Journal of Knot Theory and Its Ramifications, Vol. 5, No. 4 (1996) 441–461

ON FINITE TYPE 3-MANIFOLD INVARIANTS I

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This edition: June 10, 1996; First edition: January 28, 1995.

To appear in the *Journal of Knot Theory and its Ramifications*

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ABSTRACT. Recently Ohtsuki [Oh2], motivated by the notion of finite type knot invariants, introduced the notion of finite type invariants for oriented, integral homology 3-spheres. In the present paper we propose another definition of finite type invariants of integral homology 3-spheres and give equivalent reformulations of our notion. We show that our invariants form a filtered commutative algebra. We compare the two induced filtrations on the vector space on the set of integral homology 3-spheres. As an observation, we discover a new set of restrictions that finite type invariants in the sense of Ohtsuki satisfy and give a set of axioms that characterize the Casson invariant. Finally, we pose a set of questions relating the finite type 3-manifold invariants with the (Vassiliev) knot invariants.

CONTENTS

1. Introduction

1.1. History. In recent years there has been a lot of progress defining (geometrically and combinatorially) knot and 3-manifold invariants. A unifying approach to these invariants is the concept of a topological quantum field theory ($TQFT$ for short) in $2+1$ dimensions, [At]. Witten [Wi], using path integrals with a Chern-Simons action (a not yet defined infinite dimensional integration) gave examples of such theories depending on a semisimple compact Lie group and an integer.

Shortly afterwards, Reshetikhin-Turaev [RT1], [RT2], (and simultaneously many other authors [KM], [Koh], [Ku], [KR], [Po], [TW]), used equivalent initial data (namely a semisimple Lie algebra and a primitive complex root of unity) as in Witten's Chern-Simons theory and combinatorially defined $TQFT$ in $2+1$ dimensions. $TQFTs$ in $2+1$ dimensions give rise to (complex valued) invariants of oriented, closed 3-manifolds, and invariants of framed colored links in 3-manifolds.

The path integral approach to topological quantum field theories suggests the existence of nonperturbative and perturbative knot and 3-manifold invariants. Examples of nonperturbative knot invariants are the values at roots of unity of colored Jones polynomials of knots, [RT1]. Examples of nonperturbative 3-manifold invariants are the Reshetikhin-Turaev invariants [RT2]. Examples of perturbative (or finite type) knot invariants are the Vassiliev invariants, [B-N1], [BL], [Va]. For the Vassiliev invariants of knots in S^3 one has:

- an axiomatic definition,
- a general existence theorem [B-N1], [BT], [Ko1], [LM],

- a comparison theorem to the above mentioned nonperturbative knot invariants [B-N1], [Dr], [Ka], and to the Chern-Simons theory perturbative knot invariants [BT], and finally
- ways of calculating them, from combinatorics of chord diagrams [B-N2].

The situation with perturbative (or finite type) 3-manifold invariants is puzzling. On the one hand, perturbative Chern-Simons theory predicts the existence of invariants of a pair (M, ρ) where M is a rational homology 3-sphere and $\rho \in Hom(\pi_1(M), G)$ (G is a fixed compact semisimple Lie group here). In cases of acyclic ρ one has such invariants [AxS1], [AxS2], [Ko2]. However, these invariants do not satisfy any of the above mentioned properties, essentially due to the absence of surgery formulas.

1.2. A review of Ohtsuki's definition. However, Ohtsuki [Oh2] recently introduced the notion of finite type invariants for oriented integral homology 3-spheres. His definition was inspired by the notion of finite type knot invariants. Let us review his definition and introduce some notation. Let $\mathcal{M}$ denote the vector space (over $\mathbb{Q}$) on the set of *oriented integral homology 3-spheres.* A link $L \subseteq M$ in an integral homology 3-sphere sphere is called *algebraically split* if the linking numbers between any two components vanish. A *framing* $f = (f_1, \dots, f_n)$ for an n component link is a sequence of integers associated to each component.

Remark 1.1. Usually a framing for a link L is a choice of a simple closed curve γ_i on the boundary of a tubular neighborhood of each component L_i of L such that the intersection number between γ_i and a meridian of L_i is 1. Any two framings of a single component differ by an integer number. Since the 3-manifolds that we consider are oriented integral homology spheres, canonical (otherwise called zero) framings exist: indeed, each component represents the trivial element in the first homology (with integral coefficients) of the ambient manifold, hence it bounds a Seifert surface, and we define the canonical framing to be a parallel of the component in the surface. The result is independent of the surface chosen. The existence of canonical framings allows us to identify the set of possible framings with the integer numbers.

For a framed link (L, f) in M we denote by $M^{L,f}$ the result of doing Dehn surgery on L in M, [Ro]. A framed link (L, f) in an integral homology 3-sphere M is called *unimodular* if $f_i = \pm 1$ for all i. A framed link (L, f) is called *AS-admissible* if it is algebraically split and unimodular. Ohtsuki [Oh2] noted that if (L, f) is a framed link in an integral homology 3-sphere M then $M^{L', f|_{L'}}$ is a integral homology 3-sphere for every sublink L' of L (with the restriction $f|_{L'}$ of the framing f of L to L') if and only if (L, f) is AS-admissible. This is the reason for our interest in AS-admissible framed links.

For a framed link (L, f) in M we denote

$$(M, L, f) := \sum_{L' \subseteq L} (-1)^{|L'|} M^{L', f|_{L'}} \in \mathcal{M} \tag{1}$$

Let us define a decreasing filtration $\mathcal{F}_\star^{Oh}$ on the vector space $\mathcal{M}$ as follows: $\mathcal{F}_m^{Oh}\mathcal{M}$ is the subspace spanned by (M, L, f) for all AS-admissible links of m components. We can now state the following definition, due to Ohtsuki [Oh2]:

Definition 1.2. [Oh2] A (rationally valued) invariant λ of integral homology 3-spheres is of *type* m if $\lambda(\mathcal{F}_{m+1}^{Oh}\mathcal{M}) = 0$, i.e., if for every AS-admissible link L of $m+1$ components in an integral homology 3-sphere M, we have that

$$\sum_{L'\subseteq L} (-1)^{|L'|}\lambda(M^{L',f|_{L'}}) = 0 \tag{2}$$

Let $\mathcal{F}_m\mathcal{O}$ denote the vector space of type m invariants and let $\mathcal{O}$ be their union $\cup_{m\geq 0}\mathcal{F}_m\mathcal{O}$. It is easy to show that $\mathcal{O}$ is a filtered commutative algebra (with pointwise multiplication). Let $\mathcal{G}_\star\mathcal{O}$ (and more generally $\mathcal{G}_\star Obj$) denote the associated graded algebra of $\mathcal{O}$ (or more generally, of a filtered object $\mathcal{F}_\star Obj$).

1.3. Variations for finite type 3-manifold invariants. In the present paper we introduce another notion of finite type invariants of integral homology 3-spheres. We compare our filtration with Ohtsuki's, (theorem 2) and with the finite type knot invariants (corollary 1.4). As an observation, we discover a new set of restrictions that Ohtsuki's invariants satisfy (theorem 4). As an application, we deduce a nonexistence theorem for 3-manifold invariants (proposition 1.5) and a characterization for the Casson invariant (theorem 5).

We begin with a few definitions and some notation: A link L in an integral homology 3-sphere M is called *boundary* if each component bounds a Seifert surface, and the Seifert surfaces are disjoint from each other. A framed link (L, f) is called *B-admissible* if it is boundary and unimodular. Our interest in B-admissible links comes from the fact that such links lie as separating curves in embedded surfaces in the ambient 3-manifold M, see theorem 1. Note that B-admissible links are AS-admissible. Let us also define a decreasing filtration $\mathcal{F}_\star^B$ on the vector space $\mathcal{M}$ as follows: $\mathcal{F}_m^B\mathcal{M}$ is the subspace spanned by (M, L, f) for all B-admissible links of m components.

If $\Sigma \hookrightarrow M$ is an embedded surface and $\gamma \subseteq \Sigma$ is an oriented, simple closed curve, we denote by $M(\gamma^n)$ the 3-manifold obtained by cutting M across Σ, performing n Dehn twists along γ and gluing Σ back. Note that the resulting manifold depends only on γ and not on the surface chosen. We can now introduce the following definition.

Definition 1.3. Let $\mathcal{F}_m\mathcal{O}^B$ denote the set of all invariants λ (with values in a field, assumed to be $\mathbb{Q}$) of integral homology 3-spheres satisfying the following property:

- For every oriented embedded surface $\Sigma \hookrightarrow M$ in an integral homology 3-sphere M and every choice $\gamma_1, \dots, \gamma_{m+1}$ of oriented, separating, nonintersecting simple closed curves on Σ we have

$$\sum_{\epsilon_i\in\{0,1\}}\prod_i(-1)^{\epsilon_i}\lambda(M(\gamma_1^{\epsilon_1},\dots,\gamma_{m+1}^{\epsilon_{m+1}}))=0 \tag{3}$$

We call such λ *B-type* m invariants of integral homology 3-spheres.

Let $\mathcal{O}^B$ denote the union $\cup_{m\geq 0}\mathcal{F}_m\mathcal{O}^B$ of B-type invariants of integral homology 3-spheres. $\mathcal{O}^B$ has an increasing filtration $\mathcal{F}_\star$ and a pointwise multiplication that respects the increasing filtration and gives $\mathcal{O}^B$ the structure of a filtered commutative algebra.

1.4. Statement of the results. We begin by giving an equivalent definition of B-type m invariants of integral homology 3-spheres.

Theorem 1. *The following are equivalent:*

(1) $\lambda\in\mathcal{F}_m\mathcal{O}^B$ *(i.e., it satisfies the property of definition* 1.3*),*
(2) λ *satisfies the property of definition* 1.3 *with the extra assumption that* $\Sigma\hookrightarrow M$ *is a surface of a Heegaard splitting,*
(3) *with the notation of section* 1.2*, for every B-admissible link* $L\subseteq M$ *of* $m+1$ *components we have*

$$\sum_{L'\subseteq L}(-1)^{|L'|}\lambda(M^{L',f|_{L'}})=0 \tag{4}$$

i.e., $\lambda(\mathcal{F}^B_{m+1}\mathcal{M})=0$.

As far as comparing the two notions of finite type invariants of integral homology 3-spheres, we have the following theorem:

Theorem 2. • *The two filtrations* $\mathcal{F}^B_\star$ *and* $\mathcal{F}^{Oh}_\star$ *on* $\mathcal{M}$ *are related as follows: Fix* $m\geq 3$. *Then for every* $n\geq 4m$ *we have:*

$$\mathcal{F}^{Oh}_n\mathcal{M}\subseteq\mathcal{F}^B_m\mathcal{M}+\mathcal{F}^{Oh}_{n+1}\mathcal{M} \tag{5}$$

• *We have an inclusion map:*

$$\mathcal{F}_m\mathcal{O}^B\cap\mathcal{O}\hookrightarrow\mathcal{F}_{4m+3}\mathcal{O} \tag{6}$$

We now introduce a map from knots to (linear combinations of) 3-manifolds defined by $K\to(K,S^3)=((S^3)^{K,+1})-(S^3)$. Dually this map induces a map from 3-manifold invariants to knot invariants. This map allows us to compare finite type 3-manifold invariants and knot invariants as follows:

Proposition 1.4. *The above mentioned map descends to a map* $\psi:\mathcal{F}_m\mathcal{O}\to\mathcal{F}_{m-1}\mathcal{V}$. *where* $\mathcal{F}_m\mathcal{V}$ *is the space of type* m *knot invariants (see section* 2*).*

In particular, for $m=1,2$ we can compare finite type invariants of integral homology 3-spheres and knots as follows:

Theorem 3. *Fix* $m \in \{1, 2\}$.

- *For every* $n \geq 3m - 2$ *we have:*

(7) $$\mathcal{F}_n^{Oh}\mathcal{M} \subseteq \mathcal{F}_m^B\mathcal{M} + \mathcal{F}_{n+1}^{Oh}\mathcal{M}$$

- *The map* $\mathcal{F}_m\mathcal{O}^B \cap \mathcal{O} \to \mathcal{F}_{4m+3}\mathcal{O}$ *of theorem* 2 *factors through a map*

(8) $$\mathcal{F}_m\mathcal{O}^B \cap \mathcal{O} \to \mathcal{F}_{3m}\mathcal{O}$$

- *Furthermore, using proposition* 1.4 *the associated composite map* $\mathcal{F}_m\mathcal{O}^B \cap \mathcal{O} \hookrightarrow \mathcal{F}_{3m}\mathcal{O} \to \mathcal{F}_{3m-1}^B\mathcal{V}$ *factors through a map* $\mathcal{F}_m\mathcal{O}^B \cap \mathcal{O} \to \mathcal{V}_{2m}^{Special}$ *where* $\mathcal{V}_m^{Special}$ *is the space of special type* m *Vassiliev invariants i.e., those whose degree* m *part is a product of derivatives of the Alexander-Conway polynomial. For more information on special Vassiliev invariants see section* 2.

Ohtsuki [Oh2] gave the following dimensions for the graded vector spaces $\mathcal{G}_m\mathcal{O}$:

m	0	1	2	3
$dim\mathcal{G}_m\mathcal{O}$	1	0	0	1

We give a new set of restrictions that the type m invariants of Ohtsuki satisfy thus deducing the following:

Theorem 4. *Every type* 4 *Ohtsuki invariant is of type* 3, *i.e.* $\mathcal{G}_4\mathcal{O} = 0$.

Proposition 1.5. *If* V *is a type* 3 *knot invariant which can be extended to a invariant of integral homology 3-spheres so that it satisfies the following property:*

(9) $$V(K) = V((S^3)^{K,+1})$$

(for all knots K *in* S^3*), then* V *is a type* 2 *knot invariant.*

Theorem 4 in turn proves the following characterization of the Casson invariant:

Theorem 5. *For a finite type invariant* $\lambda \in \mathcal{O}$ *of integral homology 3-spheres the following are equivalent:*

(1) $\lambda \in \mathcal{F}_1\mathcal{O}^B$ *of definition* 1.3.
(2) *With the notation of sections* 1.2 *and* 1.3 $\lambda(\mathcal{F}_2^B\mathcal{M}) = 0$.
(3) $\lambda \in \mathcal{F}_3\mathcal{O}$.
(4) $\lambda \in \mathcal{F}_4\mathcal{O}$.
(5) *There are constants* a, b *such that* $\lambda = a + b\lambda_{Casson}$ *where* λ_{Casson} *is the Casson invariant* [AM].

Remark 1.6. Ohtsuki [Oh2] had previously proved that (3) is equivalent to (5).

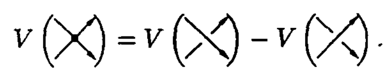
1.5. Plan of the proof. In section 2 we review the definition and a few properties of finite type knot invariants, otherwise known as Vassiliev invariants. In section 3.1 we prove proposition 1.4, thus giving a map from finite type invariants of integral homology 3-spheres to (finite type) invariants of knots in S^3. In section 3.2 we prove theorem 2. In section 4.1 we show surgery properties that our and Ohtsuki's finite type 3-manifold invariants satisfy, and in section 4.2 we prove theorem 1 that restates our definition 1.3. In section 5 we pose a set of questions relating the finite type knot and 3-manifold invariants. In section 6.1 we partially answer our questions and prove theorem 3 and in section 6.2 we give a new set of restrictions that Ohtsuki's invariants satisfy, thus showing theorem 4 and proposition 1.5. Finally, in section 7 we show theorem 5.

1.6. Acknowledgment. We wish to thank D. Auckly, L. Kauffman, R. Kirby, E. Lerman, J. Levine, K. Millett and J. Roberts for many useful comments. Especially we wish to thank D. Bar-Natan for many enlightening conversations and for his critical reading of an earlier draft of the present paper, and the anonymous referee for many useful comments.

2. Finite type knot invariants

In this section we review finite type knot invariants, otherwise known as Vassiliev invariants [B-N1], [BL], [Va]. A standard reference for the next definitions and notation is [B-N1].

A *Vassiliev invariant* of type m is a knot invariant V which vanishes whenever it is evaluated on a knot with more than m double points, where the definition of V is extended to knots with double points via the formula



The algebra $\mathcal{V}$ of all Vassiliev invariants (with values in $\mathbb{Q}$) is filtered, with the type m subspace $\mathcal{F}_m\mathcal{V}$ containing all type m Vassiliev invariants. The associated graded space of $\mathcal{V}$ is isomorphic to the space $\mathcal{W}$ of all *weight systems*. A degree m weight system is a homogeneous linear functional of degree m on the graded vector space $\mathcal{A}^r$ of *chord diagrams* as in figure 1 divided by the $4T$ and framing independence relations explained in figures 2 and 3.

Figure 1. A chord diagram with 4 chords:

$\mathcal{A}^r$ is graded by the number of chords in a chord diagram. It is a commutative and co-commutative Hopf algebra with multiplication defined by juxtaposition, and with co-multiplication Δ defined as the sum of all possible ways of 'splitting' a diagram. The co-algebra structure of $\mathcal{A}^r$ defines an algebra structure on $\mathcal{W}$.

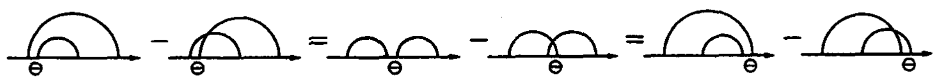


Figure 2. To get the $4T$ relations, add an arbitrary number of chords in arbitrary positions (only avoiding the short intervals marked by a 'no-entry' sign $\ominus$) to all six diagrams in exactly the same way.

Figure 3. The framing independence relation: any diagram containing a chord whose endpoints are not separated by the endpoints of other chords is equal to 0.

There are natural maps $W_m : \mathcal{F}_m\mathcal{V} \to \mathcal{G}_m\mathcal{W} = \mathcal{G}_m\mathcal{A}^{r*}$. For a type m Vassiliev invariant V it is natural to think of $W_m(V)$ as "the m'th derivative of V". W_m is not an isomorphism, however its kernel is $\mathcal{F}_{m-1}\mathcal{V}$. It therefore follows that $\mathcal{F}_m\mathcal{V}$ is a finite dimensional vector space for every m.

In the present paper we are primarily interested in type 5 knot invariants about which much more is known.

We can summarize the results in the following proposition [B-N1]:

Proposition 2.1. *The dimensions of the spaces of type m Vassiliev invariants of knots are given in the following table*

m	0	1	2	3	4	5
$dim \mathcal{G}_m\mathcal{W}$	1	0	1	1	3	4

Let us denote by $\Delta^{(m)}(K) := \frac{d^m}{dh^m}|_{h=0}\Delta(K)(e^h)$ the m^{th} derivative of the Alexander-Conway polynomial $\Delta(K)$ of a knot K [Ro] with the normalization as in [B-NG], example 2.8. It is clear that $\Delta^{(m)} \in \mathcal{F}_m\mathcal{V}$.

We also need the following lemma:

Lemma 2.2. *If a degree* 4 *weight system* $W \in \mathcal{G}_4\mathcal{W}$ *vanishes on the chord diagram* $CD[4,1]$ *of figure* 4 *then* $W = aW(\Delta^{(4)}) + bW(\Delta^{(2)} \cdot \Delta^{(2)})$ *for some constants* a, b.

Proof. A proof can be given using computer calculations of D. Bar-Natan [B-N2]. Here is a sketch of the argument. We begin by noting that there are 7 degree 4 chord diagrams without isolated chords. The vector space $\mathcal{G}_4\mathcal{A}^r$ is 3 dimensional, as follows by the program `CDReduceData.m` of [B-N2]. Therefore, its dual space $\mathcal{G}_4\mathcal{W}$ is 3 dimensional. Moreover, $W(\Delta^{(4)}), W(\Delta^{(2)} \cdot \Delta^{(2)}) \in \mathcal{G}_4\mathcal{W}$. Since W is an algebra map, we have that $W(\Delta^{(2)} \cdot \Delta^{(2)}) = W(\Delta^{(2)}) \cdot W(\Delta^{(2)})$. The Alexander-Conway weight system has been analysed in [B-NG]. Let us recall from [B-NG], section 2.3 that every diagram of m chords can be thickened to a surface and the value of $W(\Delta^{(m)})$ on a chord diagram is 1 (respectively, 0) if the associated thickened surface has one (respectively, more than one) component. Using `CDReduceData.m` of [B-N2] we can calculate the values of $W(\Delta^{(4)}), W(\Delta^{(2)} \cdot \Delta^{(2)})$ on a basis of $\mathcal{G}_4\mathcal{A}^r$ and show that the intersection of the kernels of $W(\Delta^{(4)}), W(\Delta^{(2)} \cdot \Delta^{(2)})$ is a one dimensional space spanned by the chord diagram of figure 4. □

$CD[4,1]$

Figure 4. A chord diagram with 4 chords

Lemma 2.3. *If a degree* 5 *weight system* $W \in \mathcal{W}$ *vanishes on the chord diagrams of figure* 5 *then* $W = 0$.

Proof. A proof can be given using the subprogram `CDReduce` of the program `NAT.m` of [B-N2]. `CDReduceData.m` of [B-N2] shows that the space of degree 5 chord diagrams without isolated chords is 36 dimensional, and that the quotient of it, $\mathcal{G}_5\mathcal{A}^r$ is 4 dimensional. `CDReduce` shows that the chord diagrams in figure 5 form a basis for $\mathcal{G}_5\mathcal{A}^r$. □

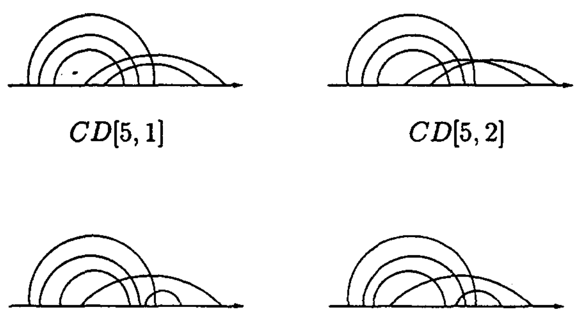


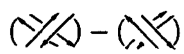
Figure 5. Some degree 5 chord diargams which form a basis for $\mathcal{G}_5\mathcal{A}^r$

3. Comparison with other approaches

3.1. From knots to 3-manifolds. In this section we prove proposition 1.4. We begin by recalling the map from (framed) links to (linear combinations of) 3-manifolds of equation (1). We can compose this map with the one that sends algebraically split links L in S^3 to a pair (S^3, L, f) where f denotes the $+1$ framing on every component of L. In the following lemma all links drawn will be algebraically split (in S^3) and will have framing +1 on each component. With this notation we have the following lemma:

Lemma 3.1. *With the notation of section* 1.2 *the following identities hold in* $\mathcal{G}^{Oh}_{\star}\mathcal{M}$*:*

(10)  $\in \mathcal{G}^{Oh}_{m+1}\mathcal{M}$

(11)  $= 0 \in \mathcal{G}^{Oh}_{m}\mathcal{M}$

$$(C_0 \cup L) = 0 \in \mathcal{G}^{Oh}_{m}\mathcal{M} \qquad (12)$$

where $C_0 \cup L$ *is the disjoint union of* L *with an unknot* C_0*. In the above equations, the left hand side represents links of* m *components. In the first equation, both strands belong to the same component, and in the second equation two strands of the same component go over/under the two strands of another component.*

Proof. A proof was first given by Ohtsuki in [Oh2]. It is a simple consequence of Kirby moves and the definition of the map (.). Note that we could also give a formula in $\mathcal{F}^{Oh}_{\star}\mathcal{M}$, rather than in the graded space $\mathcal{G}^{Oh}_{\star}\mathcal{M}$, however, the above form of the lemma suffices for our purposes. □

Proposition 1.4 now follows immediately by the first equation in (10). In the remaining part of the paper, it will be useful to describe the associated weight system of the the finite type knot invariant of proposition 1.4. This can be done as follows:

Remark 3.2. Let $\lambda \in \mathcal{F}_m\mathcal{O}$, and let ψ_λ denote the associated knot invariant $\psi(\lambda)$ of proposition 1.4. Let $W_\lambda \in \mathcal{G}_{m-1}\mathcal{W}$ be the associated degree $m-1$ weight system. One way to calculate the value of W_λ on a chord diagram CD of $m-1$ chords is as follows: represent the chord diagram in a circle, resolve the crossing points between chords in any way and replace each chord with an unknot as in figure 6. This way we get an algebraically split m component link $L(CD)$ each component of which is an unknot. By definition and using lemma 3.1 we see that

$$W_\lambda(CD) = \psi_\lambda(L(CD)) \qquad (13)$$

Note that even though $L(CD)$ depends on the way we choose to resolve the crossing points between the chords of the chord diagram, the value of ψ_λ is independent of that choice as follows by lemma 3.1.

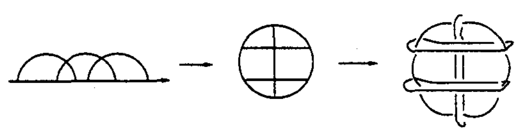


Figure 6. Reconstructing algebraically split m component links from linear chord diagrams of $m-1$ chords. Here we take $m=4$.

We believe that it is an interesting question (both for the sake of knot invariants, but also for the sake of 3-manifold invariants) to study the map of proposition 1.4.

3.2. A comparison theorem. In this section we prove theorem 2. Fix an integer $m \geq 3$. We begin with some preliminaries. In his fundamental paper [Oh2] Ohtsuki described a (finite) generating set of the vector space $\mathcal{G}_m^{Oh}\mathcal{M}$. Let us introduce some more notation before we recall his result. For $m \in \mathbb{N}$, let $G[m]$ denote the set of (possibly empty or diconnected) graphs with no Θ components,[1] whose vertices are either trivalent and oriented (i.e., one of the two possible cyclic orderings of the edges emanating from such a vertex is specified) or univalent. In analogy with the theory of Vassiliev invariants of knots [B-N1], we call such graphs *chinese manifold characters*. We denote by $v_k(\Gamma)$ the number of k-valent vertices, $e(\Gamma)$, $v(\Gamma)$ the number of edges and vertices of such a graph Γ. Let $NI(\Gamma)$ denote the maximum number of nonintersecting edges, i.e., edges with no common vertices. For such a graph $\Gamma \in G[m]$, Ohtsuki constructs an algebraically split link $L(\Gamma)$ in S^3 of m components (and framing $+1$ on each component) as follows: each edge is represented by an unknot, and each trivalent vertex is represented by a Borromean link. We denote by $[L(\Gamma)]$ the element of $\mathcal{M}$ from equation (1) (note that $L(\Gamma)$ is considered with framing $+1$ on each component). With the notation of section 3.1, we can now state the following theorem of Ohtsuki:

Theorem 6. [Oh2] *$\mathcal{G}_m^{Oh}\mathcal{M}$ is generated (as a vector space) by the set* $\{[L(\Gamma)]\}_{\Gamma \in G[m]}$.

We need the following two lemmas:

Lemma 3.3.
- *For a connected*[2] *chinese manifold character Γ one has the following lower bounds for $NI(\Gamma)$:*
  - *If $\Gamma \in G[3k+1]$ then $NI(\Gamma) \geq k+1$.*
  - *If $\Gamma \in G[3k+2]$ then $NI(\Gamma) \geq k$.*
  - *If $\Gamma \in G[3k+3]$ then $NI(\Gamma) \geq k$.*
- *Let $m \geq 3$. For any $n \geq 4m$ and for any chinese manifold character $\Gamma \in G[n]$ we have that $NI(\Gamma) \geq m$.*

Lemma 3.4. *For every chinese manifold character Γ we have that*

$$[(L(\Gamma))] \in \mathcal{F}_{NI(\Gamma)}^{Oh}\mathcal{M} \tag{14}$$

Proof. (of lemma 3.3) In order to show the first part, let Γ be a connected chinese manifold character with $e(\Gamma)$ edges and let T be a spanning tree of it. Obviously we have that $e(T) = v(T) - 1 = v_1(\Gamma) + v_3(\Gamma) - 1$. Moreover, since T has at most

[1] a Θ component is a graph with 2 vertices and 3 edges, as in the greek letter Θ.

[2] we thank the referee for pointing out the assumption of connectivity that was missing in a previous draft of this paper

trivalent vertices, an easy induction shows that $NI(T) \geq \left[\frac{e(T)}{2}\right]$ (where $[x]$ is the greatest integer smaller than or equal to x) and therefore that

$$NI(\Gamma) \geq NI(T) \geq \left[\frac{e(T)}{2}\right] = \left[\frac{v_1(\Gamma)+v_3(\Gamma)-1}{2}\right] \tag{15}$$

We need to show that $\left[\frac{v_1(\Gamma)+v_3(\Gamma)-1}{2}\right] \geq \left[\frac{e(\Gamma)-1}{3}\right] + \epsilon(e(\Gamma))$ where $\epsilon(3k+1) = 1$, $\epsilon(3k+2) = \epsilon(3k) = 0$. Since $\Gamma \in G[n]$ we have that $2e(\Gamma) = v_1(\Gamma) + 3v_3(\Gamma)$ therefore it suffices to show that

$$\left[\frac{v_1(\Gamma)+v_3(\Gamma)-1}{2}\right] \geq \left[\frac{(v_1(\Gamma)+3v_3(\Gamma))/2-1}{3}\right] + \epsilon(e(\Gamma)) \tag{16}$$

Now a case by case argument for each class of $e(\Gamma)$ mod 3 shows the result. The second part follows by applying the first part to each connected component of a chinese manifold character. □

Proof. (of lemma 3.4) We claim that the sublink L' of $L(\Gamma)$ that corresponds to a set of nonintersecting edges is a boundary sublink. In fact, we can attach discs with one handle to each unknot that corresponds to a set of nonintersecting edges, in such a way that each component of L' bounds a genus 1 or 2 surface and that every two surfaces are disjoint from each other. □

Proof. (of theorem 2) Using theorem 6 and lemmas 3.3 and 3.4 we deduce that for fixed $m \geq 3$ and for every $n \geq 4m$ equation (5) holds.

To show the second part of theorem 2, let $\lambda \in \mathcal{F}_m\mathcal{O}^B \cap \mathcal{O}$. Equation (5) implies that $\mathcal{F}_n^{Oh}\mathcal{M} \subseteq \mathcal{F}_{m+1}^B\mathcal{M} + \mathcal{F}_{n+1}^{Oh}\mathcal{M}$ for all $n \geq 4m+4$. Since $\lambda \in \mathcal{F}_m\mathcal{O}^B$ it follows that $\lambda(\mathcal{F}_n^{Oh}\mathcal{M}) \subseteq \lambda(\mathcal{F}_{n+1}^{Oh}\mathcal{M})$ for all $n \geq 4m+4$. Since $\lambda \in \mathcal{O}$, it follows that $\lambda(\mathcal{F}_{n+1}^{Oh}\mathcal{M}) = 0$ for large enough n, and therefore, $\lambda(\mathcal{F}_{4m+4}^{Oh}\mathcal{M}) = 0$, i.e., $\lambda \in \mathcal{F}_{4m+3}\mathcal{O}$. The proof of theorem 2 is complete. □

Remark 3.5. In fact, the proof of lemma 3.3 shows a bit more namely, that if Γ is a connected chinese manifold character, then

- if $\Gamma \in G[3m+2]$ satisfies $v_1(\Gamma) \neq 1$ then $NI(\Gamma) \geq m+1$, and
- if $\Gamma \in G[3m+3]$ satisfies $v_1(\Gamma) \neq 0$ then $NI(\Gamma) \geq m+1$

These lower bounds are sharp. For example the graph Γ of figure 7 shows that $\Gamma \in G[15]$ but $NI(\Gamma) = 4$ (and not 5).

4. Properties of finite type 3-manifold invariants

4.1. A surgery formula. In this section we prove a surgery formula for the invariants $\lambda \in \mathcal{F}_m\mathcal{O}^B$. Let K be a knot in an integral homology 3-sphere M, and $n \in \mathbb{N}$. Let $K_{((n))}$ denote the $(0,n)$ cable of K i.e., a link of n components parallel to K with linking numbers zero. We now have the following:

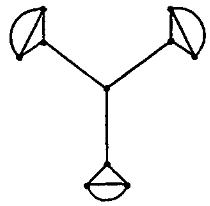


Figure 7. An annoying graph

Proposition 4.1. *If $\lambda \in \mathcal{F}_m\mathcal{O}^B$, K a knot in an integral homology 3-sphere M as above and $n \in \mathbb{N}$, then with the notation of sections* 1.2 *and* 1.3 *we have that*

$$\lambda(M^{K,1/n}) = \sum_{j=0}^{m} (-1)^j \binom{n}{j} \psi_\lambda(M, K_{((j))}) \tag{17}$$

where $\psi_\lambda(M,L)$ is defined to be $\lambda((M,L,\{+1,\ldots,+1\}))$ i.e.,
$\psi_\lambda(M,L) = \sum_{L' \subseteq L} (-1)^{|L'|} \lambda(M^{L',\{+1,\ldots,+1\}})$ (this is the analogue for links of the map ψ of proposition 1.4. *In proposition* 1.4 *we used the value of ψ_λ for knots only).*

Proof. Figure 8 shows that $M^{K,1/n}$ and $M^{K_{((n))},\{1,\ldots,1\}}$ are diffeomorphic manifolds. Furthermore, for every $j \geq 0$ it follows by the definition of ψ_λ that

$$\psi_\lambda(M, K_{((j))}) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \lambda(M^{K,1/k}) \tag{18}$$

Furthermore $K_{((j))}$ is a boundary link of j components, therefore $\psi_\lambda(M, K_{((j))}) = 0$ for $j > m$. The result now follows by solving for λ from equation (18). □

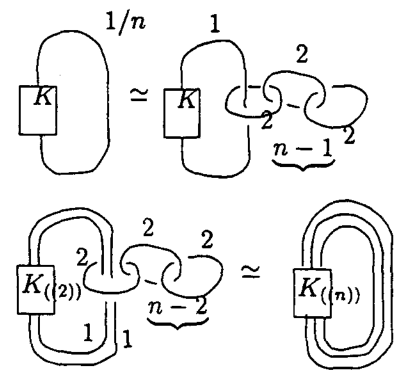


Figure 8. Some Kirby moves relating $1/n$ surgery on K

Exercise 4.2. Show that if $\lambda \in \mathcal{F}_m\mathcal{O}$, K a knot in an integral homology 3-sphere M and $n \in \mathbb{N}$ then

$$\lambda(M^{K,1/n}) = \sum_{j=0}^{3m}(-1)^j \binom{n}{j} \psi_\lambda(M, K_{((j))}) \tag{19}$$

4.2. A restatement of definition 1.3. In this section we prove theorem 1.

Proof. (of theorem 1) Obviously, (3) implies (1) which implies (2). We will show that (2) implies (3). Let $L \subseteq M$ be a boundary link of $m+1$ components in an integral homology 3-sphere M. Let E be an embedded orientable surface in M of $m+1$ components such that $\partial E = L$. Fix an identification of $E \times I$ with a bicollar of E in M. Let $\gamma := (\gamma_1, \dots, \gamma_{m+1})$ be a collar of ∂E in E. Then $E \times I$ is a (possibly disconnected) handlebody, but $\overline{M \setminus E \times I}$ need not be. In any case, attach 1-handles on $E \times I$ away from $\gamma \times I$ to construct W such that $W, \overline{M \setminus W}$ are both (connected) handlebodies. Let $\Sigma \hookrightarrow M$ be the boundary of W. Note that $\gamma \times I$ is a disjoint union of separating annuli in Σ, and $1/1$ surgery on each component of L corresponds to cutting M along Σ, performing 1 *left*-handed Dehn twist along each component of $\gamma \times I$, and glueing back. Therefore (2) of theorem 1 implies (3). □

5. Questions

5.1. A few questions. In this section we pose some questions relating our notion of B-type 3-manifold invariants with that of Ohtsuki (for 3-manifolds) and of Vassiliev (for knots).

Question 1. With the notation of section 1.2 is it true that $\mathcal{F}_m^B\mathcal{M} = \mathcal{F}_{3m}^{Oh}\mathcal{M}$?

Remark 5.1. Note that if the above question had a positive answer, it would imply that $\mathcal{F}_m^B\mathcal{M} \supseteq \mathcal{F}_{3m}^{Oh}\mathcal{M}$ and that $\mathcal{G}_m^{Oh}\mathcal{M} = 0$ for m not a multiple of 3.

Question 2. Does the map $\psi : \mathcal{F}_{3m}\mathcal{O} \to \mathcal{F}_{3m-1}\mathcal{V}$

- actually factor through a map

$$\mathcal{F}_{3m}\mathcal{O} \to \mathcal{F}_{2m}\mathcal{V} \tag{20}$$

  preserving the filtration?
- If so, is it true that the graded map

$$\mathcal{G}_{3m}\mathcal{O} \to \mathcal{G}_{2m}\mathcal{V} \tag{21}$$

  is one-to-one?
- Is it true that the image of (20) is in the space $\mathcal{V}_{2m}^{Special}$ of special Vassiliev invariants?

Question 3. Is it true for the invariants λ_m defined in [Oh2] that $\lambda_m \in \mathcal{F}_m\mathcal{O}^B$? Also that $\lambda_m \in \mathcal{F}_{3m}\mathcal{O}$?

Question 4. Do (either of the two versions of) finite type invariants of integral homology 3-spheres separate integral homology 3-spheres?

5.2. A general comment. We believe that the above mentioned questions will be helpful in understanding knot invariants as well as 3-manifold invariants. One feature of these questions is that they are (in principle) testable on a computer, which can decide about the fate of some of them. The experimental knowledge is small so far. Much remains to be done in analogy with the rather well developed theory of finite type knot invariants.

Remark 5.2. Shortly after the present paper was finished, there was a lot of progress in answering the above questions: in [GL2] it is shown that $\mathcal{F}^B_m\mathcal{M} \subseteq \mathcal{F}^{Oh}_{3m}\mathcal{M}$. In [GO1], [GL1] a positive answer to remark 5.1 was given. In [Ha], [GL2] a positive answer to the first part of question 2 was given.

6. Calculations

6.1. Questions 1, 2 for $m = 1, 2$. In this section we partially answer questions 1, 2 in the case of $m = 1, 2$ and prove theorem 3.

Fix $m = 1, 2$. We begin with the following two claims:

Claim 6.1. *For every $\Gamma \in G[3m+3]$ with $v_1(\Gamma) = 0$ we have $NI(\Gamma) \geq m+1$.*

Proof. For $m = 1$ it is easy to list all elements in $G[6]$ (see [Oh2]) and check it by hand. For $m = 2$ we could also list the relevant elements in $G[9]$ and check them by hand. Instead we prefer to give an alternative argument as follows: Let $\Gamma \in G[9]$ with $v_1(\Gamma) = 0$. Without loss of generality we may assume that Γ is connected. Then $v_3(\Gamma) = 6$ and every spanning tree T of it has 5 edges. The possibilities for a spanning tree are shown in figure 9.

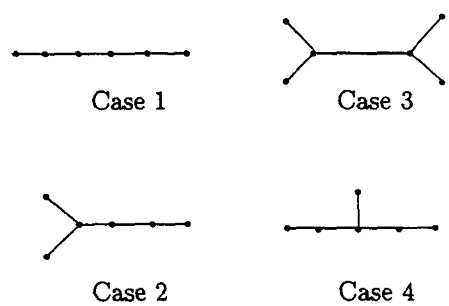


Figure 9. All possible trees with 5 edges

We now distinguish cases:

<u>Case 1</u> It is immediate since $NI(\Gamma) \geq NI(T) = 3$.

Case 2 It is easy to see that Γ has a subdiagram of the form $\Gamma_{2,1}$ or $\Gamma_{2,2}$ as in figure 10. Therefore we can choose a spanning tree of the form $T_{2,1}$ or $T_{2,2}$ and in both cases we are reduced to case 1 and the result holds.

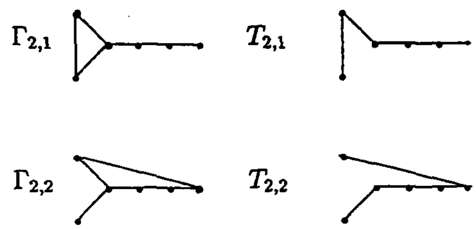


Figure 10. Subgraphs of Γ and alternative spanning trees

Case 3 It is easy to see that Γ has a subdiagram of the form $\Gamma_{3,1}$ or $\Gamma_{3,2}$ as in figure 11. Therefore we can choose a spanning tree of the form $T_{3,1}$ or $T_{3,2}$ which reduces us to case 2 or 1 and the result holds.

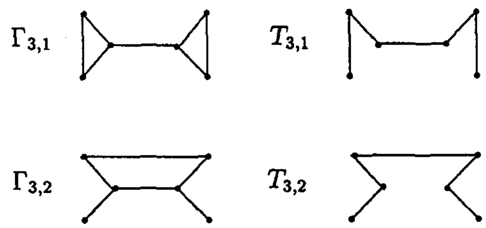


Figure 11. Subgraphs of Γ and alternative spanning trees

Case 4 It is immediate since $NI(\Gamma) \geq NI(T) = 3$. The proof of claim 6.1 is complete. □

Claim 6.2. *For every* $\Gamma \in G[3m+2]$ *with* $v_1(\Gamma) = 1$ *we have* $NI(\Gamma) \geq m+1$

Proof. The proof is analogous to claim 6.1. $m = 1$ is easy. If $m = 2$ and $\Gamma \in G[8]$ with $v_1(\Gamma) = 1$, then $v_3(\Gamma) = 5$ and a spanning tree T has 5 edges. The possibilities are shown in figure 9 and the cases are shown in figures 10, 11, 12. The proof of claim 6.2 is complete. □

Proof. (of theorem 3) Claims 6.1, 6.2, theorem 6 and lemma 3.4 imply (7). Indeed, $\mathcal{G}_n^{Oh}\mathcal{M}$ is generated by $[L(\Gamma)]$ for chinese manifold characters Γ with n edges. Claims 6.1, 6.2 and lemma 3.4 imply that for $n \geq 3m-2$, we have that $[L(\Gamma)] \in \mathcal{G}_m^{Oh}\mathcal{M}$.

The second part of the theorem 3 follows from the first the same way as in theorem 2.

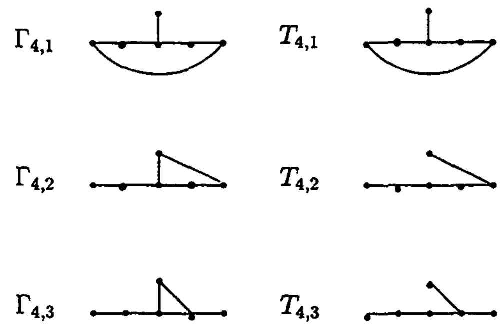


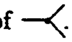
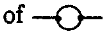
Figure 12. Subgraphs of Γ and alternative spanning trees

For the third part, if $m = 1$ it is obvious. If $m = 2$, let $\lambda \in \mathcal{F}_2\mathcal{O}^B \cap \mathcal{O} \hookrightarrow \mathcal{F}_6\mathcal{O}$, let $\psi_\lambda \in \mathcal{F}_5\mathcal{V}$ be the associated knot invariant and let W_λ be the associated degree 5 weight system, (see remark 3.2). Using remark 3.2 we see that W_λ vanishes on the chord diagrams of figure 5, since the associated 5 component links of remark 3.2 have boundary sublinks of 3 components. Therefore by lemma 2.3 we see that $W_\lambda = 0$, i.e., $\psi_\lambda \in \mathcal{F}_4\mathcal{V}$. Now letting W_λ be the associated degree 4 weight system, arguing as above, we see that it vanishes on the chord diagram of figure 4 and therefore by lemma 2.2 we see that the image of ψ_λ is in the space $\mathcal{V}_4^{Special}$ of special type 4 knot invariants. □

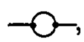
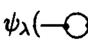
6.2. A new set of restrictions for Ohtsuki's invariants. In this section we prove theorem 4 and proposition 1.5 by introducing a new set of restrictions that the finite type invariants of Ohtsuki have to satisfy. The main idea is to study the map ψ of proposition 1.4.

Proof. (of theorem 4) Let us assume that $\lambda \in \mathcal{G}_4\mathcal{O}$. Let ψ_λ be the associated type 3 knot invariant as in remark 3.2 and in proposition 1.4. It follows from the main result of Ohtsuki [Oh2] (quoted as theorem 6 in the present paper) that $\lambda \in \mathcal{G}_4\mathcal{O}$ is determined by its value on the graph $-\!\!\bigcirc\!\!-$ (with the two possible vertex orientations). Choosing the counter clockwise orientation on each vertex of it, and recalling the discussion of section 3.2 (in particular, chinese manifold characters correspond to algebraically split links in S^3 which correspond to linear combinations of integral homology 3-spheres, using +1 framing on each component) we obtain that $\lambda(-\!\!\bigcirc\!\!-)$ is a sum of 16 terms, each of which is the image under ψ_λ of an appropriate link. Since all links containing an unlinked unknot have zero image under ψ_λ, the sum simplifies to the following equation:

(22)
$$\lambda(-\!\!\bigcirc\!\!-) = \psi_\lambda(K_0) - 2\psi_\lambda(T_+)$$

where K_0 denotes the knot (in S^3) obtained by blowing down the three components  of  and T_+ denotes the knot in S^3 obtained by blowing down (with $+1$ framing) any two components of . Here blowing down refers to the first Kirby move [Ro], i.e., the result of erasing an unknotted component of a link (with framing ± 1) and replacing the rest of the components of the link by a full twist around a disc that the unknotted component bounds. Note that each component of  is unknotted (with linking numbers zero with the other components and with framing $+1$) and remains unknotted after blowing down the other components. This shows that T_+ exists. In fact, T_+ is the right handed trefoil.

In other words, λ is determined by the type 3 knot invariant ψ_λ. We can now finish the proof of theorem 4 as follows: A basis for type 3 knot invariants is $1, J^{(2)}, J^{(3)}$ (where $J^{(m)}(K) := \frac{d^m}{dh^m}|_{h=0} J(K)(e^h)$ is the m^{th} derivative of the Jones polynomial). A calculation shows that $J^{(m)}(K_0) = 2J^{(m)}(T_+)$ for $m = 2, 3$.[3]

Therefore, $\psi_\lambda($$) = 0$. Similarly, had we chosen a different vertex orientation of the graph , K_0 and T_+ would be replaced by their mirror image and still $\psi_\lambda($$) = 0$. Therefore, $\mathcal{G}_4 \mathcal{L}^{Oh} = 0$. $\square$

Proof. (of proposition 1.5) Let $V = aJ^{(2)} + bJ^{(3)} \in \mathcal{F}_3\mathcal{V}$ be a type 3 knot invariant satisfying the assumptions of proposition 1.5. Figure 14 shows two knots K_3 and K_4 with the property that -1 surgery on them gives diffeomorphic integral homology 3-spheres. The knots appear in [Li] as an example of distinct knots in S^3 whose -1 surgery gives diffeomorphic integral homology 3-spheres. For convenience of the reader, the proof that the above mentioned integral homology 3-spheres are diffeomorphic is included in figure 13, taken from [Li]. We are indebted to R. Kirby for pointing out this reference to us. Since $(S^3)^{K_3,-1}$ and $(S^3)^{K_4,-1}$ are diffeomorphic integral homology 3-spheres, after a change of the orientation we obtain that $(S^3)^{\tau K_3,+1}$ and $(S^3)^{\tau K_4,+1}$ are diffeomorphic integral homology 3-spheres, where τK is the mirror image of a knot K in S^3. Therefore we have that

$$aJ^{(2)}(\tau K_3) + bJ^{(3)}(\tau K_3) = aJ^{(2)}(\tau K_4) + bJ^{(3)}(\tau K_4) \tag{23}$$

The Jones polynomials of them are given as follows:

$$J(\tau K_3)(q) = -q^{-3} + 2q^{-2} - 2q^{-1} + 3 - 2q + 2q^2 - q^3 \tag{24}$$

$$J(\tau K_4)(q) = q^{-5} - q^{-4} - q^{-1} + 1 + q^2 + q^3 - q^4 + q^5 - q^6 \tag{25}$$

from which we can deduce that $J^{(2)}(K_3) = J^{(2)}(K_4) = -6$ (this is not a surprise, since the Casson invariant exists!) but $J^{(3)}(K_3) = 0 \neq J^{(3)}(K_4) = -180$. Therefore,

[3] There are various programs [B-N2], [EM], [Och] that calculate the Jones polynomial of knots. As a check, we used all of the above mentioned and got the same results. We thank D. Dar-Natan, L. Kauffman, K. Millet and M. Ochiai for their help in distributing and running the programs.

$b = 0$ and V is a type 2 knot invariant. Needless to say, we do not understand why this happens. □

Remark 6.3. Note that proposition 1.5 implies in order to show theorem 4 it suffices to check that $J^{(2)}(K_0) = 2J^{(2)}(T_+)$.

Exercise 6.4. (after a conversation with L. Kauffman) Show that $K_3 = \tau K_3$, which actually explains why $J(\tau K_3)(q) = J(\tau K_3)(q^{-1})$ and therefore that $J^{(3)}(\tau K_3) = 0$.

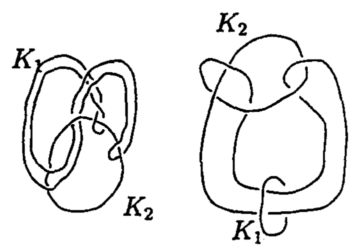


Figure 13. Two different views of the same two component link

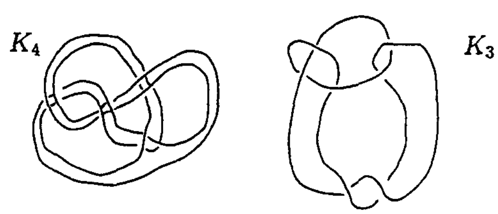


Figure 14. The result of figure 13 after blowing down K_2 (left) or K_1 (right)

7. Uniqueness of the Casson invariant

In this section we collect our results to show theorem 5. The first statement (1) is equivalent to (2) because of theorem 1. (2) implies (3) (by theorem 3) which implies (4). (4) implies (5) (by theorem 4) and finally (5) implies (1) by Casson, [AM].

Remark 7.1. It is surprising that we only used *nonintersecting,* bounding, simply closed curves in surfaces to characterize the Casson invariant.

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