbb{Q}[P_2^{\mathbb{SE}}(n)] \to \mathcal{G}_n^{Oh}\mathcal{M}$. By Corollary 2.8 we can regard $P_2^{\mathbb{SE}}(n)$ as a free abelian group on generators α_I for $I \in I_n := \{(i,j,k) | 1 \leq i,j,k \leq n\}$ (where all i,j,k are distinct) with identifications $\alpha_I = \mathrm{sgn}(\sigma)\alpha_{\sigma(I)}$, where σ is any permutation of I. Therefore, $\mathbb{Q}[P_2^{\mathbb{SE}}(n)]$ is the Laurent polynomial ring in the commuting variables α_I.

Remark 3.2. Using the fundamental relation 17 and the notation of equation 16, we see that if an n-component AS-admissible link L' is obtained by twisting, along an admissible unknot γ, an AS-admissible n component link L, then we have that

$$[L \cup \gamma] = [L] - [L'] \tag{20}$$

Therefore, surgical equivalence relates n component AS- admissible links in S^3 modulo $n+1$ component AS-admissible ones. Since $P^{\mathbb{SE}}(n)$ is not a finite group (instead, it is a finitely generated free abelian group) at this point it is not clear why $\mathcal{G}_n^{Oh}\mathcal{M}$ is a finite dimensional $\mathbb{Q}$ vector space.

Our next task is to introduce a finite dimensional quotient of $\mathbb{Q}[P_2^{\mathbb{SE}}(n)]$ that maps onto $\mathcal{G}_n^{Oh}\mathcal{M}$. This will involve looking at n and $n-1$ component AS-admissible links in S^3.

We begin by translating equation 17 into a *graphical form*. In Fig. 7 we give the drawing conventions for pieces of AS-admissible links in S^3.

It is easy to see that Figs. 8, 9 and 10 are special cases of the fundamental equation 17.

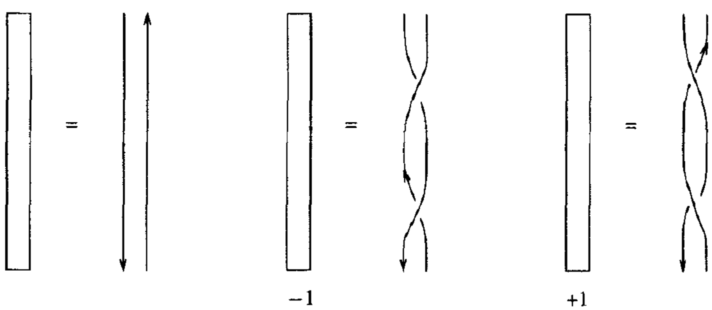

Fig. 7. Some drawing conventions for bands. Shown here are ribbon parts of AS-admissible links that represent (linear combinations of) $\mathbb{Z}HS$ under the map 15. The numbers in the bottom of each band indicate the number of twists that we put in the band

Let us now prepare some notation that will be used in Proposition 3.4 below. Let L be a $(m-1)$-component link in S^3 and α be an element of $\pi = \pi_1(S^3 - L)$. Then we can consider the m-component link $L(\alpha)$ defined by adding to L a new component which represents α. Note that $L(\alpha)$ is well-defined up to homotopy

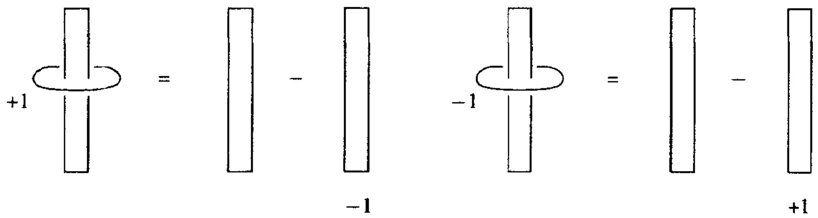

Fig. 8. A special case of equation 17 in a pictorial way

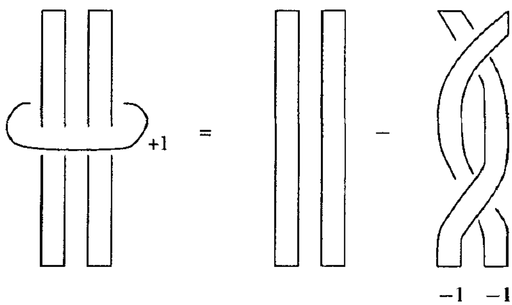

Fig. 9. Another special case of equation 17 in a pictorial way

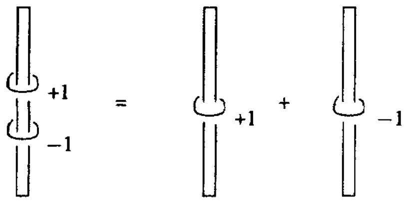

Fig. 10. Yet another special case of equation 17 in a pictorial way

and, therefore, up to surgical equivalence. Note also that $L(\alpha)$ is algebraically split if and only if L is algebraically split and $\alpha \in \pi_2$, the commutator subgroup of π.

Proposition 3.4. *Suppose that L is an algebraically split $(k-1)$- component link and $\alpha_1, \ldots, \alpha_n \in \pi_2$ and $n \geq 3$. Then, with the notation of equation 16, we have the following identities in $\mathcal{G}_k^{Oh}\mathcal{M}$:*

$$(21)\qquad [L(\alpha_1 \ldots \alpha_n)] = \sum_{1\leq i<j\leq n}[L(\alpha_i\alpha_j)] - (n-2)\sum_{1\leq i\leq n}[L(\alpha_i)]$$

$$(22)\qquad [L(\alpha^{-1})] = [L(\alpha)]$$

Proof. Equation 22 follows from the fact that surgery along a simple closed curve is independent of the orientation of the curve. Equation 21 will follow from a more general relation in $\mathcal{M}$ (Theorem 4) stated below. □

Before we state Theorem 4, we need to fix some notation: Suppose M is $\mathbb{Z}HS$ containing an AS-admissible link L with a +1-framing on each component. Let D be a 3-ball imbedded in M such that $D \cap M$ consists of n parallel bundles of strands of L, such that, among the strands of any one component of L in any of the bundles, there are an equal number going in each of the two directions. For any sequence $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, let $x_{i_1\cdots i_k}$ denote the circle in D, with a +1-framing, which encloses the $i_1 \cdots i_k$ bundles and passes *above* the other bundles (see Fig. 11) A *monomial* $x_{i_1\cdots i_k}x_{j_1\cdots j_l}\cdots x_{r_1\cdots r_s}$ will denote the union of the circles given by the terms of the monomial placed in *descending* order as read from left to right (see Fig. 12). Note that this multiplication is not commutative in general but that x_i commutes with any monomial. For any monomial m, $L(m)$ will denote the link obtained by adjoining to L the circles denoted by m. $L(m)$ will denote the link $L \subseteq M_m$ but since $M_m \cong M$ this can be alternatively described as the link in M obtained by giving a full clockwise twist to the bundles which pass through each of the components of m (see Fig. 13).

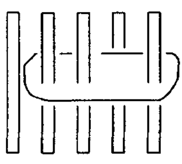

Fig. 11. A graphical representation of the monomial x_{235}

Lemma 3.5. *With the above notation we have the following identity in $\mathcal{G}_k^{Oh}\mathcal{M}$:*

$$L((x_1\cdots x_n)^{n-2}x_{1\cdots n}) = L(\prod_{1\leq i<j\leq n} x_{ij})$$

where the x_{ij} appear in lexicographic order from left to right, i.e. x_{ij} is to the left of x_{rs} if $i<r$ or $i=r$ and $j<s$.

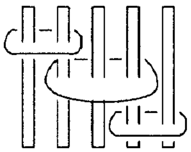

Fig. 12. A graphical representation of the product $x_{12} \cdot x_{234} \cdot x_{45}$

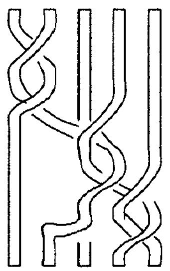

Fig. 13. The bands have $-1, -2, 0, -2, -1$ twists

Proof. The proof is a refinement and generalization of Ohtsuki's idea of resolving a full twist of three bands into a sequence of twists of pairs and individual bands. We proceed by induction on n. The result is obvious for $n = 2$. The inductive step is the equation:

$$L(x_1^{n-2} x_2 \cdots x_n x_{1\cdots n}) = L(x_{12} \cdots x_{1n} x_{2\cdots n}) \tag{23}$$

To prove this we observe that a full clockwise twist of L in D can be decomposed into a full twist of the first bundle followed by winding the first bundle around the other bundles once in a clockwise direction and finally by a full twist of the other bundles. This is illustrated in Fig. 14. By using the relation given in Fig. 15, we obtain a product of simple twists of two bundles which however force us to insert counterclockwise twists of each bundle to compensate. The result is pictured in Fig. 16. Premultiplying $x_{1\cdots n}$ by $x_1^{n-2} x_2 \cdots x_n$ eliminates the counterclockwise twists of the individual bundles and results in the desired element $L(x_{12} \cdots x_{1n} x_{2\cdots n})$. □

We want to use Lemma 3.5 to "solve" for $[M; L, x_{1\cdots n}]$. We first need another lemma.

Lemma 3.6. *Let $L' \cup L$ be an AS link in M. Then:*

$$[M_L, L'] = \sum_{L'' \subseteq L} (-1)^{|L''|} [M, L'' \cup L'] \tag{24}$$

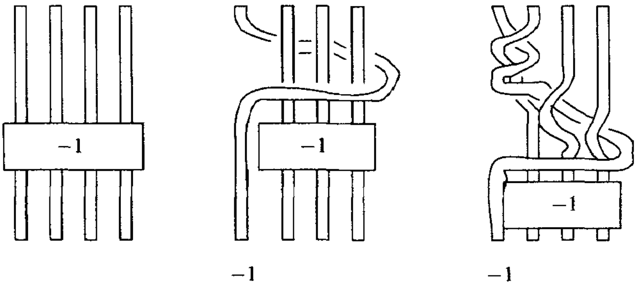

Fig. 14. A full clockwise twist on 4 bands

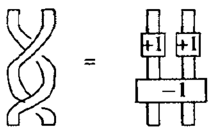

Fig. 15. A full clockwise twist on 2 bands

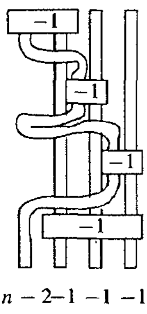

Fig. 16. More twists. Note that the number of individual twists in the bands are $n-2, -1, -1, -1$

Proof. This is essentially an inversion of equation 15. According to 15

$$[M_L, L'] = \sum_{K \subseteq L'} (-1)^{|K|} M_{K \cup L}$$

while the right hand side of equation 24 is:

$$\sum_{\substack{K' \subseteq L'' \subseteq L \\ K \subseteq L'}} (-1)^{|L''|+|K \cup K'|} M_{K \cup K'}$$

Thus it suffices to show that, for any *non-empty* link $l \subseteq M$, $\sum_{l' \subseteq l} (-1)^{|l'|} = 0$. But, if $|l|=p$, then the left-hand side of this equation is just $\sum_{i=0}^{p} \binom{p}{i} (-1)^{p-i}$, which is zero by the binomial theorem. □

We now use the following notation. If m is any monomial, then let $\overline{m} = [M; L, m]$. Now it follows from Lemma 3.6 that:

$$[M, L(m)] = \sum_{m' \leq m} (-1)^{d(m')} \overline{m}' \tag{25}$$

where m' ranges over all submonomials of m, i.e. those obtained by deleting zero or more terms from m, and $d(m')$ is the number of terms in m'. If $m = y_1 \cdots y_k$, then the right side of equation 25 is just $(1 - \overline{y_1}) \cdots (1 - \overline{y_k})$. So when $m = (x_1 \cdots x_n)^{n-2} x_{1 \cdots n}$ the right side of 25 is:

$$\prod_{i=1}^{n} (1 - \overline{x_i})^{n-2} (1 - \overline{x_{1 \cdots n}})$$

and when $m = \prod_{1 \leq i < j \leq n} x_{ij}$ the right side is $\prod_{1 \leq i < j \leq n} (1 - \overline{x_{ij}})$. Combining these observations with Lemma 3.5 proves:

Theorem 4.

$$1 - \overline{x_{1 \cdots n}} = \frac{\prod_{1 \leq i < j \leq n} (1 - \overline{x_{ij}})}{\prod_{i=1}^{n} (1 - \overline{x_i})^{n-2}} \tag{26}$$

To correctly interpret this theorem recall that x_i commutes with every monomial and then equation 26 should be viewed as an equation in the completion of $\mathcal{M}$ or in $\mathcal{M}/\mathcal{F}_k^{Oh}\mathcal{M}$ for any k. Note that the denominator is invertible.

The proof of Theorem 4, and therefore of Proposition 3.4, is now complete.

Let us now come back to the problem (stated in the beginning of the present section) of showing that $\mathcal{G}_n^{Oh}\mathcal{M}$ is a finite dimensional $\mathbb{Q}$ vector space. Motivated by Proposition3.4 and Theorem 4 we define, for each n a quotient $\mathbb{Q}[P_2^{\mathbb{SE}}(n)]/\mathcal{R}_n$. Using the notation of Corollary 2.8 we define a monomial $a = \prod_{I \in I_n} \alpha_I^{a(I)} \in P_2^{\mathbb{SE}}(n)$ to be *i-trivial* (for $1 \leq i \leq n$) if $a(I) \neq 0$ implies $i \in I$ and to be *i-disjoint* if $a(I) \neq 0$ implies $i \notin I$. (If we look at the canonical braid representative of an i-trivial monomial, it has the property that removing the i-th strand leaves a trivial braid. If we look at the canonical braid representative of an i-disjoint monomial, it has the property that the i-th strand

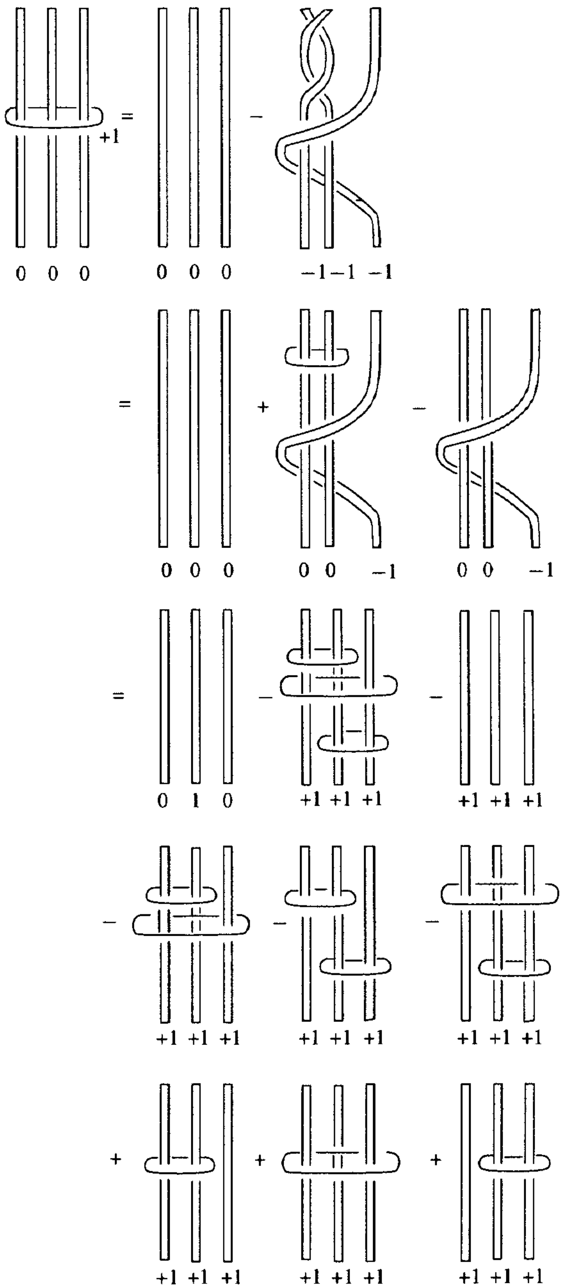

Fig. 17. The proof of Theorem 4 if $n = 3$. The two equalities here follow from repeated applications of Figs. 7 and 9. The framings in all horizontal components is +1

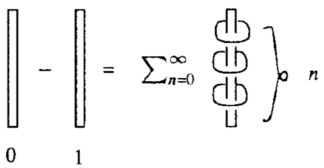

Fig. 18. The end of the proof of Theorem 4. The identity in the present figure follows from Fig. 10. The framings of the horizontal components represented by unknots is +1

separates from the remaining strands.) Now we define our set of relations to be all those obtained by taking linear combinations with rational coefficients of the following elementary relations:

(27)	$a =$	0	if a is i-disjoint for some i
(28)	$d(a-1)(b-1)(c-1) =$	$-d$	if a, b, c are i-trivial and d is i- disjoint for some i
(29)	$d(a-a^{-1}) =$	0	if a is i-trivial and d is i-disjoint for some i

We will denote by $\mathscr{R}_n$ the subspace of $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)]$ generated by equations 27, 28 and 29.

Remark 3.3. A few comments are in order: recall that Proposition 3.4 and Theorem 4 are given in *graphical notation*, whereas our relations $\mathscr{R}_n$ are given in *algebraic notation*. For a precise comparison of the relations $\mathscr{R}_n$ and the statements of Proposition 3.4 and Theorem 4 see the proof of Theorem 5.

We record here some useful particular consequences of our algebraic relations:

(30)	$1 =$	0	
(31)	$da^2 =$	$d(4a-2)$	if a is i-trivial and d is i-disjoint for some i
(32)	$d(ab^{-1}+ab) =$	$d(2a+2b-1)$	if a, b are i-trivial and d is i-disjoint for some i

Indeed, setting $a = 1$ in 27 gives 30. Setting $b = a, c = a^{-1}$ in 28 and using 27 gives 31. Setting $c = b^{-1}$ in 28 and using 27 gives 32. □

Theorem 5. *[Oh] The map $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)] \to \mathscr{G}_n^{Oh}\mathscr{M}$ is onto. Furthermore, it factors through a (necessarily onto) map $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)]/\mathscr{R}_n \to \mathscr{G}_n^{Oh}\mathscr{M}$.*

Proof of Theorem 5. We first show ontoness. Recall that $\mathscr{G}_n^{Oh}\mathscr{M}$ is generated by triples $[M, L, f]$ where L is an AS-admissible link of n components in a $\mathbb{Z}HS$ M. Every 3-manifold can be obtained from S^3 by surgery on a framed link and, if the manifold is a $\mathbb{Z}HS$, this means that the matrix of linking numbers of the link is unimodular. By a sequence of handle slides we can diagonalize the linking matrix and so we can assume that M is obtained by surgery on an

AS-admissible link in S^3. Using our fundamental relation 17 we can assume that $M = S^3$. Ontoness now follows once we can show that we can assume that the framing f_i is equal to $+1$, for all $1 \leq i \leq n$. This follows by an argument due to Ohtsuki [Oh]. Let L be any AS-admissible n component link with two framings f and f' so that, if L' is the sublink obtained by deleting the n-th component, then $f|L' = f'|L'$ and $f_n = -f'_n$. Let $(\tilde{L},\tilde{f})$ be obtained from (L,f) by replacing L_n by two parallel non-linking copies of L_n, and with $\tilde{f}|L' = f|L'$, $\tilde{f}_n = f_n$ and $\tilde{f}_{n+1} = -f_n$. Then we have $[M,\tilde{L},\tilde{f}] = [M,L,f] - [M^-,L,f]$ where $M^\epsilon = M_{(L_n,\epsilon f_n)}$. So it suffices to show that $[M^-,L,f] = -[M,L,f']$. But we have $[M,L,f'] = [M,L',f'|L'] - [M^-,L',f'|L']$ and $[M^-,L,f] = [M^-,L',f'|L'] - [M^{-+},L',f'|L']$. Since $M^{-+} = M$ and $f|L' = f'|L'$, we are done. The same proof is represented in Fig. 10.

To prove that the map $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{I}}(n)] \to \mathscr{G}_n^{Oh}\mathscr{M}$ factors through a map $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{I}}(n)]/\mathscr{R}_n \to \mathscr{G}_n^{Oh}\mathscr{M}$ we recast the relations in $\mathscr{R}_n$ in terms of links.

To prove equation 27, we map a to (S^3,L,f) where the i-th component l of L bounds a disk disjoint from the remaining components L'. Since surgery on l does not change L' this relation follows from the fundamental relation 17.

To prove equation 28, after closing up the braid $abcd$ to a link L, the i-th component l represents the element $\alpha\beta\gamma$ in the fundamental group of the complement of the remaining components L' and the surgical equivalence class of L only depends on this element. Furthermore we can assume that l is unknotted since this can be achieved by crossing changes which do not change the surgical equivalence class of L. In this way, with the notation preceding Proposition 3.4, we have that $dabc \simeq_s L'(\alpha\beta\gamma)$, $dab \simeq_s L'(\alpha\beta)$, $da \simeq_s L'(\alpha)$, where $\cong_s$ denotes surgically equivalent links. Therefore, 28 and 29 follow (and in fact are equivalent) by Proposition 3.4. This completes the proof of Theorem 5. □

3.3. A vanishing theorem for finite type invariants of $\mathbb{Z}HS$

In this section we prove a vanishing theorem (Theorem 6 and Corollary 3.8) for the graded vector space of finite type invariants of $\mathbb{Z}HS$. The proof will exploit the algebraic form of the relations $\mathscr{R}_n$. Thus, for the results in the present section we do not need to draw any links, bands or braids. With the notation of the introduction (see Sections 1.1, 1.2), recall that $\mathscr{M}$ is the vector space (over $\mathbb{Q}$) on the set of oriented $\mathbb{Z}HS$, and elements in $\mathscr{F}_m\mathscr{O}$ are thought of as polynomial functions of degree m on $\mathscr{M}$.

Consider the following three properties of a graph G:

(a) Every edge has two distinct vertices.
(b) Every vertex is either trivalent and oriented (i.e., one of the two possible cyclic orders for the edges emanating from it has been chosen), or univalent.
(c) Every component of G is either a Y-graph (i.e., has exactly one trivalent vertex, three univalent vertices and three edges) or has every vertex trivalent.
(d) G contains no Θ component, i.e. a trivalent graph with 3 edges and 2 vertices.

We will say G is a *UT graph* if it satisfies (a) and (b), and a *Chinese manifold character* (or Ohtsuki graph) if it also satisfies (c) and (d). The *degree* of G is the number of edges. It is easy to see that any Chinese manifold character has degree a multiple of 3.

We can now state the following theorem:[1]

Theorem 6. *For every $n \geq 0$, the vector space $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)]/\mathscr{R}_n$ has a basis in one-one correspondence with the set of (unoriented) Ohtsuki graphs of order n.*

Proof of Theorem 6. First, we define a correspondence between Ohtsuki graphs of order n and a set of elements that generates (over $\mathbb{Q}$) the vector space $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)]/\mathscr{R}_n$. For a monomial $a = \prod_{I \in I_n} \alpha_I^{a(I)}$ and for $1 \leq i \leq n$ we define the *i-order* of a, $o_i(a)$, to be $\sum_{i \in I} |a(I)|$ and the *total order* of a to be $\sum_I |a(I)|$. We show that $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)]/\mathscr{R}_n$ is generated by those a which satisfy: $o_i(a) \leq 2$ for every i, and $a(I) = \pm 1$ for every I- we call such a *admissible*. If $o_i(a) > 2$ then we may write $a = \alpha_{I_1}^{\epsilon_1}\alpha_{I_2}^{\epsilon_2}bc$, for some $i \in I_1 \cap I_2$, where b is i-trivial and c is i-disjoint. By equation 28 we see that a can be written as a linear combination of monomials with strictly smaller i-order and no larger total order. So we can assume $o_i(a) \leq 2$ for every i. Now suppose $a(I) = \pm 2$ for some I. Then $a = \alpha_I^{a(I)}b$ where b is i-disjoint and $i \in I$. We can apply equation 30 and 31 to express a as a linear combination of monomials with strictly smaller i-order and no larger total order.

We can, following Ohtsuki, associate to any admissible monomial a a graph $G(a)$ as follows. For every $i = 1, \ldots, n$ we associate an edge $e(i)$ with two distinct vertices. For every I with $a(I) \neq 0$ we define a trivalent vertex $v(I)$ incident to each $e(i)$ with $i \in I$. We can also give each trivalent vertex $v(I)$ an *orientation*, i.e. a cyclic ordering to its incident edges, by choosing the ordering given by those I for which $a(I) = +1$. It is easy to see that $G(a)$ is a *UT* graph with no Θ components. Furthermore, $G(a)$ determines a. We can use this graphical interpretation to describe further reductions to the generating set of monomials. We will say an edge is interior if both its vertices are trivalent and exterior otherwise.

We make the following observations:

(a) If $G(a)$ has an edge with both vertices univalent, which will be true if the order of a is small enough, then $a = 0$; this follows from 27.
(b) Suppose e is any edge in $G(a)$. Define a' so that $G(a) = G(a')$ except that the trivalent vertices of e have orientations reversed. Then 29 implies $a = a'$.
(c) If e is an interior edge and we define a'' by changing the orientation of only one of the vertices, then 32 says that $a + a''$ is a linear combination of monomials with smaller total order.

Suppose now that $G(a)$ has two edges, one interior and the other exterior, which share a common vertex. Then (b) and (c) imply that $2a$ is a linear combination of monomials with smaller total order. But if $G(a)$ contains no such edges it is

[1] A more refined version of Theorem 6 will appear in [GO].

easy to see that it must be an Ohtsuki graph. Thus the set of all such graphs *with oriented vertices* form a generating set. To complete the proof we choose some preferred orientation for each Ohtsuki graph and show that, if a is any monomial corresponding to this graph with *another* orientation then it is equal to $\pm$ the preferred one plus monomials of lower total order. To see this let $G(a)$ be the associated graph and let $G(a')$ be produced by changing the orientation of one of the trivalent vertices v. Then $a' = \pm a +$ monomials of lower total order. In fact, if v lies in a Y component then, by (a), $a = a'$ while if v lies in a completely trivalent component then, by (b), $a' = -a +$ monomials of lower total order.

Second, we prove the linear independence of the set of admissible monomials $\{G(a)\}$ where G lies in the set of Ohtsuki graphs of order n. To prove linear independence we reinterpret $\mathbb{Q}[P_2^{\S\mathbb{Z}}(n)]/\mathscr{R}_n$ as the quotient of the vector space spanned by *unreduced* but commuting monomials in the $\{\alpha_I\}$, i.e. α_I and α_I^{-1} can both occur in a monomial, by the relations defining $\mathscr{R}_n$ and the cancellation relation:

$$aa^{-1}bd = d \quad \text{if } a, b \text{ are } i\text{-trivial and } d \text{ is } i\text{-disjoint for some } i \tag{33}$$

We write an unreduced monomial uniquely in the form $\alpha = \alpha_{I_1}^{e_1} \cdots \alpha_{I_n}^{e_n}$, where each $I_r = ijk$ with $i < j < k$ and $e_r = \pm 1$. Then we define the *multiplicity* $m_i = \sum_{i \in I_r} |e_r|$, for each $i = 1, \dots, n$. We say i is *multiple* if $m_i > 1$ and *singular* if $m_i = 1$. We now define a canonical reduction of α in five steps.

Step 1. Write:

$$\alpha = \sum \alpha_{I_{i_1}}^{e_{i_1}} \cdots \alpha_{I_{i_k}}^{e_{i_k}} - \sum c_{i_1 \cdots i_k} \alpha_{I_{i_1}}^{e_{i_1}} \cdots \alpha_{I_{i_k}}^{e_{i_k}}$$

where the first summation is over all subsequences of $I_1, \dots, I_n$ such that the multiplicities m_i' of $\alpha_{I_{i_1}}^{e_{i_1}} \cdots \alpha_{I_{i_k}}^{e_{i_k}}$ satisfy $m_i' = \min_{\{m_i, 2\}}$ and the second sum is over all remaining subsequences such that $m_i' = m_i$ if $m_i \leq 1$, $m_i' = 2$ if $m_i = 2$ and $m_i' = 1$ or 2 if $m_i > 2$. The coefficients are defined by the formula: $c_{i_1 \cdots i_k} = \prod_r (m_{i_r} - 2)$, where the product is over all r such that $m_{i_r} > 2$ and $m_{i_r}' = 1$. Note that there must be at least one such r.

Step 2. After Step 1 we can assume every $m_i \leq 2$. Now we replace every occurrence of $\alpha_I^{\pm 2}$ by $4\alpha_I^{\pm 1}$ and any occurrence of $\alpha_I \alpha_I^{-1}$ by 1.

Step 3. After Step 2 we can assume, in addition, that $I_i = I_j$ if and only if $i = j$. At this point any such monomial is determined by its corresponding *UT* graph. We describe the next two reductions in terms of these graphs and linear combinations of them. First we describe a preliminary modification. A *connected UT* graph G will be called *even* if every cycle has an even number of edges. In this case the vertices of G can be divided into two classes: any two vertices in the same class are connected by a path with an even number of edges. If we choose one of these classes and remove each trivalent vertex in the class from the graph, replacing it by three univalent vertices, we obtain a new graph which is a union of Y graphs and isolated edges. We carry over the orientations of G to the new graph. Now define G' to be the sum $G_1 + G_2$ of these new graphs, if G is even, and 0 otherwise. We can now describe Step 3 by replacing every component G with at least one univalent vertex, in a UT graph, by G'.

Step 4. After Step 3 we have a linear combination of Chinese manifold characters (with, perhaps, some additional isolated edges). This step will reduce it to a linear combination of chinese manifold characters. For example, we can always prefer the orientation given by i, j, k where $i < j < k$. We describe Step 4 as follows. Suppose G is a component all of whose vertices are trivalent and k of its vertices have the wrong orientation. Then we replace G by $(-1)^k \tilde{G} + 2kG'$, where $\tilde{G} = G$ except that the orientations are now all the preferred ones. If G is a Y graph, then we replace G by $\tilde{G}$.

Step 5. The final step is to eliminate any graph which has an isolated edge.

We leave as a straightforward, if, perhaps, lengthy exercise to show that these five steps can be achieved by using the relations given by equations 27-32 and 33. The important thing is to show that the result of this reduction depends only on the class of the monomial in $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{I}}(n)]/\mathscr{R}_n$. But this can be achieved by checking that, for each of the equations 27-29 and 33, reducing both sides of the equation gives the same result. □

Corollary 3.8. *$\mathscr{G}_m\mathscr{C}$ is a finite dimensional vector space and is nonzero only if m is a multiple of* 3.

Remark 3.9. This partially answers question 1 of [Ga].

Remark 3.10. In [GO] the notion of manifold weight systems for finite type invariants of $\mathbb{Z}HS$ will be introduced.

3.4. From knots to 3-manifolds

In this section we prove a vanishing theorem (Theorem 7) for type $5m + 1$ invariants of $\mathbb{Z}HS$. The proof exploits the graphical, as well as the algebraic notation from the previous chapters. As a corollary, in Proposition 3.12 we make some progress on question 2 of [Ga].

With the notation of 16, we can state the following theorem:

Theorem 7. *If L is an AS-admissible link in S^3 with a $(4m+1)$-component trivial sublink, then $[L] \in \mathscr{F}_{5m+2}\mathscr{M}$.*

Proof. If L has $4m + 1 + r$ components, we proceed by downward induction on r. Obviously if $r > m$, there is nothing to prove. We record the following consequence of the defining relations in $\mathscr{M}$.

Lemma 3.11. *If $\tilde{L}$ is obtained from L by changing a crossing of two bands, then $[\tilde{L}] - [L]$ is a linear combination of $[L_i]$, where each L_i contains L as a* proper *sublink.*

A *band* means a collection of parallel subarcs of components of L, and we assume, after choosing some oriention for the components of L and a direction for each band, that each component of L has an equal number of subarcs travelling in the positive direction in the two bands as in the negative direction. (Note that this condition is independent of the orientations chosen but does depend on the directions of the bands) (see Figs. 4 and 5).

Proof. This follows from Figs. 4, 5 and equations (15) and (20). □

Suppose we write $L = L_0 \cup L'$, where L_0 is the trivial sublink. If we perform a crossing change, as in Lemma 3.11 where at most one of the two bands contains arcs from L_0, then $\tilde{L}$ and each L_i also contains L_0 as a sublink and so, by the inductive assumption, $[L] \in \mathcal{F}_{5m+2}\mathcal{M}$ if and only if $[\tilde{L}] \in \mathcal{F}_{5m+2}\mathcal{M}$. So, for example, we are free to change any component of L' within its homotopy class in the complement of L_0. Furthermore, if $\pi = \pi_1(S^3 - L_0)$ then we can change a component l by any element of π_3, by the argument in [Le1, pages 58-9], using band crossings in which one band consists of arcs from L_0 and the other consists of arcs from l.

As a consequence of these observations, we may assume that L' consists of components l_k so that each l_k represents a product of commutators of degree two $\prod_{i<j}[x_i, x_j]^{e^k_{ij}}$. Now each commutator $[x_i, x_j]$ can be represented by a curve σ_{ij} which intersects only two of the disjoint disks D_j bounded by the components of L_0 (see Fig. 19). We may therefore assume that each l_k is a band sum of a number of copies of the σ_{ij}, slightly translated so that they are all disjoint. We now want to apply Theorem 4. This tells us that $[L]$ is a linear combination of $[L_i]$, where L_i coincides with L except that each l_k has been replaced by either one of the σ_{ij} or a band sum of two of the σ_{ij}, or several of these. We can ignore any terms in which any l_k has been replaced by more than one component, by the inductive assumption. In each of the L_i that remain, each new l_k intersects at most four of the D_j and so L'_i intersects at most $4r$ of the D_j. Therefore if $r \leq m$, there will be at least one of the D_j not intersected by any of the component of L. But it then follows from equation 27 that $[L] = 0$.

As an application of Theorem 7, we study the map from knots to (linear combinations of) 3-manifolds defined by $K \to [S^3, K, +1] := S^3_{(K,+1)} - S^3$. Dually this map induces a map from 3-manifold invariants to knot invariants. In [Ga] it was shown that this descends to a map

$$\mathcal{F}_n\mathcal{C} \longrightarrow \mathcal{F}_{n-1}\mathcal{V} \tag{34}$$

where $\mathcal{F}_n\mathcal{V}$ is the space of type n knot invariants [B-N1], [BL], [Va]. In [Ga] it is conjectured that the above map actually descends to a map:

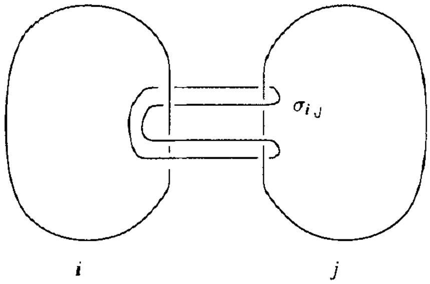

Fig. 19. A curve σ_{ij} that represents a commutator $[x_i, x_j]$ in the fundamental group of the complement of a trivial 2 component link. Compare this to Fig. 3

$$\mathcal{F}_{3m}\mathcal{G} \longrightarrow \mathcal{F}_{2m}\mathcal{V} \tag{35}$$

Proposition 3.12. *The above map descends to a map:*

$$\mathcal{F}_{5m+1}\mathcal{G} \longrightarrow \mathcal{F}_{4m}\mathcal{V} \tag{36}$$

Proof. Let $\lambda \in \mathcal{F}_{5m+1}\mathcal{G}$ be a type $5m$ invariant of $\mathbb{Z}HS$, and let ψ_λ be the associated knot invariant of equation 34. Let K be an immersed knot in S^3 with $4m+1$ double points. Let $K \cup L_0$ denote the AS-admissible link in S^3 of $4m+2$ components obtained by replacing each double point with the left hand of Fig. 20. Note that $K \cup L_0$ contains an unlink L_0 of $4m+1$ components. Using the equality of Fig. 20 and the definition of the associated knot invariant ψ_λ we obtain the following equality

$$\psi_\lambda(K) = \lambda([K \cup L_0]) \tag{37}$$

Using Theorem 7 we have that $[K \cup L_0] \in \mathcal{F}_{5m+2}\mathcal{M}$ therefore $\lambda([K \cup L_0]) = 0$. This shows that ψ_λ is a type $4m$ invariant of knots in S^3. □

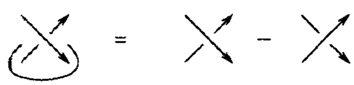

Fig. 20. Another special case of equation 17 in graphical notation

Remark 3.13. Proposition 3.12 is a $\frac{4m}{5m+1} \simeq \frac{4}{5}$ result, whereas question 2 of [Ga] asks for a $\frac{2}{3} < \frac{4}{5}$ result. In a recent preprint M. Greenwood and Xiao-Song Lin [GrLi] have shown a $\frac{n-2}{n} \simeq 1$ result, which is a weaker statement than Proposition 3.12 if $n > 6$.

Addendum 3.14. N. Habegger [Ha] has recently given a proof of the $\frac{2}{3}$ result:

$\star$ If L is an AS-admissible link in S^3 with a $(2m-1)$-component trivial sublink, then $[L] \in \mathcal{F}_{3m}\mathcal{M}$.

A corollary of this is that the map $\mathcal{F}_n\mathcal{C} \to \mathcal{F}_{n-1}\mathcal{V}$ in Equation (30) descends to a map $\mathcal{F}_{3m}\mathcal{C} \to \mathcal{F}_{2m}\mathcal{V}$, as conjectured by Garoufalidis in [Ga]. We thank Habegger for sending us a copy of [Ha]. After reading Habegger's proof we saw how to refine our proof of Theorem 7 to give another proof of Theorem $\star$. Our proof of Theorem $\star$ is rather different from Habegger's and will be included in a future work [GaLe].

4. A philosophical comment

We feel that we owe a word on the appearance of trivalent graphs. Trivalent graphs appear both in Sect. 2 and in Sect. 3. As mentioned in the introduction, a motivation for the notion of finite type invariants of $\mathbb{Z}HS$ is Chern-Simons theory, exploited by Witten [Wi2]. Chern-Simons thoery is a *topological quantum field theory* with a topological Lagrangian containing a quadratic and a cubic term. The asymptotic expansion of the associated path integral (over the space of connections) as the coupling parameter goes to infinity can be approximated by a power series sum, each term of which is a finite sum over trivalent graphs (Feynman diagrams). This is the reason that trivalent graphs appear in Chern-Simons theory.

In the theory of finite type knot invariants, trivalent graphs appear either as triple point degenerations of knots, or as Feynman diagrams of an associated conformal field theory (governed by the KZ equation), see [Dr], [B-N1] and [Ko].

In the theory of finite type invariants of $\mathbb{Z}HS$ trivalent graphs appear because of the presence of the Kirby moves, an intrinsic 3-dimensional property of space.

Finally, in the notion of surgical equivalence of links trivalent graphs appear because of the association of Fig. 5 with triple commutators $[a,[b,c]]$ in fundamental groups.

In motivic cohomology, trivalent graphs appear because of the algebraic fundamental group $\pi_1^{alg}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\})$.

The use of trivalent graphs, whether they come from the topology of 3-dimensional space, or the algebra (commutator groups, or cubic interaction terms in path integrals) is a unifying approach, and as such, it can be a source of inspiration, or confusion. We will let the reader decide which.

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