Communications in
Mathematical
Physics

CrossMark

Knots, BPS States, and Algebraic Curves

Stavros Garoufalidis[1]**, Piotr Kucharski**[2]**, Piotr Sułkowski**[2,3]

[1] School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
[2] Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland
[3] Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena,
   CA 91125, USA. E-mail: psulkows@theory.caltech.edu

Abstract: We analyze relations between BPS degeneracies related to Labastida-Mariño-Ooguri-Vafa (LMOV) invariants and algebraic curves associated to knots. We introduce a new class of such curves, which we call extremal A-polynomials, discuss their special properties, and determine exact and asymptotic formulas for the corresponding (extremal) BPS degeneracies. These formulas lead to nontrivial integrality statements in number theory, as well as to an improved integrality conjecture, which is stronger than the known M-theory integrality predictions. Furthermore, we determine the BPS degeneracies encoded in augmentation polynomials and show their consistency with known colored HOMFLY polynomials. Finally, we consider refined BPS degeneracies for knots, determine them from the knowledge of super-A-polynomials, and verify their integrality. We illustrate our results with twist knots, torus knots, and various other knots with up to 10 crossings.

Contents

1. Introduction

There are many profound relations between quantum field theory, string theory and knot theory. This paper focuses on two aspects of the polynomial knot invariants—in particular HOMFLY polynomials, superpolynomials, and their specializations—colored by symmetric powers of the fundamental representation:

- The special geometry of algebraic curves of a knot
- The integrality of the BPS invariants of a knot.

By (affine) algebraic curves we mean curves obtained as classical limits of recursions of knot polynomials. Such recursions are known to exist (for colored Jones polynomials [GL05,Hik04], or colored HOMFLY and superpolynomials of twist or torus knots) and conjectured to exist for all colored HOMFLY or superpolynomials [AV12,AGSF12, FGS13,NRZS12]. For the colored Jones polynomial, the above algebraic curve agrees with the A-polynomial [GL05,Hik04,Guk05]. For the colored HOMFLY polynomial, the above algebraic curve is conjectured [AV12,FGS13] to agree with the augmentation polynomial of a knot [Ng08]. The curve arising as the classical limit of recursions for colored superpolynomials is called the super-A-polynomial [FGS13,FS15].

On the other hand, by BPS invariants of a knot we mean the Labastida-Mariño-Ooguri-Vafa (LMOV) invariants of a knot [OV00,LM01,LMV00,LM02], or certain combinations thereof.

Our aim is to give exact formulas for a certain class of BPS invariants, as well as their asymptotic expansions to all orders, using the corresponding algebraic curve. One motivation of our work is as follows. On one hand, as stated above, various algebraic curves associated to knots arise as classical limits of recursion relations for knot polynomials (Jones, HOMFLY, or superpolynomials) colored by symmetric representations [GL05,Hik04,AV12,AGSF12,FGS13,FGSS13,NRZS12]. On the other hand, as we will show, one can restrict the defining relations between HOMFLY polynomials and LMOV invariants to the case of symmetric representations only. This implies that recursion relations for knots should encode information about LMOV invariants labeled by symmetric representations, and classical limits of these recursions should still capture some of this information. In this paper we make these statements precise. Our main results are the following.

Proposition 1.1. (a) Fix a knot K, a natural number r and an integer i. Then the BPS invariants $b_{r,i}$ can be read from

$$x\frac{\partial_x y(x,a)}{y(x,a)} = -\frac{1}{2}\sum_{s,j} r^2 b_{s,j}\frac{x^s a^j}{1 - x^s a^j}, \tag{1.1}$$

where $y = y(x, a) \in 1 + \mathbb{Q}[[x, a]]$ is an algebraic function of (x, a) that satisfies a polynomial equation

$$\mathcal{A}(x, y, a) = 0. \tag{1.2}$$

(b) Explicitly, $x\partial_x y / y$ is an algebraic function of (x, a), and if $x\partial_x y / y = \sum_{n,m \geq 0} a_{n,m} x^n a^m$, and $\mu(d)$ denotes the Möbius function, then

$$b_{r,i} = \frac{2}{r^2} \sum_{d|r,i} \mu(d) a_{\frac{r}{d}, \frac{i}{d}}. \tag{1.3}$$

The BPS invariants $b_{r,i}$ introduced above are certain combinations of LMOV invariants. In the string theory interpretation they are encoded in holomorphic disk amplitudes or superpotentials [OV00, AV00, AKV02] for D-branes conjecturally associated to knots, and more accurately they could be referred to as classical BPS invariants; however, in this paper, unless otherwise stated, we simply call them BPS invariants or degeneracies. The definition of $b_{r,i}$ is given in Sect. 2.1. The polynomial $\mathcal{A}(x, y, a)$, sometimes referred to as the dual A-polynomial, is defined in Sect. 2.2. Equation (1.1) gives exact formulas for the BPS invariants $b_{r,i}$ for arbitrary r, i, as well as asymptotic expansions to all orders of r when (r, i) are along a ray. In particular, one obtains exact formulas and asymptotic expansions for the BPS invariants $b_r^{\pm} = b_{r, r \cdot c_{\pm}}$ along the two extreme rays, which we call extremal BPS invariants. The extremal BPS invariants $b_r^{\pm}$ can also be expressed in terms of coefficients of, respectively, maximal and minimal powers of a in HOMFLY polynomials. The coefficients of these extremal powers are also referred to as top-row and bottom-row HOMFLY polynomials (*rows* refer to the components of the diagram representing HOMFLY homology, see e.g. [GGS13]).

We call the algebraic curves that encode extremal BPS degeneracies the extremal A-polynomials, and for a knot K we denote them $\mathcal{A}_K^{\pm}(x, y)$. We also refer to $\mathcal{A}_K^+(x, y)$ and $\mathcal{A}_K^-(x, y)$ respectively as top and bottom A-polynomials. In this work, among the others, we determine extremal A-polynomials for various knots. We also discuss their properties; we note here that in particular they are tempered, i.e., the roots of their face polynomials are roots of unity, which is also referred to as the quantizability condition [FGS13, FS15] and is the manifestation of the so-called K_2 condition [GS12]. The extremal A-polynomials for twist knots, for a family of $(2, 2p + 1)$ torus knots, and for various knots with up to 10 crossings are given in Appendix A. The extremal A-polynomials for twist knots are also given below.

Fix an integer $p \neq 0, 1$ and consider the family of hyperbolic twist knots K_p. For $p = -1, -2, -3, \ldots$ these are $4_1, 6_1, 8_1, \ldots$ knots; for $p = 2, 3, 4 \ldots$ these are $5_2, 7_2, 9_2, \ldots$ knots. Much more can be said for this family of knots.

Proposition 1.2. *The extremal BPS invariants of twist knots are given by*

$$b_{K_p, r}^- = -\frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right) \binom{3d - 1}{d - 1}, \quad b_{K_p, r}^+ = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right) \binom{(2|p| + 1)d - 1}{d - 1} \tag{1.4}$$

for $p \leq -1$ and

$$b_{K_p, r}^- = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)(-1)^d \binom{2d - 1}{d - 1}, \quad b_{K_p, r}^+ = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)(-1)^d \binom{(2p + 2)d - 1}{d - 1} \tag{1.5}$$

for $p \geq 2$.

Our formulas and the integrality of the BPS invariants lead to nontrivial integrality statements among sequences of rational numbers. In particular, it implies that $b^{\pm}_{K_p,r}$ are integers for all natural numbers r and all integers $p \neq 0, 1$. Note that this implies a nontrivial statement, that for fixed r the sums in expressions (1.4) and (1.5) are divisible by r^2. Also note that the above BPS degeneracies are determined explicitly for an infinite range of r; this is in contrast to other LMOV invariants in the literature, which were determined explicitly only for some finite range of labeling representations (Young diagrams consisting up to several boxes) [LM01,LMV00,LM02,RS01,ZR13,ZR12,IMMM12].

What's more, we experimentally discover an Improved Integrality for the BPS invariants, observed by Kontsevich for algebraic curves satisfying the K_2 condition [Kon] (which is the same condition as already mentioned above [GS12]).

Conjecture 1.3 (*Improved Integrality*). *Given a knot there exist nonzero integers $\gamma^{\pm}$ such that for any $r \in \mathbb{N}$*

$$\frac{1}{r}\gamma^{\pm}b^{\pm}_r \in \mathbb{Z}. \tag{1.6}$$

We checked the above conjecture for twist knots, torus knots and several knots with up to 10 crossings. The values of $\gamma^{\pm}$ for various knots are given in Table 1. Note some resemblance of the above conjecture to the Wolstenholme's theorem, as well as Gauss congruences in number theory. Also note that this integrality conjecture is more general than the integrality of LMOV invariants [OV00,LM01,LMV00,LM02], which only implies integrality of b_r. In fact the divisibility by an additional factor of r would be a property of Gopakumar-Vafa invariants, rather than LMOV invariants; this suggests that there are some hidden Calabi-Yau manifolds that could be associated to knots. It would be interesting to give a physical interpretation of this property and of the conjectured integer knot invariants $\gamma^{\pm}$.

Our next proposition illustrates the special geometry of the extremal A-polynomials of the twist knots.

Proposition 1.4. *(a) The extremal A-polynomials of twist knots are given by*

$$\mathcal{A}^-_{K_p}(x, y) = x - y^4 + y^6, \qquad \mathcal{A}^+_{K_p}(x, y) = 1 - y^2 + xy^{4|p|+2}, \quad p \leq -1, \tag{1.7}$$

$$\mathcal{A}^-_{K_p}(x, y) = 1 - y^2 - xy^4, \quad \mathcal{A}^+_{K_p}(x, y) = 1 - y^2 - xy^{4p+4}, \qquad p \geq 2. \tag{1.8}$$

(b) The algebraic curves $\mathcal{A}^{\pm}_{K_p}(x, y^{1/2})$ have a distinguished solution $y = y(x) \in 1 + x\mathbb{Z}[[x]]$ such that

$$\boxed{y \text{ and } xy'/y \text{ are algebraic hypergeometric functions}} \tag{1.9}$$

Explicit formulas for those solutions can be found in Sect. 4. Note that extremal A-polynomials for twist knots (and as we will see, also for some other knots) coincide with framed mirror curves of $\mathbb{C}^3$, and for this reason the corresponding extremal BPS invariants (1.4) and (1.5) coincide with framed $\mathbb{C}^3$ BPS invariants [AV00,AKV02].

Note that if y is algebraic, so is xy'/y. The converse nearly holds. The next theorem was communicated to us by Kontsevich in the 2011 Arbeitstagung talk. A proof, using the solution to the Grothendieck-Katz conjecture, was written in Kassel-Reutenauer [KR14].

Theorem 1.1. *If $f \in \mathbb{Z}[[x]]$ is a formal power series with integer coefficients such that $g = xf'/f$ is algebraic, then f is algebraic.*

With the above notation, the converse holds trivially: if f is algebraic, so is g, without any integrality assumption on the coefficients of f. The integrality hypothesis is required in Theorem 1.1: if $f = e^x$ then $g = x$. Note finally that the class of algebraic hypergeometric functions has been completely classified in [BH89,Beu10]. Using this, one can also determine the functions that satisfy (1.9). We will not pursue this here.

The special geometry property (1.9) does not seem to hold for non-twist hyperbolic knots.

Next, we point out more information encoded in extremal A-polynomials.

Lemma 1.5. *The exponential growth rates of $b_r^{\pm}$ are given by the zeros of y-discriminant of the corresponding extremal A-polynomials $\mathcal{A}_K^{\pm}(x, y)$*

$$x_0 = \lim_{r \to \infty} \frac{b_r^{\pm}}{b_{r+1}^{\pm}} \quad \Rightarrow \quad \mathrm{Disc}_y \mathcal{A}_K^{\pm}(x_0, y) = 0 \tag{1.10}$$

For example, for K_p torus knots with $p \leq -1$, the y-discriminant of $\mathcal{A}_{K_p}^{-}(x, y)$ in (1.7) is given by

$$\mathrm{Disc}_y \mathcal{A}_{K_p}^{-}(x, y) = -64x^3(-4 + 27x)^2$$

and its zero $x_0 = \frac{4}{27}$ matches the exponential growth rate of $b_{K_p,r}^{-}$ in (1.4), which can be obtained from Stirling formula.

Furthermore, we generalize the above analysis of extremal A-polynomials and extremal BPS states by introducing the dependence on variables a and t. As conjectured in [AV12,FGS13], the augmentation polynomials [Ng11,Ng08], which depend on variable a, should agree with Q-deformed polynomials obtained as classical limits of recursions for colored HOMFLY polynomials [AV12]. We verify that this is indeed the case by computing corresponding BPS invariants from augmentation polynomials, and verifying that they are consistent with LMOV invariants determined from known colored HOMFLY polynomials for various knots with up to 10 crossings. While using our method we can determine BPS invariants for arbitrarily large representations S^r from augmentation polynomials, for more complicated knots the colored HOMFLY polynomials are known explicitly only for several values of r, see e.g., [NRZ13,Wed]. Nonetheless, this is already quite a non-trivial check.

Finally, we introduce the dependence on the parameter t, consider refined BPS degeneracies arising from appropriate redefinitions of super-A-polynomials, and show their integrality.

We note that apart from the relation to the LMOV invariants our results have an interpretation also from other physical perspectives. In particular, recently a lot of attention has been devoted to the so-called 3d-3d duality, which relates knot invariants to 3-dimensional $\mathcal{N} = 2$ theories [DGG14,FGS13,CDGS14]. In this context the A-polynomial curves, such as (1.2) or extremal A-polynomials, represent the moduli space of vacua of the corresponding $\mathcal{N} = 2$ theories. Furthermore, the Eq. (1.1) can be interpreted as imposing (order by order) relations between BPS degeneracies, which should arise from relations in the corresponding chiral rings. We also note that in the corresponding brane system extremal invariants arise from the $\mathbb{C}^3$ limit of the underlying resolved conifold geometry, and the precise way of taking this limit is encoded in the integers $c_{\pm}$ mentioned below (1.3) that specify the extreme rays. We comment on these and other physical interpretations in sect. 5 and plan to analyze them further in future work.

2. BPS Invariants of Knots From Algebraic Curves

2.1. BPS invariants for knots. In this section we recall the formulation of Labastida-Mariño-Ooguri-Vafa (LMOV) invariants and discuss its form in case of S^r-colored HOMFLY polynomials. The starting point is to consider the Ooguri-Vafa generating function [OV00,LM01,LMV00,LM02]

$$Z(U, V) = \sum_R \text{Tr}_R U \, \text{Tr}_R V = \exp \Big(\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \, U^n \text{Tr} \, V^n \Big), \qquad (2.1)$$

where $U = P \exp \oint_K A$ is the holonomy of $U(N)$ Chern-Simons gauge field along a knot K, V can be interpreted as a source, and the sum runs over all representations R, i.e., all two-dimensional partitions. The LMOV conjecture states that the expectation value of the above expression takes the following form

$$\langle Z(U, V) \rangle = \sum_R P_{K,R}(a, q) \text{Tr}_R V = \exp \Big(\sum_{n=1}^{\infty} \sum_R \frac{1}{n} f_{K,R}(a^n, q^n) \text{Tr}_R V^n \Big), \quad (2.2)$$

where the expectation value of the holonomy is identified with the unnormalized HOMFLY polynomial of a knot K, $\langle \text{Tr}_R U \rangle = P_{K,R}(a, q)$, and the functions $f_{K,R}(a, q)$ take form

$$f_{K,R}(a, q) = \sum_{i,j} \frac{N_{R,i,j} a^i q^j}{q - q^{-1}}, \qquad (2.3)$$

where $N_{R,i,j}$ are famous BPS degeneracies, or LMOV invariants, a term that we will use interchangably; in particular they are conjectured to be integer. In string theory interpretation they count D2-branes ending on D4-branes that wrap a Lagrangian submanifold associated to a given knot K. From two-dimensional space-time perspective D2-branes are interpreted as particles with charge i, spin j, and magnetic charge R. For a fixed R there is a finite range of i and j for which $N_{R,i,j}$ are non-zero.

In what follows we are interested in the case of one-dimensional source $V = x$. In this case $\text{Tr}_R V \neq 0$ only for symmetric representations $R = S^r$ (labeled by partitions with a single row with r boxes [AM98]), so that $\text{Tr}_{S^r}(x) = x^r$. For a knot K, let us denote $P_K(x, a, q) = \langle Z(U, x) \rangle$, and let $P_{K,r}(a, q) \in \mathbb{Q}(a, q)$ denote the S^r-colored HOMFLY polynomial of K. For a detailed definition of the latter, see for example [AM98]. In this setting (2.2) reduces to the following expression

$$P_K(x, a, q) = \sum_{r=0}^{\infty} P_{K,r}(a, q) x^r = \exp \Big(\sum_{r,n \geq 1} \frac{1}{n} f_{K,r}(a^n, q^n) x^{nr} \Big). \qquad (2.4)$$

Note that we use the unnormalized (or unreduced) HOMFLY polynomials, so that the unknot is normalized as $P_{\mathbf{0_1},1}(a, q) = (a - a^{-1})/(q - q^{-1})$. Often, we will drop the knot K from the notation. Note that $f_r(a, q)$ is a universal polynomial (with rational coefficients) of $P_{r/d}(a^k, q^k)$ for all divisors d of r. For instance, we have:

$$f_1(a, q) = P_1(a, q),$$

$$f_2(a, q) = P_2(a, q) - \frac{1}{2} P_1(a, q)^2 - \frac{1}{2} P_1(a^2, q^2),$$

$$f_3(a, q) = P_3(a, q) - P_1(a, q) P_2(a, q) + \frac{1}{3} P_1(a, q)^3 - \frac{1}{3} P_1(a^3, q^3) \qquad (2.5)$$

$$f_4(a, q) = P_4(a, q) - P_1(a, q) P_3(a, q) - \frac{1}{2} P_2(a, q)^2 + P_1(a, q)^2 P_2(a, q)$$

$$- \frac{1}{4} P_1(a, q)^4 - \frac{1}{2} P_2(a^2, q^2) + \frac{1}{4} P_1(a^2, q^2)^2.$$

It follows that $f_r(a, q) \in \mathbb{Q}(a, q)$. The LMOV conjecture asserts that $f_r(a, q)$ can be expressed as a finite sum

$$f_r(a, q) = \sum_{i,j} \frac{N_{r,i,j} a^i q^j}{q - q^{-1}}, \qquad N_{r,i,j} \in \mathbb{Z}.$$

and in this case the BPS degeneracies $N_{r,i,j}$ are labeled by a natural number r.

We now explain how to extract BPS degeneracies from the generating function (2.4). First we write it in product form

$$P(x, a, q) = \sum_r P_r(a, q) x^r = \exp \Big(\sum_{r,n \geq 1; i,j} \frac{1}{n} \frac{N_{r,i,j} (x^r a^i q^j)^n}{q^n - q^{-n}} \Big)$$

$$= \exp \Big(\sum_{r \geq 1; i,j; k \geq 0} N_{r,i,j} \log(1 - x^r a^i q^{j+2k+1}) \Big)$$

$$= \prod_{r \geq 1; i,j; k \geq 0} \Big(1 - x^r a^i q^{j+2k+1} \Big)^{N_{r,i,j}}$$

$$= \prod_{r \geq 1; i,j} \Big(x^r a^i q^{j+1}; q^2 \Big)_\infty^{N_{r,i,j}} \qquad (2.6)$$

where the q-Pochhammer symbol (or quantum dilogarithm) notation is used

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - x q^k). \qquad (2.7)$$

Then, in the limit $q = e^\hbar \to 1$, using well known asymptotic expansion of the quantum dilogarithm (see e.g., [GS12]), we get the following asymptotic expansion of $P(x, a, e^\hbar)$

$$P(x, a, e^\hbar) = \exp \Big(\sum_{r,i,j} N_{r,i,j} \Big(\frac{1}{2\hbar} \mathrm{Li}_2(x^r a^i) - \frac{j}{2} \log(1 - x^r a^i) + O(\hbar) \Big) \Big)$$

$$= \exp \Big(\frac{1}{2\hbar} \sum_{r,i} b_{r,i} \mathrm{Li}_2(x^r a^i) - \sum_{r,i,j} \frac{j}{2} N_{r,i,j} \log(1 - x^r a^i) + O(\hbar) \Big)$$

$$= \exp \Big(\frac{1}{\hbar} S_0(x, a) + S_1(x, a) + O(\hbar) \Big), \qquad (2.8)$$

where

$$S_0(x, a) = \frac{1}{2} \sum_{r,i} b_{r,i} \text{Li}_2(x^r a^i), \tag{2.9}$$

$$S_1(x, a) = -\frac{1}{2} j \sum_{r,i,j} \log(1 - x^r a^i). \tag{2.10}$$

Above we introduced

$$b_{r,i} = \sum_j N_{r,i,j} \tag{2.11}$$

that appear at the lowest order in $\hbar$ expansion in the exponent of (2.8) and can be interpreted as the classical BPS degeneracies. These degeneracies are of our main concern. In the string theory interpretation they determine holomorphic disk amplitudes or superpotentials [OV00, AV00, AKV02] for D-branes conjecturally associated to knots. In what follows, unless otherwise stated, by BPS degeneracies we mean these numbers.

Our next task is to compute $S_0(x, a)$. To do so, we use a linear q-difference equation for $P(x, a, q)$ reviewed in the next section.

2.2. Difference equations and algebraic curves. In this section we introduce various algebraic curves associated to knots. First, recall that the colored Jones polynomial $J_{K,r}(q) \in \mathbb{Z}[q^{\pm 1}]$ of a knot K can be defined as a specialization of the colored HOMFLY polynomial:

$$J_{K,r}(q) = P_{K,r}(q^2, q).$$

It is known [GL05] that the colored Jones polynomial satisfies a linear q-difference equation of the form

$$\widehat{A}_K(\widehat{M}, \widehat{L}, q) J_{K,r}(q) = 0, \tag{2.12}$$

where $\widehat{A}_K$ is a polynomial in all its arguments, and $\widehat{M}$ and $\widehat{L}$ are operators that satisfy the relation $\widehat{M}\widehat{L} = q\widehat{L}\widehat{M}$ and act on colored Jones polynomials by

$$\widehat{M} J_{K,r}(q) = q^r J_{K,r}(q), \qquad \widehat{L} J_{K,r}(q) = J_{K,r+1}(q). \tag{2.13}$$

The AJ Conjecture [Gar04] states that

$$\widehat{A}_K(\widehat{M}, \widehat{L}, 1) = A_K(M, L) \tag{2.14}$$

where $A_K(M, L)$ is the A-polynomial of K [CCG$^+$94]. The meaning of the above equality is the following: express $\widehat{A}_K(\widehat{M}, \widehat{L}, q)$ as a $\mathbb{Z}[q^{\pm 1}]$-linear combination of a finite set of monomials in $\widehat{M}$ and $\widehat{L}$. Then, set $q = 1$ and replace $\widehat{M}$ and $\widehat{L}$ by M and L respectively. The result should agree with the right hand side, up to multiplication with a polynomial in M.

It is also known [GLL] that the colored HOMFLY polynomial of a knot satisfies a linear q-difference equation of the form

$$\widehat{A}(\widehat{M}, \widehat{L}, a, q) P_r(a, q) = 0. \tag{2.15}$$

The corresponding 3-variable polynomial $A(M, L, a) = A(M, L, a, 1)$ defines a family of algebraic curves parametrized by a. A further conjecture [AV12, FGS13] identifies the 3-variable polynomial $A(M, L, a)$ with the augmentation polynomial of knot contact homology [Ng11, Ng08].

We further assume the existence of the super-A-polynomial of a knot. In other words, we assume the existence of the refined colored HOMFLY polynomial $P_r(a, q, t)$ of a knot (that specializes at $t = -1$ to the usual colored HOMFLY polynomial), and we also assume that $P_r(a, q, t)$ satisfies a linear q-difference equation [FGS13, FGSS13]

$$\widehat{A}^{\text{super}}(\widehat{M}, \widehat{L}, a, q, t) P_r(a, q, t) = 0. \tag{2.16}$$

The specialization $\widehat{A}^{\text{super}}(M, L, a, 1, t)$ can be thought of as an (a, t)-family of A-polynomials of a knot.

In the remainder of this section we discuss a dual version $\mathcal{A}(x, y, a)$ of the algebraic curve $A(M, L, a)$.

Lemma 2.1. *Fix a sequence $P_r(a, q)$ which is annihilated by an operator $\widehat{A}(\widehat{M}, \widehat{L}, a, q)$ and consider the generating function $P(x, a, q) = \sum_{r=0}^{\infty} P_r(a, q) x^r$. Then,*

$$\widehat{A}(\widehat{y}, \widehat{x}^{-1}, a, q) P(x, a, q) = const, \tag{2.17}$$

where

$$\widehat{x} P(x, a, q) = x P(x, a, q), \qquad \widehat{y} P(x, a, q) = P(qx, a, q) \tag{2.18}$$

satisfy $\widehat{y}\widehat{x} = q\widehat{x}\widehat{y}$, and const is an (a, q)-dependent term.

Proof. We have

$$\widehat{A}(\widehat{M}, \widehat{L}) P(x, a, q) = \sum_r x^r \widehat{A}(\widehat{M}, \widehat{L}) P_r(a, q) = 0. \tag{2.19}$$

On the other hand, acting with $\widehat{M}$ and $\widehat{L}$ on this generating function (and taking care of the boundary terms) we get

$$\widehat{M} P(x, a, q) = \sum_{r=0}^{\infty} P_r(a, q)(qx)^r = P(qx, a, q), \tag{2.20}$$

$$\widehat{L} P(x, a, q) = \sum_{r=0}^{\infty} P_{r+1}(a, q) x^r = \frac{1}{x}\Big(P(x, a, q) - P_0(a, q)\Big).$$

Therefore the action of $\widehat{M}$ and $\widehat{L}$ on $P(x)$ can be identified, respectively, with the action of operators $\widehat{y}$ and $\widehat{x}^{-1}$, up to the boundary term arising from $r = 0$. From the property of the recursion relations for the HOMFLY polynomial the result follows. □

Applying the above lemma to the colored HOMFLY polynomial $P_r(a, q)$, this motivates us to introduce the operator

$$\widehat{\mathcal{A}}(\widehat{x}, \widehat{y}, a, q) = \widehat{A}(\widehat{y}, \widehat{x}^{-1}, a, q). \tag{2.21}$$

Moreover it can be shown that (2.16) holds also for negative r (with $P_r(a, q) = 0$ for negative r) [GLL], and in consequence (2.17) can be rewritten as

$$\widehat{\mathcal{A}}(\widehat{x}, \widehat{y}, a, q) P(x, a, q) = 0. \tag{2.22}$$

In the limit $q \to 1$ we can consider the algebraic curve

$$\mathcal{A}(x, y, a) = A(y, x^{-1}, a) = 0. \tag{2.23}$$

2.3. The Lambert transform. In this section we recall the Lambert transform of two sequences (a_n) and (b_n) which is useful in the proof of Proposition 1.1.

Lemma 2.2. *(a) Consider two sequences (a_n) and (b_n) for $n = 1, 2, 3, \ldots$ that satisfy the relation*

$$a_n = \sum_{d|n} b_d \tag{2.24}$$

for all positive natural numbers n. Then we have:

$$b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d \tag{2.25}$$

where μ is the Möbius function. Moreover, we have the Lambert transformation property

$$\sum_{n=1}^{\infty} a_n q^n = \sum_{n=1}^{\infty} b_n \frac{q^n}{1 - q^n} \tag{2.26}$$

and the Dirichlet series property

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \tag{2.27}$$

(b) If (a_n) has an asymptotic expansion

$$a_n \sim \sum_{(\lambda,\alpha)} \lambda^n n^\alpha \left(c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right) \tag{2.28}$$

where the first sum is a finite sum of pairs (λ, α) such that $|\lambda|$ is fixed, then so does (b_n) and vice-versa.

Proof. Part (b) follows from Eq. (2.27) easily. For a detailed discussion, see the appendix to [Zei06] by Zagier. □

Lemma 2.3. *Suppose $y \in \mathbb{Z}[[x]]$ is algebraic with constant term $y(0) = 1$. Write*

$$y = 1 + \sum_{n=1}^{\infty} c_n x^n = \prod_{n=1}^{\infty} (1 - x^n)^{b_n}.$$

If y has a singularity in the interior of the unit circle then (b_n) has an asymptotic expansion of the form (2.28). Moreover, the singularities of the multivalued function $y = y(x)$ are the complex roots of the discriminant of $p(x, y)$ with respect to y, where $p(x, y) = 0$ is a polynomial equation.

Proof. We have:

$$\log y = \sum_{n=1}^{\infty} b_n \log(1 - x^n)$$

thus if $z = xd \log y = xy'/y$, then we have

$$z = -\sum_{n=1}^{\infty} nb_n \frac{x^n}{1 - x^n}.$$

Now z is algebraic by the easy converse to Theorem 1.1. It follows that the coefficients (a_n) of its Taylor series

$$z = \sum_{n=1}^{\infty} a_n x^n$$

is a sequence of Nilsson type [Gar11b]. Since z is algebraic, there are no $\log n$ terms in the asymptotic expansion. Moreover, the exponential growth rate is bigger than 1, in absolute value. Part (b) of Lemma 2.2 concludes the proof. $\square$

2.4. Proof of Proposition 1.1. Let us define

$$y(x, a) = \lim_{q \to 1} \frac{P(qx, a, q)}{P(x, a, q)} = \lim_{q \to 1} \prod_{r \geq 1; i, j; k \geq 0} \left(\frac{1 - x^r a^i q^{r+j+2k+1}}{1 - x^r a^i q^{j+2k+1}} \right)^{N_{r,i,j}}$$

$$= \prod_{r \geq 1; i} (1 - x^r a^i)^{-rb_{r,i}/2}. \tag{2.29}$$

If $P(x, a, q)$ is annihilated by $\widehat{\mathcal{A}}(\widehat{x}, \widehat{y}, a, q)$, it follows that $y = y(a, x)$ is a solution of the polynomial equation

$$\mathcal{A}(x, y, a) = 0. \tag{2.30}$$

Indeed, divide the recursion

$$\widehat{\mathcal{A}}(\widehat{x}, \widehat{y}, a, q) P(x, a, q) = 0$$

by $P(x, a, q)$, and observe that

$$\lim_{q \to 1} \frac{P(q^j x, a, q)}{P(x, a, q)} = \prod_{l=1}^{j} \lim_{q \to 1} \frac{P(q^l x, a, q)}{P(q^{l-1} x, a, q)} = y(x, a)^j.$$

Taking the logarithm and then differentiating (2.29) concludes Equation (1.1). Part (b) of Proposition 1.1 follows from Lemma 2.2. $\square$

2.5. Refined BPS invariants. In this section we discuss refined BPS invariants $N_{r,i,j,k}$. In full generality, we can consider the generating function of superpolynomials $P_r(a, q, t)$; suppose that it has the product structure analogous to (2.6), however with an additional t-dependence

$$P(x, a, q, t) = \sum_{r=0}^{\infty} P_r(a, q, t) x^r = \prod_{r \geq 1; i, j, k; n \geq 0} \left(1 - x^r a^i t^j q^{k+2n+1} \right)^{N_{r,i,j,k}} \tag{2.31}$$

We conjecture that the refined BPS numbers $N_{r,i,j,k}$ encoded in this expression are integers. As in this work we are mainly interested in invariants in the $q \to 1$ limit, encoded in (classical) algebraic curves, let us denote them as

$$b_{r,i,j} = \sum_k N_{r,i,j,k}. \tag{2.32}$$

In this case the curves in question are of course the dual versions of super-A-polynomial $A^{\text{super}}(M, L, a, t)$, with arguments transformed as in (2.23), i.e.,

$$\mathcal{A}(x, y, a, t) = A^{\text{super}}(y, x^{-1}, a, t) = 0. \tag{2.33}$$

Solving this equation for $y = y(x, a, t)$ and following steps that led to (1.1), we find now

$$x \frac{\partial_x y(x, a, t)}{y(x, a, t)} = \frac{1}{2} \sum_{r,i,j} r^2 b_{r,i,j} \frac{x^r a^i t^j}{1 - x^r a^i t^j}, \tag{2.34}$$

and from such an expansion $b_{r,i,j}$ can be determined. Conjecturally these should be integer numbers; as we will see in several examples this turns out to be true.

3. Extremal Invariants

3.1. Extremal BPS invariants. In this section we define extremal BPS invariants of knots. If $P_r(a, q)$ is a q-holonomic sequence, it follows that the minimal and maximal exponent with respect to a is a quasi-linear function of r, for large enough r. This follows easily from the Lech-Mahler-Skolem theorem as used in [Gar11a] and is also discussed in detail in [VdV]. We now restrict our attention to knots that satisfy

$$P_r(a, q) = \sum_{i=r \cdot c_-}^{r \cdot c_+} a^i p_{r,i}(q) \tag{3.1}$$

for some integers $c_\pm$ and for every natural number r, where $p_{r,r \cdot c_\pm}(q) \neq 0$. This is a large class of knots – in particular two-bridge knots and torus knots have this property. For such knots, we can consider the extremal parts of the colored HOMFLY polynomials (i.e. their top and bottom rows, that is the coefficients of maximal and minimal powers of a), defined as one-variable polynomials

$$P_r^\pm(q) = p_{r,r \cdot c_\pm}(q). \tag{3.2}$$

Likewise, we define the extremal LMOV invariants by

$$f_r^\pm(q) = \sum_j \frac{N_{r,r \cdot c_\pm, j} q^j}{q - q^{-1}}, \tag{3.3}$$

and the extremal BPS invariants by

$$b_r^\pm = b_{r,r \cdot c_\pm} = \sum_j N_{r,r \cdot c_\pm, j}. \tag{3.4}$$

Table 1. Improved Integrality: values of γ^- and γ^+ for various knots

Knot	γ^-	γ^+	Knot	γ^-	γ^+	Knot	γ^-	γ^+
K_{-1-6k}	2	2	K_{2+6k}	6	2	6_2	3	30
K_{-2-6k}	2	3	K_{3+6k}	6	3	6_3	6	6
K_{-3-6k}	2	2	K_{4+6k}	6	2	7_3	1	10
K_{-4-6k}	2	1	K_{5+6k}	6	1	7_5	3	30
K_{-5-6k}	2	6	K_{6+6k}	6	6	8_{19}	7	1
K_{-6-6k}	2	1	K_{7+6k}	6	1	10_{124}	2	8
$T_{2,2p+1}$	$2p+3$	$2p-1$						

For twist knots K_p the range of the subscript is labeled by $k = 0, 1, 2, \ldots$, and $T_{2,2p+1}$ denotes $(2, 2p + 1)$ torus knot

We also refer to b_r^+ and b_r^- as, respectively, top and bottom BPS invariants. Finally, we define the extremal part of the generating series $P(x, a, q)$ by

$$P^\pm(x, q) = \sum_{r=0}^\infty P_r^\pm(q)x^r = \prod_{r\geq 1; j; k\geq 0} \left(1 - x^r q^{j+2k+1}\right)^{N_{r, r \cdot c_\pm, j}}. \tag{3.5}$$

The analogue of Eq. (2.8) is

$$P^\pm(x, e^\hbar) = \exp\left(\frac{1}{2\hbar} \sum_r b_r^\pm \mathrm{Li}_2(x^r) - \sum_{r, j} \frac{j}{2} N_{r, r \cdot c_\pm, j} \log(1 - x^r) + O(\hbar)\right). \tag{3.6}$$

It follows from the LMOV conjecture that $b_r^\pm$, as combinations of LMOV invariants $N_{r, r \cdot c_\pm, j}$, are integer. Moreover, according to the Improved Integrality conjecutre 1.6, for each knot one can find integer numbers $\gamma^\pm$, such that

$$\frac{1}{r}\gamma^\pm b_r^\pm \in \mathbb{Z}. \tag{3.7}$$

The numbers $\gamma^\pm$ can be regarded as new invariants of a knot. We compute these numbers for various knots in Sect. 4, with the results summarized in Table 1.

3.2. Extremal A-polynomials. It is easy to see that if $P_r(a, q)$ is annihilated by $\widehat{A}(\widehat{M}, \widehat{L}, a, q)$, then its extremal part $P_r^\pm(q)$ is annihilated by the operator $\widehat{A}^\pm(\widehat{M}, \widehat{L}, q)$ obtained by multiplying $\widehat{A}(\widehat{M}, \widehat{L}, a^{\mp 1}, q)$ by $a^{\pm r c_\pm}$ (to make every power of a nonnegative), and then setting $a = 0$. This allows us to introduce the extremal analogues of the curve (2.23) defined as distinguished, irreducible factors in

$$\mathcal{A}(xa^{-c_\pm}, y, a)|_{a^{\mp 1}\to 0} \tag{3.8}$$

that determine the extremal BPS degeneracies. We call these curves extremal A-polynomials and denote them $\mathcal{A}^\pm(x, y)$. We also refer to $\mathcal{A}^+(x, y)$ and $\mathcal{A}^-(x, y)$ as top and bottom A-polynomials respectively. The extremal A-polynomials that we determine in this work are listed in Appendix A.

Among various interesting properties of extremal A-polynomials we note that they are tempered, i.e., the roots of their face polynomials are roots of unity. This is a manifestation of their quantizability and the so-called K_2 condition [GS12, FGS13, Kon], and presumably is related to the Improved Integrality of the corresponding extremal BPS states.

3.3. Extremal BPS invariants from extremal A-polynomials. In this section we give the analogue of Proposition 1.1 for extremal BPS invariants.

Proposition 3.1. *(a) Fix a knot K and a natural number r. Then the extremal BPS invariants $b_r^\pm$ are given by*

$$x \frac{(y^\pm)'(x)}{y^\pm(x)} = \frac{1}{2} \sum_{r \geq 1} r^2 b_r^\pm \frac{x^r}{1 - x^r}, \tag{3.9}$$

where $y^\pm = y^\pm(x) \in 1 + \mathbb{Q}[[x]]$ is an algebraic function of x that satisfies a polynomial equation

$$\mathcal{A}^\pm(x, y^\pm) = 0.$$

(b) Explicitly, $x \partial_x y^\pm / y^\pm$ is an algebraic function of x and if

$$x(y^\pm)'(x)/y^\pm(x) = \sum_{n \geq 0} a_n^\pm x^n, \tag{3.10}$$

then

$$b_r^\pm = \frac{2}{r^2} \sum_{d|r} \mu(d) a_{\frac{r}{d}}^\pm. \tag{3.11}$$

Proof. We define

$$y^\pm(x) = \lim_{q \to 1} \frac{P^\pm(qx, q)}{P^\pm(x, q)} = \lim_{q \to 1} \prod_{r \geq 1; i, j; k \geq 0} \left(\frac{1 - x^r q^{r+j+2k+1}}{1 - x^r q^{j+2k+1}} \right)^{N_{r, r \cdot c_\pm, j}}$$

$$= \prod_{r \geq 1; i} (1 - x^r)^{-r b_r^\pm / 2}. \tag{3.12}$$

As in the proof of Proposition 1.1, it follows that $y^\pm(x)$ satisfies the polynomial equation

$$\mathcal{A}^\pm(x, y) = 0.$$

This concludes the first part. The second part follows just as in Proposition 1.1.   □

4. Examples and Computations

In this section we illustrate the claims and ideas presented earlier in many examples. First of all, LMOV invariants arise from redefinition of unnormalized knot polynomials. Therefore we recall that the unnormalized superpolynomial for the unknot reads [FGS13]

$$P_{\mathbf{0_1}, r}(a, q, t) = (-1)^{\frac{r}{2}} a^{-r} q^r t^{-\frac{3r}{2}} \frac{(-a^2 t^3; q^2)_r}{(q^2; q^2)_r}, \tag{4.1}$$

and the unnormalized HOMFLY polynomial arises from $t = -1$ specialization of this expression. In what follows we often take advantage of the results for normalized superpolynomials $P_r^{norm}(K, a, q, t)$ for various knots K, derived in [FGS13, FGSS13, NRZS12].

Then the unnormalized superpolynomials that we need from the present perspective differ simply by the unknot contribution

$$P_{K,r}(a, q, t) = P_{\mathbf{0_1},r}(a, q, t) P_{K,r}^{norm}(a, q, t). \tag{4.2}$$

Let us stress some subtleties related to various variable redefinitions. Super-A-polynomials for various knots, corresponding to the normalized superpolynomials $P_r^{norm}(K, a, q, t)$, were determined in [FGS13, FGSS13, NRZS12]; in the current notation we would write those polynomials using variables M and L, as $A^{super}(M, L, a, t)$. Super-A-polynomials in the unnormalized case (i.e. encoding asymptotics of unnormalized superpolynomials), which are relevant for our considerations, arise from $A^{super}(M, L, a, t)$ by the substitution

$$M \mapsto M, \qquad L \mapsto (-a^2 t^3)^{-1/2} \frac{1 + a^2 t^3 M}{1 - M} L. \tag{4.3}$$

In what follows we often consider a-deformed polynomials, which for knots considered in this paper are again simply obtained by setting $t = -1$ in $A^{super}(M, L, a, t)$. These a-deformed polynomials are not yet identical, however closely related (by a simple change of variables) to augmentation polynomials or Q-deformed polynomials; we will present these relations in detail in some examples.

4.1. The unknot. Let us illustrate first how our formalism works for the unknot. From the analysis of asymptotics, or recursion relations satisfied by (4.1), the following super-A-polynomial is determined[1] [FGS13]

$$A(M, L, a, t) = (-a^{-2} t^{-3})^{1/2} (1 + a^2 t^3 M^2) - (1 - M^2) L. \tag{4.4}$$

The analysis of the refined case essentially is the same as the unrefined, as the dependence on t can be absorbed by a redefinition of a. Therefore let us focus on the unrefined case. From (2.23) we find that, up to an irrelevant overall factor, the dual A-polynomial reads

$$\mathcal{A}(x, y, a) = x - a^2 x y^2 - a + a y^2. \tag{4.5}$$

From this expression we can immediately determine

$$y^2 = \frac{1 - x a^{-1}}{1 - x a}, \tag{4.6}$$

and comparing with (2.29) we find only two non-zero BPS invariants $b_{1,\pm 1} = \pm 1$ (and from LMOV formulas (2.5) one can check that there are also only two non-zero LMOV invariants $N_{1,\pm 1, j}$). These invariants represent of course two open M2-branes wrapping $\mathbb{P}^1$ in the conifold geometry [OV00, AV00] (for the refined case we also find just two refined BPS invariants). Furthermore, from (4.1) we find $c_{\pm} = \pm 1$, and therefore from (3.8) we determine the following extremal A-polynomials

$$\mathcal{A}^-(x, y) = 1 - x - y^2, \qquad \mathcal{A}^+(x, y) = 1 + x y^2 - y^2, \tag{4.7}$$

which represent two $\mathbb{C}^3$ limits of the resolved conifold geometry (in terms of $Y = y^2$, the curve (4.5) and the above extremal A-polynomials are the usual B-model curves for the conifold and $\mathbb{C}^3$ respectively).

---

[1] More precisely, due to present conventions, one has to substitute $M^2 \mapsto x, L \mapsto y, a^2 \mapsto a$ to obtain the curve from [FGS13] on the nose.
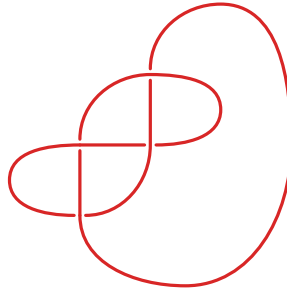
Fig. 1. The 4_1 knot

4.2. The 4_1 knot. As the second example we consider the 4_1 (figure-8) knot, see Fig. 1. The (normalized) superpolynomial for this knots reads [FGS13]

$$P_r^{norm}(a, q, t) = \sum_{k=0}^{\infty} (-1)^k a^{-2k} t^{-2k} q^{-k(k-3)} \frac{(-a^2 t q^{-2}, q^2)_k}{(q^2, q^2)_k}$$
$$\times (q^{-2r}, q^2)_k (-a^2 t^3 q^{2r}, q^2)_k. \quad (4.8)$$

From this formula, after setting $t = -1$, it is immediate to determine $N_{r,i,j}$ LMOV invariants using the explicit relations (2.5), up to some particular value of r. Instead, using the knowledge of associated algebraic curves, we will explicitly determine the whole family of these invariants, labeled by arbitrary r.

First of all, by considering recursion relations satisfied by P_r^{norm}, or from the analysis of its asymptotic behavior for large r, the following (normalized) super-A-polynomial is determined[2] in [FGS13]

$$A^{super}(M, L, a, t) = a^4 t^5 (M^2 - 1)^2 M^4 + a^2 t^2 M^4 (1 + a^2 t^3 M^2)^2 L^3 +$$
$$+ a^2 t (M^2 - 1)(1 + t(1 - t)M^2 + 2a^2 t^3 (t + 1)M^4 - 2a^2 t^4 (t + 1)M^6$$
$$+ a^4 t^6 (1 - t)M^8 - a^4 t^8 M^{10})L - (1 + a^2 t^3 M^2)(1 + a^2 t (1 - t)M^2$$
$$+ 2a^2 t^2 (t + 1)M^4 + 2a^4 t^4 (t + 1)M^6 + a^4 t^5 (t - 1)M^8 + a^6 t^7 M^{10})L^2. \quad (4.9)$$

We will use this formula when we consider refined BPS states; however at this moment let us consider its unrefined (i.e. $t = -1$) version. With the notation of Sect. 2.2 and Eq. (2.23) we find the dual A-polynomial

$$\mathcal{A}(x, y, a) = a^3 \left(y^6 - y^4 \right) + x \left(-a^6 y^{10} + 2a^4 y^8 - 2a^2 y^2 + 1 \right) \quad (4.10)$$
$$+ ax^2 \left(a^4 y^{10} - 2a^4 y^8 + 2y^2 - 1 \right) + x^3 \left(a^2 y^4 - a^4 y^6 \right).$$

From (4.2) and (4.8) we find that the HOMFLY polynomial in the fundamental ($r = 1$) representation is given by

$$P_1(a, q) = a^{-3} \frac{q}{(1 - q^2)} + a^{-1} \frac{q^4 + 1}{q(q^2 - 1)} + a \frac{(q^4 + 1)}{q(1 - q^2)} + a^3 \frac{q}{q^2 - 1}. \quad (4.11)$$

---

[2] Again, one has to substitute $M^2 \mapsto x$, $L \mapsto y$, $a^2 \mapsto a$ to obtain the curve from [FGS13] on the nose.

Comparing this with (3.1) we determine the value of $c_\pm$

$$c_- = -3, \qquad c_+ = 3, \tag{4.12}$$

so the extremal A-polynomials following from the definition (3.8) and the result (4.10) are given by

$$A^-(x, y) = x - y^4 + y^6, \qquad A^+(x, y) = 1 - y^2 + xy^6. \tag{4.13}$$

Note that $A^+(x, y^{-1}) = y^{-6}A^-(x, y)$, i.e. these curves agree up to $y \mapsto y^{-1}$ (and multiplication by an overall monomial factor) – this reflects the fact that the 4_1 knot is amphicheiral.

We can now extract the extremal BPS invariants from the curves (4.13). As these curves are cubic in terms of $Y = y^2$ variable, we can determine explicit solutions of the corresponding cubic equations. We will use two fortunate coincidences.

The first coincidence is that the unique solution $Y(x) = 1 + O(x)$ to the equation $A^-(x, Y) = x - Y^2 + Y^3 = 0$ is an algebraic hypergeometric function. Explicitly, we have

$$\begin{aligned}
Y(x) = Y^-(x) &= \frac{1}{3} + \frac{2}{3} \cos\left[\frac{2}{3} \arcsin\left(\sqrt{\frac{3^3 x}{2^2}}\right)\right] \\
&= \frac{1}{3}\left[1 - \sum_{n=0}^{\infty} \frac{2}{3n-1}\binom{3n}{n}x^n\right] \\
&= \frac{1}{3} + \frac{2}{3}\, {}_2F_1\left(-\frac{1}{3}, \frac{1}{3}; \frac{1}{2}; \frac{3^3 x}{2^2}\right).
\end{aligned} \tag{4.14}$$

The second coincidence is that $x\, \partial_x Y^-/Y^-$ is not only algebraic, but also hypergeometric. Explicitly, we have:

$$\begin{aligned}
x\frac{\partial_x Y^-(x)}{Y^-(x)} &= \frac{1}{3} - \frac{2}{3}\frac{\cos\left[\frac{1}{6}\arccos\left(1 - \frac{27x}{2}\right)\right]}{\sqrt{4 - 27x}} \\
&= -\sum_{n=1}^{\infty}\binom{3n-1}{n-1}x^n \tag{4.15} \\
&= \frac{1}{3} - \frac{1}{3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{3^3 x}{2^2}\right). \tag{4.16}
\end{aligned}$$

Recalling (3.11), we find that

$$x\frac{\partial_x Y^-(x)}{Y^-(x)} = \sum_{n=1}^{\infty} 2a_n^- x^n, \qquad a_n^- = -\frac{1}{2}\binom{3n-1}{n-1} \tag{4.17}$$

so that the extremal bottom BPS degeneracies (3.11) are given by

$$b_r^- = -\frac{1}{r^2}\sum_{d|r} \mu\left(\frac{r}{d}\right)\binom{3d-1}{d-1}. \tag{4.18}$$

Table 2. Extremal BPS invariants and their Improved Integrality for the 4_1 knot

r	$b_r^- = -b_r^+$	$2\frac{b_r^-}{r}$
1	-1	-2
2	-1	-1
3	-3	-2
4	-10	-5
5	-40	-16
6	-171	-57
7	-791	-226
8	$-3,828$	-957
9	$-19,287$	-4286
10	$-100,140$	$-20,028$
11	$-533,159$	$-96,938$
12	$-2,897,358$	$-482,893$
13	$-16,020,563$	$-2,464,702$
14	$-89,898,151$	$-12,842,593$
15	$-510,914,700$	$-68,121,960$

Several values of b_r^- are given in Table 2. Note that the integrality of b_r^- implies a nontrivial statement, that for each r the sum in (4.18) must be divisible by r^2.

In an analogous way we determine top BPS invariants. The above mentioned two coincidences persist. The solution $Y^+(x) = 1/Y^-(x) = 1 + O(x)$ of the equation $\mathcal{A}^+(x, Y) = 1 - Y + xY^3 = 0$ is given by

$$Y^+(x) = \frac{2}{\sqrt{3x}} \sin\left[\frac{1}{3} \arcsin\left(\sqrt{\frac{3^3 x}{2^2}}\right)\right] = \sum_{n=0}^{\infty} \frac{x^n}{2n+1}\binom{3n}{n} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \frac{3^3 x}{2^2}\right)$$

(4.19)

so that

$$x\frac{\partial_x Y^+(x)}{Y^+(x)} = -x\frac{\partial_x Y^-(x)}{Y^-(x)}.$$

(4.20)

Therefore $a_n^+ = -a_n^-$ and $b_r^+ = -b_r^-$, which is a manifestation of the amphicheirality of the 4_1 knot. The above results illustrate Proposition 1.2 for the $4_1 = K_{-1}$ twist knot.

Experimentally, it also appears that the Improved Integrality holds (1.6) with $\gamma^\pm = 2$; see Table 2.

Next, we discuss the asymptotics of $b_r^\pm$ for large r. Stirling's formula gives the asymptotics of a_r^-, and part (b) of Lemma 2.2 concludes that the asymptotics of b_r^- are given by

$$b_r^- = -\frac{1}{28\sqrt{3\pi}}\left(\frac{27}{4}\right)^r r^{-1/2}\left(1 - \frac{7}{72r} + \frac{49}{10368r^2} + \frac{6425}{2239488r^3}\right.$$
$$\left. - \frac{187103}{644972544r^4} + O\left(\frac{1}{r}\right)^5\right).$$

Note that the Y-discriminant of $\mathcal{A}^-(x, Y)$ is given by

$$\text{Disc}_Y \mathcal{A}^-(x, Y) = -x(-4 + 27x)$$

Table 3. BPS invariants $b_{r,i}$ for the 4_1 knot

$r \setminus i$	0	1	2	3	4	5	6	7	8
1	-1	1	1	-2	1	0	0	0	0
2	-1	2	1	-4	3	-6	11	-8	2
3	-3	7	2	-18	21	-23	34	-48	82
4	-10	30	-2	-88	134	-122	150	-234	384
5	-40	143	-55	-451	889	-797	664	-978	1716
6	-171	728	-525	-2346	5944	-5822	3134	-2862	6196
7	-791	3876	-4080	$-12{,}172$	39,751	$-44{,}657$	17,210	4958	4071
8	$-38{,}28$	21,318	$-29{,}562$	$-62{,}016$	264,684	$-347{,}256$	121,276	191,744	$-263{,}282$
9	$-19{,}287$	120,175	$-206{,}701$	$-303{,}910$	1,751,401	$-2{,}692{,}471$	1,053,774	2,299,115	$-4{,}105{,}859$
10	$-100{,}140$	690,690	$-1{,}418{,}417$	$-1{,}381{,}380$	11,503,987	$-20{,}672{,}858$	10,012,945	21,567,000	$-46{,}462{,}399$

Fig. 2. The 5_2 knot

and its root $x = \frac{4}{27}$ matches the exponential growth rate of b_r^-, as asserted in Lemma 2.3.

Finally, we discuss all BPS invariants $b_{r,i}$, not just the extremal ones, i.e. we turn on the a-deformation. To this end it is useful to rescale the variable x in (4.10) by $c_- = -3$, so that

$$
\mathcal{A}(a^3 x, y, a) = (x - y^4 + y^6) - 2axy^2 + a^2 \left(2x^2 y^2 - x^2 + 2xy^8\right)
$$
$$
- a^3 xy^{10} + a^4 \left(x^3 y^4 + x^2 y^{10} - 2x^2 y^8\right) - a^5 x^3 y^6 \qquad (4.21)
$$

contains $\mathcal{A}^-(x, y)$ at its lowest order in a. Then, from (1.1) and (1.3) we can determine the invariants $b_{r,i}$ for this curve; we list some of them in Table 3, whose first column of course agrees with the extremal BPS invariants b_r^- given by (4.18).

4.3. 5_2 knot and Catalan numbers. We can analyze the $K_2 = 5_2$ knot, see Fig. 2, similarly as we did for the figure-8 knot. Starting from the super-A-polynomial derived in [FGSS13] and performing redefinitions discussed above, we get the following a-deformed algebraic curve (dual A-polynomial)

$$
\mathcal{A}(x, y, a) = a^{10} x^2 y^{16} - 2a^9 x^3 y^{16} + a^8 x^2 y^{12} \left(x^2 y^4 - 3y^2 - 4\right) + a^7 xy^{12}
$$
$$
\times \left(x^2 \left(4y^2 + 3\right) - 1\right) + a^6 x^2 y^8 \left(-x^2 y^6 + 5y^4 + 3y^2 + 6\right)
$$
$$
+ a^5 xy^{10} \left(2 - x^2 \left(3y^2 + 2\right)\right) - a^4 x^2 \left(3y^6 + 4y^4 - 3y^2 + 4\right) y^4
$$

$$-a^3 x \left(y^2 - 1 \right) y^4 \left(x^2 \left(y^2 - 1 \right) + y^2 + 3 \right)$$
$$+ a^2 x^2 \left(-3 y^6 + 5 y^4 - 3 y^2 + 1 \right) + ax \left(-3 y^4 + 4 y^2 - 2 \right) - y^2 + 1.$$

$$(4.22)$$

The unnormalized HOMFLY polynomial is given by

$$P_1(a, q) = a \frac{(q^4 - q^2 + 1)}{q \left(1 - q^2 \right)} + a^5 \frac{(q^4 + 1)}{q \left(q^2 - 1 \right)} + a^7 \frac{q}{1 - q^2}, \qquad (4.23)$$

so that

$$c^- = 1, \qquad c^+ = 7. \qquad (4.24)$$

It follows that the extremal A-polynomials of (3.8) are given by

$$\mathcal{A}^-(x, y) = 1 - y^2 - xy^4, \qquad \mathcal{A}^+(x, y) = 1 - y^2 - xy^{12}. \qquad (4.25)$$

The two fortunate coincidences of the 4_1 knot persist for the 5_2 knot as well. It is again convenient to use a rescaled variable $Y = y^2$. In particular note that the curve $\mathcal{A}^-(x, y)$, presented as $1 - y^2 - xy^4 = 1 - Y - xY^2$, is the curve that encodes the Catalan numbers. The latter are the coefficients in the series expansion

$$Y(x) = Y^-(x) = \frac{-1 + \sqrt{1 + 4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} (-x)^n. \qquad (4.26)$$

Therefore we have found a new role of Catalan numbers – they encode BPS numbers for 5_2 knot (and as we will see, also for other twist knots K_p for $p > 1$).

Now we get

$$x \frac{\partial_x Y^-(x)}{Y^-(x)} = -\frac{1}{2} + \frac{1}{2} \frac{1}{\sqrt{1 + 4x}} = \sum_{n=1}^{\infty} \binom{2n-1}{n-1} (-x)^n$$

$$= -\frac{1}{2} + \frac{1}{2} \, {}_1 F_0 \left(\frac{1}{2}; \ -\frac{2^2 x}{1^1} \right), \qquad (4.27)$$

so that

$$b_r^- = \frac{1}{r^2} \sum_{d|r} \mu \left(\frac{r}{d} \right) (-1)^d \binom{2d-1}{d-1}. \qquad (4.28)$$

Several values of b_r^- are given in Table 4.

In an analogous way for $\mathcal{A}^+(x, Y) = 1 - Y - xY^6$ we get more involved solution

$$Y(x)^+ = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \binom{6n}{n-1} (-x)^n = {}_5 F_4 \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}; \frac{6}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}; \ -\frac{6^6 x}{5^5} \right),$$

$$(4.29)$$

Table 4. Extremal BPS invariants and their Improved Integrality for 5_2 knot

r	b_r^-	b_r^+	$6\frac{b_r^-}{r}$	$2\frac{b_r^+}{r}$
1	-1	-1	-6	-2
2	1	3	3	3
3	-1	-15	-2	-10
4	2	110	3	55
5	-5	-950	-6	-380
6	13	9021	13	3007
7	-35	$-91{,}763$	-30	$-26{,}218$
8	100	982,652	75	245,663
9	-300	$-10{,}942{,}254$	-200	$-2{,}431{,}612$
10	925	125,656,950	555	25,131,390
11	-2915	$-1{,}479{,}452{,}887$	-1590	$-268{,}991{,}434$
12	9386	17,781,576,786	4693	2,963,596,131
13	$-30{,}771$	$-217{,}451{,}355{,}316$	$-14{,}202$	$-33{,}454{,}054{,}664$
14	102,347	2,698,753,797,201	43,863	385,536,256,743
15	$-344{,}705$	$-33{,}922{,}721{,}455{,}050$	$-137{,}882$	$-4{,}523{,}029{,}527{,}340$

so that

$$x\frac{\partial_x Y^+(x)}{Y^+(x)} = \sum_{n=1}^{\infty}\binom{6n-1}{n-1}(-x)^n$$

$$= -\frac{1}{6} + \frac{1}{6}\,{}_5F_4\left(\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6};\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5};-\frac{6^6 x}{5^5}\right), \qquad (4.30)$$

and in consequence

$$b_r^+ = \frac{1}{r^2}\sum_{d|r}\mu\left(\frac{r}{d}\right)(-1)^d\binom{6d-1}{d-1}. \qquad (4.31)$$

Several values of b_r^+ are also given in Table 4.

Our discussion illustrates Proposition 1.2 for the $5_2 = K_2$ twist knot. Experimentally—see Table 4—it appears that the Improved Integrality (1.6) holds with

$$\gamma^- = 6, \qquad \gamma^+ = 2. \qquad (4.32)$$

Next, we consider the asymptotics of the extremal BPS numbers. Using Lemma 2.2 and Stirling formula for the binomials in (4.28) and (4.31) we find respectively

$$\lim_{r\to\infty}\frac{b_r^-}{b_{r+1}^-} = -\frac{1}{4}, \qquad \lim_{r\to\infty}\frac{b_r^+}{b_{r+1}^+} = -\frac{5^5}{6^6}. \qquad (4.33)$$

This matches with Lemma 2.3, since the Y-discriminants of $\mathcal{A}^\pm(x, Y)$ are given by

$$\text{Disc}_Y\mathcal{A}^-(x, Y) = 1 + 4x,$$
$$\text{Disc}_Y\mathcal{A}^+(x, Y) = x^4(5^5 + 6^6 x).$$

Finally, we also consider the a-deformation of the extremal curves. Rescaling $x \mapsto a^{-c_-}x$ in (4.22) with $c_- = 1$ we get a curve that contains $\mathcal{A}^-(x, y)$ at its lowest order in a. Then, from (1.1) and (1.3), the find integral invariants $b_{r,i}$ given in Table 5. The first column of Table 5 agrees with the values of (4.28).

Table 5. BPS invariants $b_{r,i}$ for the 5_2 knot

$r \setminus i$	0	1	2	3	4	5	6	7	8
1	-1	0	2	-1	0	0	0	0	0
2	1	-2	1	-4	11	-10	3	0	0
3	-1	4	0	-12	23	-71	154	-162	80
4	2	-12	14	8	40	-226	594	-1542	2944
5	-5	36	-66	-30	132	-184	1550	-6108	16,525
6	13	-114	302	-94	-419	-660	3387	$-12,042$	56,209
7	-35	372	-1296	1168	1843	-1400	-1372	$-27,398$	135,350
8	100	-1244	5382	-8014	-4222	12,390	23,462	$-42,626$	163,928
9	-300	4240	$-21,932$	45,383	-6044	$-79,628$	$-31,246$	116,368	589,812
10	925	$-14,676$	88,457	$-234,124$	160,251	359,514	$-192,014$	$-1,022,096$	$-86,854$

4.4. Twist knots. The 4_1 and 5_2 knots are special cases, corresponding respectively to $p = -1$ and $p = 2$, in a series of twist knots K_p, labeled by an integer p. Apart from a special case $p = 0$ which is the unknot and $p = 1$ which is the trefoil knot 3_1, all other twist knots are hyperbolic. In this section we analyze their BPS invariants. The two coincidences of the 4_1 knot persist for all twist knots. The formulas for $p > 1$ are somewhat different from those for $p < 0$, so we analyze them separately.

We start with $p < 0$. In this case the bottom A-polynomial turns out to be the same for all p, however the top A-polynomial depends on p,

$$\boxed{A_{K_p}^-(x, y) = x - y^4 + y^6, \qquad A_{K_p}^+(x, y) = 1 - y^2 + xy^{4|p|+2}} \tag{4.34}$$

For $p = -1$ these curves of course reduce to those for the 4_1 knot (4.13). For all $p < 0$ the bottom BPS invariants are the same as for the 4_1 knot (4.18)

$$b_{K_p,r}^- = -\frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)\binom{3d-1}{d-1}, \qquad p < 0. \tag{4.35}$$

To get top invariants it is convenient to introduce $Y = y^2$ and consider the equation $A_{K_p}^+(x, Y) = 1 - y^2 + xY^{2|p|+1} = 0$, whose solution of interest reads

$$
\begin{aligned}
Y(x) &= \sum_{n=0}^{\infty} \frac{1}{2|p|n + 1}\binom{(2|p|+1)n}{n}x^n \\
&= {}_{2|p|}F_{2|p|-1}\left(\frac{1}{2|p|+1}, \dots, \frac{2|p|}{2|p|+1}; \frac{2|p|+1}{2|p|}, \frac{2|p|-1}{2|p|}, \right. \\
&\quad \left. \times \frac{2|p|-2}{2|p|}, \dots, \frac{2}{2|p|}; \frac{(2|p|+1)^{2|p|+1}x}{(2|p|)^{2|p|}}\right)
\end{aligned}
\tag{4.36}
$$

where ${}_{2|p|}F_{2|p|-1}$ is the generalized hypergeometric function. Then

$$
\begin{aligned}
x\frac{Y'(x)}{Y(x)} &= \sum_{n=1}^{\infty}\binom{(2|p|+1)n-1}{n-1}x^n \\
&= \frac{1}{2|p|+1}\left({}_{2|p|}F_{2|p|-1}\left(\frac{1}{2|p|+1}, \dots, \frac{2|p|}{2|p|+1};\right.\right. \\
&\quad \left.\left. \times \frac{2|p|-1}{2|p|}, \dots, \frac{1}{2|p|}; \frac{(2|p|+1)^{2|p|+1}x}{(2|p|)^{2|p|}}\right) - 1\right)
\end{aligned}
\tag{4.37}
$$

From this expression we find top BPS invariants

$$b^+_{K_p,r} = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)\binom{(2|p|+1)d-1}{d-1}, \qquad p < 0. \tag{4.38}$$

Next, we consider the case $p > 1$. Their bottom A-polynomials are the same for all p, however top A-polynomials depend on p,

$$\boxed{A^-_{K_p}(x,y) = 1 - y^2 - xy^4, \qquad A^+_{K_p}(x,y) = 1 - y^2 - xy^{4p+4}} \tag{4.39}$$

For $p = 2$ these curves reduce to the results for 5_2 knot (4.25). For all $p > 1$ the bottom BPS invariants are the same as for 5_2 knot (4.28)

$$b^-_{K_p,r} = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)(-1)^d\binom{2d-1}{d-1}, \qquad p > 1 \tag{4.40}$$

which means that Catalan numbers encode these BPS invariants for all $K_{p>1}$ twist knots.

To get top invariants we again introduce $Y = y^2$ and consider the equation $A^+_{K_p}(x,Y)$ $= 1 - Y - xY^{2p+2} = 0$, whose solution of interest reads

$$Y(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n}\binom{(2p+2)n}{n-1}(-x)^n =_{2p+1} F_{2p}$$

$$\times\left(\frac{1}{2p+2},\ldots,\frac{2p+1}{2p+2};\frac{2p+2}{2p+1},\frac{2p}{2p+1},\frac{2p-1}{2p+1},\ldots,\frac{2}{2p+1};-\frac{(2p+2)^{2p+2}x}{(2p+1)^{2p+1}}\right) \tag{4.41}$$

Then

$$x\frac{Y'(x)}{Y(x)} = \sum_{n=1}^{\infty}\binom{(2p+2)n-1}{n-1}(-x)^n = \frac{1}{2p+2}$$

$$\times\left(_{2p+1}F_{2p}\left(\frac{1}{2p+2},\ldots,\frac{2p+1}{2p+2};\frac{2p}{2p+1},\ldots,\frac{1}{2p+1};-\frac{(2p+2)^{2p+2}x}{(2p+1)^{2p+1}}\right)-1\right) \tag{4.42}$$

From this expression we find top BPS invariants

$$b^+_{K_p,r} = \frac{1}{r^2} \sum_{d|r} \mu\left(\frac{r}{d}\right)(-1)^d\binom{(2p+2)d-1}{d-1}, \qquad p > 1. \tag{4.43}$$

The Improved Integrality holds for twist knots K_p and the values of $\gamma^{\pm}$ are given in Table 1. Note that these invariants repeat periodically with period 6 for both positive and negative p.

Finally, turning on a-deformation also leads to integral invariants $b_{r,i}$, as an example see the results for $6_1 = K_{-2}$ knot in Table 6.

Table 6. BPS invariants $b_{r,i}$ for the 6_1 knot

$r \setminus i$	0	1	2	3	4	5	6	7	8
1	-1	1	1	-2	1	0	0	0	0
2	-1	2	1	-4	3	-6	11	-8	2
3	-3	7	2	-18	21	-23	34	-48	82
4	-10	30	-2	-88	134	-122	150	-234	384
5	-40	143	-55	-451	889	-797	664	-978	1716
6	-171	728	-525	-2346	5944	-5822	3134	-2862	6196
7	-791	3876	-4080	$-12,172$	39,751	$-44,657$	17,210	4958	4071
8	-3828	21,318	$-29,562$	$-62,016$	264,684	$-347,256$	121,276	191,744	$-263,282$
9	$-19,287$	120,175	$-206,701$	$-303,910$	1,751,401	$-2,692,471$	1,053,774	2,299,115	$-4,105,859$
10	$-100,140$	690,690	$-1,418,417$	$-1,381,380$	11,503,987	$-20,672,858$	10,012,945	21,567,000	$-46,462,399$

Fig. 3. The 3_1 knot

4.5. Torus knots. We can analyze BPS degeneracies for torus knots in the same way as we did for twist knots. Let us focus on the series of $(2, 2p + 1) \equiv (2p + 1)_1$ knots and present several examples. For the trefoil, 3_1, see Fig. 3, the (a-deformed) dual A-polynomial is given by

$$
A(x, y, a) = -1 + y^2 + a^2 x^2 y^6 + a^5 xy^8 + ax(1 - y^2 + 2y^4) \\
- a^3 xy^4 (2 + y^2) - a^4 x^2 y^8. \tag{4.44}
$$

From the form of the HOMFLY polynomial we find $c_- = 1$, so after appropriate rescaling of the above result we get

$$
A(a^{-1}x, y, a) = (-1 + x + y^2 - xy^2 + 2xy^4 + x^2 y^6) \\
- a^2 xy^4 (2 + y^2 + xy^4) + a^4 xy^8. \tag{4.45}
$$

The lowest term in a in this expression represents the extremal A-polynomial

$$
A^-(x, y) = -1 + x + y^2 - xy^2 + 2xy^4 + x^2 y^6. \tag{4.46}
$$

The corresponding BPS invariants are given in Table 7.

For the 5_1 knot we find that $c_- = 3$ and the rescaled dual A-polynomial takes form

$$
A(a^{-3}x, y, a) = -1 + x + y^2 + 2x^2 y^{10} + x^3 y^{20} \tag{4.47}
$$
$$
+ xy^2(-1 + 2y^2) + xy^6(-2 + 3y^2) + x^2 y^{12}(-1 + 3y^2)
$$
$$
+ a^2 \left(-x^3 y^{22} - 2x^2 y^{16} - x^2 \left(4y^2 + 1\right) y^{12} - xy^{10} - 2xy^4 + 2x \left(1 - 2y^2\right) y^6 \right)
$$
$$
+ a^4 \left(x^2 y^{14} + 2x^2 \left(y^2 + 1\right) y^{16} + xy^8 + x \left(2y^2 - 1\right) y^{10} \right)
$$
$$
+ a^6 (-2x^2 y^{18} - x^2 y^{20}) + a^8 x^2 y^{22}.
$$

Table 7. BPS invariants $b_{r,i}$ for the 3_1 knot

$r \setminus i$	0	1	2	3	4	5	6	7
1	−2	3	−1	0	0	0	0	0
2	2	−8	12	−8	2	0	0	0
3	−3	27	−84	126	−99	39	−6	0
4	8	−102	488	−12,14	1764	−1554	816	−234
5	−26	413	−2682	9559	−20,969	29,871	−28,203	17,537
6	90	−1752	14,484	−67,788	201,810	−405,888	569,322	−564,192
7	−329	7686	−77,473	451,308	−1,711,497	4,499,696	−8,504,476	11,792,571
8	1272	−34,584	411,948	−2,882,152	13,350,352	−43,658,370	104,759,240	−188,904,738
9	−5130	158,730	−2,183,805	17,877,558	−98,157,150	385,713,186	−1,128,850,632	2,524,827,921
10	21,312	−740,220	11,560,150	−108,550,256	690,760,044	−3,179,915,704	11,028,120,884	−29,597,042,376

Table 8. BPS invariants $b_{r,i}$ for the 5_1 knot

$r \setminus i$	0	1	2	3
1	−3	5	−2	0
2	10	−40	60	−40
3	−66	451	−1235	1750
4	628	−5890	23,440	−51,978
5	−7040	83,725	−438,045	1,330,465
6	87,066	−1,257,460	8,165,806	−31,571,080
7	−1,154,696	19,630,040	−152,346,325	716,238,720
8	16,124,704	−315,349,528	2,847,909,900	−15,779,484,560
9	−234,198,693	5,179,144,365	−53,361,940,365	340,607,862,518
10	3,508,592,570	−86,566,211,200	1,002,184,712,130	−7,243,117,544,640

Table 9. BPS invariants $b_{r,i}$ for the 7_1 knot

$r \setminus i$	0	1	2	3	4	5	6	7
1	−4	7	−3	0	0	0	0	0
2	28	−112	168	−112	28	0	0	0
3	−406	2618	−6916	9604	−7406	3010	−504	0
4	8168	−71,588	270,928	−579,124	765,576	−641,452	332,864	−97,852
5	−193,170	2,139,333	−10,554,173	30,562,838	−57,563,814	73,721,676	−65,048,368	39,063,778

The terms in the first two lines above constitute the extremal A-polynomial $\mathcal{A}^-(x, y)$. Several corresponding BPS invariants are given in Table 8.

For the 7_1 knot the a-deformed A-polynomial takes form of a quite lengthy expression, so we list just some BPS invariants—with a-deformation taken into account—in the Table 9.

Similarly we can determine top A-polynomials. We present the list of extremal A-polynomials for several torus knots in Table 12. Note that for various knots these polynomials have the same terms of lower degree, and for higher degrees more terms appear for more complicated knots. It would be interesting to understand this pattern and its manifestation on the level of BPS numbers.

Furthermore, we observe that the Improved Integrality also holds for this family of torus knots, with the values of $\gamma^\pm$ given in Table 1. Note that the values of $\gamma^\pm$ grow linearly with p.

4.6. BPS invariants from augmentation polynomials: 6_2, 6_3, 7_3, 7_5, 8_{19}, 8_{20}, 8_{21}, 10_{124}, 10_{132}, and 10_{139} knots.

Using the methods presented above we can easily compute BPS invariants for knots with known augmentation polynomials. Moreover, in many non-
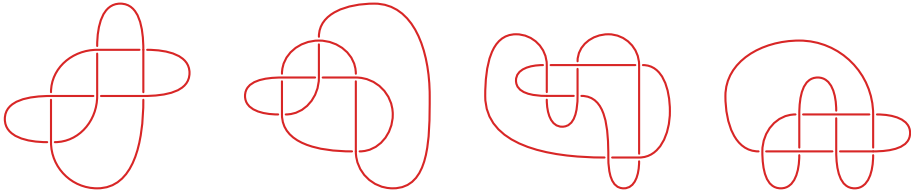
Fig. 4. The 6_2, 6_3, 7_3 and 7_5 knots

trivial cases we can confirm the conjecture that augmentation polynomials agree with Q-deformed polynomials (defined as the classical limit of recursion relations satisfied by colored HOMFLY polynomials) and $t = -1$ limit of super-A-polynomials. This conjecture has been explicitly verified for several torus and twist knots in [AV12, FGS13, FGSS13], where it was shown that appropriate change of variables relates the two algebraic curves. For example, starting with the super-A-polynomial for figure-8 knot (4.9), setting $t = -1$ and changing variables [note that it is not simply (4.3)] [FGS13]

$$Q = a, \qquad \beta = M, \qquad \alpha = L \frac{1 - \beta Q}{Q(1 - \beta)}, \qquad (4.48)$$

we obtain (up to some irrelevant simple factor) the Q-deformed polynomial in the same form as in [AV12]

$$\begin{aligned} A^{\text{Q-def}}(\alpha, \beta, Q) &= (\beta^2 - Q\beta^3) + (2\beta - 2Q^2\beta^4 + Q^2\beta^5 - 1)\alpha \\ &\quad + (1 - 2Q\beta + 2Q^2\beta^4 - Q^3\beta^5)\alpha^2 + Q^2(\beta - 1)\beta^2\alpha^3 , \end{aligned}$$

where it was shown to match the augmentation polynomial.

Now we show that this conjecture can be verified for many non-trivial knots, even if an explicit form of Q-deformed polynomials or super-A-polynomial is not known. Let us consider the following knots: 6_2, 6_3, 7_3, 7_5, 8_{19}, 8_{20}, 8_{21}, 10_{124}, 10_{132}, and 10_{139}, shown in Figs. 4, 5, 6, for which augmentation polynomials are determined in [Ng11, Ng08]. Changing the variables into M and L relevant for A-polynomials, and further into x and y relevant for our considerations, and from appropriate rescalings (3.8) we obtain extremal A-polynomials. They are presented in Table 13, and constitute one of our main results.

From extremal polynomials in Table 13 we can determine the extremal BPS invariants $b_r^{\pm}$ for arbitrary r; some results are presented in Tables in Appendix B. We also experimentally confirm the Improved Integrality – the constants $\gamma^{\pm}$ for some knots are shown in Table 1. However, unfortunately, the corresponding functions $Y^{\pm}(x)$ no longer satisfy the fortunate condition (1.9).

Even though the Q-deformed or super-A-polynomials are not known for knots listed in Table 13, colored HOMFLY polynomials for those knots for several values of r have been explicitly determined in [NRZ13, Wed]. We can therefore determine the corresponding invariants $N_{r,i,j}$ using LMOV formulas (2.5)—we list some of these invariants in tables in appendix C. On the other hand, from the known augmentation polynomials we can compute some BPS invariants $b_{r,i}$ using our techniques. In all cases [apart from 10_{132} knot, which does not belong to a class of knots for which (3.1) holds] we find the agreement between these two computations, as the reader can also verify by comparing tables in appendices B and C. This is quite a nontrivial test of the (still conjectural) relation between augmentation polynomials and colored HOMFLY polynomials.
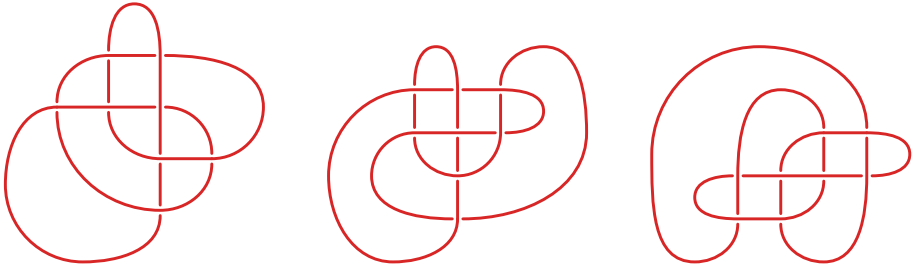
Fig. 5. The 8_{19}, 8_{20} and 8_{21} knots
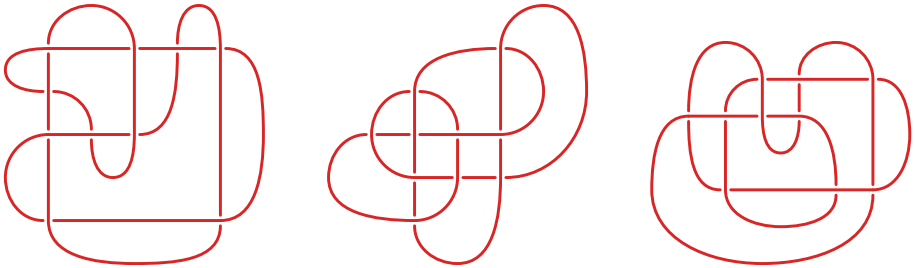
Fig. 6. The 10_{124}, 10_{132} and 10_{139} knots

For example, consider the LMOV invariants $N_{r,i,j}$ of the 6_2 knot for $r = 1, 2, 3$, given in tables 17, 18, 19. We determined these invariants from the knowledge of HOMFLY polynomials, determined up to $r = 4$ in [NRZ13], and applying formulas (2.5). To obtain b_r^- and b_r^+ we need to resum, respectively, the first and the last row in those tables (corresponding to minimal and maximal power of a). For the minimal case (first rows in the tables) from the resummation we obtain the number $-1, -2, -10$, and for the maximal case (last rows in the tables) we find the numbers $2, 2, 7$. These results indeed agree with values of $b_r^\pm$ for 6_2 knot given in Table 14, which are determined from its augmentation polynomial (more precisely, the corresponding extremal A-polynomials given in Table 13). We verified such an agreement for other knots discussed in this section.

However note that 10_{132} knot, which is not two-bridge and the condition (3.1) does not hold for it, needs more careful treatment. More precisely, everything works in the minimal case with $c_- = -7$ (resummation of the first rows in Tables 29 and 30 properly reproduces $b_1^- = 2, b_2^- = -13$). However we encounter a problem when we try to find c_+; considering even r leads to $c_+ = 0$, but for odd r the highest power of a is -1. Therefore for even r we get the proper result (the resummation of the last row in Table 30 properly reproduces $b_2^+ = -1$), and to restore the agreement for odd r we may include an additional row filled only with zeros in corresponding Tables of LMOV invariants; resummation of this additional row reproduces then $b_r^+ = 0$ (for odd r). In other words our method can describe BPS invariants along a given extremal ray.

4.7. Refined BPS invariants from super-A-polynomials.
Beyond the a-dependence we can consider further deformation of A-polynomials, in parameter t, that leads to super-A-polynomials. It is natural to ask if super-A-polynomials encode refined BPS degeneracies $b_{r,i,j}$. Such degeneracies should be identified with generalized LMOV invariants,

Table 10. Generating functions of refined degeneracies $\tilde{b}_{r,i,j}$ for several knots. Integrality of coefficients confirms the refined version of the LMOV conjecture

Knot	$\sum_{r,i,j} \tilde{b}_{r,i,j} x^r a^i T^j$
$\mathbf{4_1}$	$(-1 + aT - aT^2 + 2aT^3 - 2a^2T^4 + a^2T^5)x + (-1 + 2aT + (-a - a^2)T^2 + (3a + a^2)T^3$
	$-4a^2T^4 + (a^2 + a^3)T^5)x^2 + (-3 + 7aT + (-3a - 5a^2)T^2 + (10a + 5a^2 + a^3)T^3$
	$+(-21a^2 - 2a^3)T^4 + (6a^2 + 13a^3)T^5)x^3 + (-10 + 30aT + (-12a - 32a^2)T^2$
	$+(42a + 28a^2 + 14a^3)T^3 + (-117a^2 - 21a^3 - 2a^4)T^4 + (35a^2 + 114a^3 + 5a^4)T^5)x^4$
	$+(-40 + 143aT + (-55a - 198a^2)T^2 + (198a + 165a^2 + 132a^3)T^3$
	$+(-690a^2 - 180a^3 - 42a^4)T^4 + (210a^2 + 912a^3 + 84a^4 + 5a^5)T^5)x^5 + \ldots$
$\mathbf{6_1}$	$(-1 + aT - 2aT^2)x + (-1 + 2aT + (-3a - a^2)T^2)x^2 + (-3 + 7aT + (-10a - 5a^2)T^2)x^3$
	$+(-10 + 30aT + (-42a - 32a^2)T^2)x^4 + (-40 + 143aT + (-198a - 198a^2)T^2)x^5 \ldots$
$\mathbf{5_2}$	$(-1 + T + (-1 - a)T^2 + 2aT^3 - 2aT^4 + (a + 2a^2)T^5)x + (T^2 + (-1 - 2a)T^3$
	$+(1 + 5a + a^2)T^4 + (-8a - 4a^2)T^5)x^2 + (T^3 + (-3 - 4a)T^4 + (2 + 16a + 4a^2)T^5)x^3$
	$+(2T^4 + (-6 - 10a)T^5)x^4 + 4T^5x^5 + \ldots$
$\mathbf{3_1}$	$(-1 - T^2 + 2aT^3 + aT^5)x + (T^2 - aT^3 + T^4 - 5aT^5)x^2$
	$+(-2T^4 + 7aT^5)x^3 + (T^4 - 3aT^5)x^4 + \ldots$
$\mathbf{5_1}$	$(-1 - T^2 + 2aT^3 - T^4)x + (T^2 - aT^3 + 4T^4)x^2 - 5T^4x^3 + 2T^4x^4 + \ldots$

defined by relations (2.31), and could be determined from the knowledge of super-A-polynomials, using the relation (2.34). This conjecture would be confirmed if $b_{r,i,j}$ would turn out to be integer. In this section we extract such invariants from the known super-A-polynomials. To this end it is convenient to consider super-A-polynomials as T-deformation and a-deformation of bottom A-polynomials (that arise for $a = 0$ and $T = 1$), where $t = -T^2$. This results in slightly redefined degeneracies $\tilde{b}_{r,i,j}$, which can be combined into the generating functions

$$\sum_{r,i,j} \tilde{b}_{r,i,j} x^r a^i T^j. \tag{4.49}$$

We present such generating functions in Table 10. Clearly all coefficients in these generating functions are integer, and therefore capture putative refined BPS degeneracies. We plan to analyze these refined BPS invariants for knots in more detail in future work.

5. Conclusions and Discussion

The results of this work deserve further studies, which we plan to undertake. On one hand they should inspire mathematical research. We have formulated and tested various conjectures, in particular divisibility by r^2 following from the conjectured LMOV integrality, and Improved Integrality of extremal BPS degeneracies. These statements should hold for all knots and proving them is an important task (even proofs of divisibility by r^2 in various specific cases, in particular (1.4) and (1.5), are challenging); note that proofs of integrality of Gopakumar-Vafa in certain cases were given in [KSV06, Vol07, SV09], and presumably these techniques could be generalized to the case of knots. We also associated new integer invariants $\gamma^{\pm}$, related to Improved Integrality, to all knots – it is important to understand deeper their mathematical meaning and, possibly, relation to other characteristics of knots. Furthermore, for some knots we observed that the solutions of extremal A-polynomial equations are given by hypergeometric functions. It is important to understand for which knots such algebraic hypergeometric functions arise and if they have any further meaning. Understanding which knots have this property

could lead to a new method to determine many other algebraic hypergeometric functions (associated to various knots).

It would also be interesting to understand our results from the perspective of knot homologies. Currently, the most powerful method to determine—at least conjecturally—colored HOMFLY homologies is the formalism of (colored) differentials [DGR06, GS11]. These differentials reveal intricate structure not only of colored homologies, but also of ordinary HOMFLY invariants. Therefore they should capture some essential information about unrefined and refined BPS degeneracies that we consider. Note that the "bottom row" structure of HOMFLY homologies (i.e., corresponding to the minimal power of a) was analyzed, e.g., in [GGS13]; it would be interesting to relate it to BPS degeneracies determined here.

Our results also raise further interesting questions on the physics side. First, in the introduction we already mentioned their intimate connection to 3d-3d duality, which relates knot invariants to 3-dimensional $\mathcal{N} = 2$ theories [DGG14, FGS13]. Various objects in our analysis, such as colored knot polynomials, super-A-polynomials, etc., play an important role in this duality. Therefore, all the new objects and statements that we consider should also find its interpretation on the $\mathcal{N} = 2$ side of this duality. In particular, note that extremal A-polynomials that we consider arise from (a rescaling of) a-deformed A-polynomials, and therefore automatically encode information about ($y - 1$) branch encoding abelian connections in the $a = 1$ case; therefore the problems of the missing abelian branch (in case of ordinary A-polynomial) in the 3d-3d correspondence, discussed e.g. in [CDGS14], do not appear in the case of extremal A-polynomials or a-deformed A-polynomials.

Second, the results such as Improved Integrality or formulation of refined BPS invariants for knots generalize the statements of the original LMOV conjectures [OV00, LM01, LMV00]. It is desirable to understand in more detail the M-theory interpretation of these results. In particular we obtain the BPS degeneracies in an analogous way as has been done for D-branes in [AV00, AKV02]. Furthermore, it has been conjectured in [OV00, AV12] that all knots should be mirror to Lagrangian branes in the conifold geometry, and for some knots such Lagrangian branes have been constructed [DSV13, JKS14]. It would be amusing to construct such Lagrangian branes for other knots that we consider, and compare the degeneracies they encode with our computations. Moreover, the divisibility by an additional factor of r suggests that the invariants that we consider are in fact (related to) Gopakumar-Vafa invariants of some Calabi-Yau manifolds that could be assigned to knots.

Third, our results concern primarily the classical algebraic curves, i.e. the $q \to 1$ limit of recursion relations for knot polynomials. It is desirable to introduce the dependence on the parameter q and determine corresponding BPS degeneracies directly from the knowledge of those recursion relations. This would also be related to the notion of quantum curves and quantization of A-polynomials [GS12]; in particular, it is interesting to see if there is a relation between quantization of Catalan numbers encoded in extremal A-polynomials for twist knots, and generalization of Catalan numbers arising from the matrix model perspective [MS15].

Our results can also be further generalized to higher-dimensional varieties generalizing algebraic curves, and correspondingly to links or knots labeled by more general (multi-row) representations.

Fourth, an important challenge is to understand refined open BPS states that we compute for several knots, based on the known super-A-polynomials. Recently various formulations of closed refined BPS states have been considered, see e.g., [CKK14,

HKP13]. It would be nice to make contact between these various approaches involving open and closed BPS states.

Yet another intriguing direction of research relating algebraic curves and knot invariants has to do with the topological recursion. It has been conjectured in [DF09] and further analyzed in [DFM11, BE12, GS12, BEM12, GJKS15] that the asymptotic expansion of colored Jones or HOMFLY polynomials can be reconstructed from the topological recursion for the A-polynomial curve. This conjecture has been tested in a very limited number of cases and it still seems poorly understood. The new algebraic curves that we consider in this paper, in particular extremal A-polynomials, should provide a simpler setup in which this conjecture can be analyzed. It also appears that it is more natural to apply the topological recursion to the dual curves that we consider here, that depend on x and y variables, rather than to the original A-polynomials expressed in terms of M and L. In particular, recall that various extremal A-polynomials that we find coincide with framed mirror curves of $\mathbb{C}^3$ (see Appendix A); in this case the application of the topological recursion reduces to the remodeling conjecture. It is important to understand these relations in more detail and beyond framed $\mathbb{C}^3$ curves.

A. Extremal *A*-polynomials for Various Knots

See Tables 11, 12, 13.

Table 11. Extremal A-polynomials for twist knots K_p (including the unknot $0_1 = K_0$)

Knot	$\mathcal{A}^-(x, y)$	$\mathcal{A}^+(x, y)$
$\mathbf{0_1} = K_0$	$1 - x - y^2$	$1 + xy^2 - y^2$
$K_p, \; p \leq -1$	$x - y^4 + y^6$	$1 - y^2 + xy^{4\lvert p\rvert+2}$
$K_p, \; p \geq 2$	$1 - y^2 - xy^4$	$1 - y^2 - xy^{4p+4}$

Table 12. Extremal A-polynomials for torus knots

Knot	$\mathcal{A}^-(x, y)$	$\mathcal{A}^+(x, y)$
3_1	$-1 + x + y^2 - xy^2 + 2xy^4 + x^2y^6$	$-1 + y^2 + xy^8$
5_1	$-1 + x + y^2 - xy^2 + 2xy^4 - 2xy^6$ $+ 3xy^8 + 2x^2y^{10} - x^2y^{12} + 3x^2y^{14} + x^3y^{20}$	$-1 + y^2 + xy^8 - xy^{10} + 2xy^{12} + x^2y^{22}$
7_1	$-1 + x + y^2 - xy^2 + 2xy^4 - 2xy^6$ $+ 3xy^8 - 3xy^{10} + 4xy^{12} + 3x^2y^{14} - 2x^2y^{16}$ $+ 6x^2y^{18} - 3x^2y^{20} + 6x^2y^{22} + 3x^3y^{28}$ $- x^3y^{30} + 4x^3y^{32} + x^4y^{42}$	$-1 + y^2 + xy^8 - xy^{10} + 2xy^{12} - 2xy^{14}$ $+ 3xy^{16} + 2x^2y^{26} - x^2y^{28} + 3x^2y^{30} + x^3y^{44}$
9_1	$-1 + x + y^2 - xy^2 + 2xy^4 - 2xy^6$ $+ 3xy^8 - 3xy^{10} + 4xy^{12} - 4xy^{14} + 5xy^{16}$ $+ 4x^2y^{18} - 3x^2y^{20} + 9x^2y^{22} - 6x^2y^{24}$ $+ 12x^2y^{26} - 6x^2y^{28} + 10x^2y^{30} + 6x^3y^{36}$ $- 3x^3y^{38} + 12x^3y^{40} - 4x^3y^{42} + 10x^3y^{44}$ $+ 4x^4y^{54} - x^4y^{56} + 5x^4y^{58} + x^5y^{72}$	$-1 + y^2 + xy^8 - xy^{10} + 2xy^{12} - 2xy^{14}$ $+ 3xy^{16} - 3xy^{18} + 4xy^{20} + 3x^2y^{30}$ $- 2x^2y^{32} + 6x^2y^{34} - 3x^2y^{36} + 6x^2y^{38}$ $+ 3x^3y^{52} - x^3y^{54} + 4x^3y^{56} + x^4y^{74}$
11_1	$-1 + x + y^2 - xy^2 + 2xy^4 - 2xy^6$ $+ 3xy^8 - 3xy^{10} + 4xy^{12} - 4xy^{14}$ $+ 5xy^{16} - 5xy^{18} + 6xy^{20} + 5x^2y^{22}$ $- 4x^2y^{24} + 12x^2y^{26} - 9x^2y^{28} + 18x^2y^{30}$ $- 12x^2y^{32} + 20x^2y^{34} - 10x^2y^{36} + 15x^2y^{38}$ $+ 10x^3y^{44} - 6x^3y^{46} + 24x^3y^{48} - 12x^3y^{50}$ $+ 30x^3y^{52} - 10x^3y^{54} + 20x^3y^{56} + 10x^4y^{66}$ $- 4x^4y^{68} + 20x^4y^{70} - 5x^4y^{72} + 15x^4y^{74}$ $+ 5x^5y^{88} - x^5y^{90} + 6x^5y^{92} + x^6y^{110}$	$-1 + y^2 + xy^8 - xy^{10} + 2xy^{12} - 2xy^{14}$ $+ 3xy^{16} - 3xy^{18} + 4xy^{20} - 4xy^{22}$ $+ 5xy^{24} + 4x^2y^{34} - 3x^2y^{36} + 9x^2y^{38}$ $- 6x^2y^{40} + 12x^2y^{42} - 6x^2y^{44} + 10x^2y^{46}$ $+ 6x^3y^{60} - 3x^3y^{62} + 12x^3y^{64} - 4x^3y^{66}$ $+ 10x^3y^{68} + 4x^4y^{86} - x^4y^{88} + 5x^4y^{90} + x^5y^{112}$

Table 13. Extremal A-polynomials for various knots

Knot	$\mathcal{A}^-(x, y)$	$\mathcal{A}^+(x, y)$
6_2	$x - y^8 + y^{10}$	$1 - y^2 + 2xy^2 - xy^4 + x^2y^4 + xy^6$
6_3	$x + y^2 - y^4$	$-1 + y^2 + xy^4$
7_3	$-1 + y^2 + xy^8$	$-1 + y^2 + xy^{12} - xy^{14} + 2xy^{16} + x^2y^{30}$
7_5	$x + y^{14} - y^{16}$	$x^2 + 2xy^6 - xy^8 + xy^{10} + y^{12} - y^{14}$
8_{19}	$-1 + x + y^2 - xy^2 + 3xy^6 + xy^8 - 4xy^{10}$ $+5xy^{12} + x^2y^{12} - x^2y^{14} + 5x^2y^{16} + 3x^2y^{18}$ $- 6x^2y^{20} + 10x^2y^{22} - x^3y^{24} + x^3y^{26}$ $+ 3x^3y^{28} - 4x^3y^{30} + 10x^3y^{32} - x^4y^{36}$ $+ x^4y^{38} - x^4y^{40} + 5x^4y^{42} + x^5y^{52}$	$-1 + y^2 - xy^{18}$
8_{20}	$x^2 - xy^4 + xy^6 - 2xy^8 - y^{14} + y^{16}$	$-x + x^2 - y^2 + 2xy^2 - 5x^2y^2 + 4x^3y^2$ $+ y^4 - xy^4 + 11x^2y^4 - 9x^3y^4 + 6x^4y^4$ $+ xy^6 - 7x^2y^6 + 17x^3y^6 - 7x^4y^6 + 4x^5y^6$ $+ 4x^2y^8 - 12x^3y^8 + 9x^4y^8 - 2x^5y^8 + x^6y^8$ $+ 5x^3y^{10} - 6x^4y^{10} + x^5y^{10} - x^3y^{12} + 2x^4y^{12} - x^5y^{12}$
8_{21}	$x - y^8 + y^{10}$	$x^4 + 3x^3y^2 + 3x^2y^4 - 7x^3y^4 + xy^6 - 10x^2y^6$ $+ 2x^3y^6 - 3xy^8 + 17x^2y^8 - x^3y^8 - y^{10} + 2xy^{10}$ $- 11x^2y^{10} + y^{12} - 3xy^{12} + 6x^2y^{12} - x^2y^{14}$
10_{124}	$x^2 - 2xy^{20} + xy^{22} - xy^{24} + y^{40} - y^{42}$	$x^7 + 7x^6y^{14} - x^6y^{16} + x^6y^{18} - 2x^6y^{20}$ $+ 21x^5y^{28} - 6x^5y^{30} + 5x^5y^{32} - 3x^5y^{34} - x^5y^{38}$ $+ x^5y^{40} + 35x^4y^{42} - 15x^4y^{44} + 10x^4y^{46} + 8x^4y^{48}$ $- 3x^4y^{50} - 3x^4y^{52} + 35x^3y^{56} - 20x^3y^{58} + 10x^3y^{60}$ $+ 22x^3y^{62} - 9x^3y^{64} + 2x^3y^{68} + 21x^2y^{70} - 2x^3y^{70}$ $- 15x^2y^{72} + 5x^2y^{74} + 18x^2y^{76} - 9x^2y^{78} + 5x^2y^{80}$ $+ 3x^2y^{82} + 7xy^{84} - 6xy^{86} + xy^{88} + 5xy^{90}$ $- 3xy^{92} + 3xy^{94} + y^{98} - xy^{98} - y^{100} + xy^{100}$
10_{132}	$x^2 - xy^8 + xy^{10} - 2xy^{12} - y^{22} + y^{24}$	$1 - 4x^2 + 6x^4 - 4x^6 + x^8 - y^2 + 6x^2y^2 - 12x^4y^2$ $+ 6x^6y^2 - 4x^2y^4 + 13x^4y^4 - 2x^6y^4 - 7x^4y^6 + x^4y^8$
10_{139}	$1 - x - y^2 + xy^2 - 6xy^6 + 10xy^8$ $- 9xy^{10} + 2x^2y^{12} + 8xy^{14} - 9x^2y^{14} - 9xy^{16}$ $+ 17x^2y^{16} - 19x^2y^{18} - 16x^2y^{20} + 45x^2y^{22}$ $- 51x^2y^{24} - x^3y^{24} + 8x^3y^{26} + 28x^2y^{28}$ $- 26x^3y^{28} - 36x^2y^{30} + 52x^3y^{30} - 61x^3y^{32}$ $- 3x^3y^{34} + 75x^3y^{36} - 117x^3y^{38} - x^4y^{38}$ $+ 13x^4y^{40} + 56x^3y^{42} - 39x^4y^{42} - 84x^3y^{44}$ $+ 65x^4y^{44} - 72x^4y^{46} + 28x^4y^{48} + 50x^4y^{50}$ $+ x^5y^{50} - 135x^4y^{52} - 4x^5y^{52} + 12x^5y^{54}$ $+ 70x^4y^{56} - 23x^5y^{56} - 126x^4y^{58} + 37x^5y^{58}$ $- 40x^5y^{60} + 32x^5y^{62} - 75x^5y^{66} + x^6y^{68}$ $+ 56x^5y^{70} - 2x^6y^{70} - 126x^5y^{72} + 8x^6y^{72}$ $- 13x^6y^{74} + 12x^6y^{76} - 15x^6y^{78} - 9x^6y^{80}$ $+ 28x^6y^{84} - 84x^6y^{86} - 3x^7y^{88} + x^7y^{90}$ $- 5x^7y^{92} + 9x^7y^{94} + 8x^7y^{98} - 36x^7y^{100}$ $+ 3x^8y^{108} + x^8y^{112} - 9x^8y^{114} - x^9y^{128}$	$1 - y^2 + xy^{22}$

B. Extremal BPS Invariants for Various Knots

See Tables 14, 15, 16.

Table 14. Extremal BPS invariants for the 6_2, 6_3, 7_3 and 7_5 knots

r	$b^-_{6_2,r}$	$b^+_{6_2,r}$	$b^-_{6_3,r}$	$b^+_{6_3,r}$	$b^-_{7_3,r}$	$b^+_{7_3,r}$	$b^-_{7_5,r}$	$b^+_{7_5,r}$
1	-1	2	1	-1	-1	-2	1	2
2	-2	2	-1	1	2	14	-4	-6
3	-10	7	1	-1	-6	-183	28	31
4	-60	28	-2	2	28	3,316	-280	-236
5	-425	134	5	-5	-155	$-70,502$	3,290	2,118
6	$-3,296$	695	-13	13	936	1,656,429	$-42,616$	$-20,923$
7	$-27,447$	3,892	35	-35	$-6,041$	$-41,726,720$	591,626	221,522
8	$-240,312$	22,888	-100	100	41,080	1,106,293,848	$-8,644,784$	$-2,469,560$
9	$-2,188,056$	140,139	300	-300	$-290,565$	$-30,500,237,331$	131,347,227	28,632,747
10	$-20,544,450$	885,014	-925	925	2,119,190	867,235,597,322	$-2,058,115,960$	$-342,395,810$

Table 15. Extremal BPS invariants for the 8_{19}, 8_{20} and 8_{21} knots

r	$b^-_{8_{19},r}$	$b^+_{8_{19},r}$	$b^-_{8_{20},r}$	$b^+_{8_{20},r}$	$b^-_{8_{21},r}$	$b^+_{8_{21},r}$
1	-5	1	2	-1	-1	3
2	44	4	-9	-1	-2	-2
3	-795	36	73	2	-10	5
4	19,828	408	-842	0	-60	-10
5	$-581,230$	5,430	11,421	-9	-425	44
6	18,855,192	79,704	$-171,043$	22	$-3,296$	-141
7	$-656,198,799$	1,254,582	2,746,933	14	$-27,447$	605
8	24,043,189,592	20,779,440	$-46,443,364$	-228	$-240,312$	$-2,372$
9	$-916,260,658,326$	357,870,825	816,652,818	474	$-2,188,056$	10,601
10	36,017,403,983,220	6,356,271,400	$-14,811,255,141,$	1,008	$-20,544,450$	$-46,225$

Table 16. Extremal BPS invariant for the 10_{124}, 10_{132} and 10_{139} knots

r	$b^-_{10_{124},r}$	$b^+_{10_{124},r}$	$b^-_{10_{132},r}$	$b^+_{10_{132},r}$	$b^-_{10_{139},r}$	$b^+_{10_{139},r}$
1	2	7	2	0	-6	1
2	-13	-119	-13	-1	87	5
3	157	4,194	157	0	$-2,635$	55
4	$-2,638$	$-201,271$	$-2,638$	1	108,874	770
5	52,029	11,347,905	52,029	0	$-5,285,162$	12,650
6	$-1,133,950$	$-708,286,791$	$-1,133,950$	-2	284,005,066	229,427
7	26499872	47,426,316,644	26,499,872	0	$-16,372,244,568$	4,461,611
8	$-651,831,508$	$-3,343,261,431,701$	$-651,831,508$	7	993,643,037,662	91,302,244
9	16,673,241,018	245,124,281,115,981	16,673,241,018	0	$-62,721,577,459,427$	1,942,795,668

C. LMOV Invariants for Various Knots

See Tables 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32.

$i \backslash j$	-4	-2	0	2	4
-5	0	-1	1	-1	0
-3	1	0	1	0	1
-1	-1	0	-2	0	-1
1	0	1	0	1	0

(C.1)

Table 17. LMOV invariants $N_{1,i,j}$ for the 6_2 knot

$i \backslash j$	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9
-10	0	-1	1	-1	-1	1	-1	0	0	0	0	0
-8	1	1	0	3	1	1	1	2	-1	1	0	0
-6	-2	0	-3	-3	-2	-3	-2	-2	0	-1	0	0
-4	1	1	2	2	4	0	4	-1	1	-1	0	-1
-2	0	-1	0	-2	-1	0	-1	1	2	1	1	2
0	0	0	0	1	-1	1	-1	-1	-2	0	-2	-1
2	0	0	0	0	0	0	0	1	0	0	1	0

(C.2)

Table 18. LMOV invariants $N_{2,i,j}$ for the 6_2 knot.

$i \backslash j$	-30	-28	-26	-24	-22	-20	-18	-16	-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14	16	18
-15	0	-1	1	-1	-1	-1	0	-1	-1	-1	-2	0	0	-2	1	-1	0	0	0	0	0	0	0	0	0
-13	1	1	1	2	5	4	4	5	7	8	7	3	6	4	3	0	3	-1	1	0	0	0	0	0	0
-11	-2	-1	-4	-4	-10	-9	-13	-14	-19	-21	-16	-16	-16	-10	-9	-3	-4	-1	-1	0	0	0	0	0	0
-9	1	3	2	7	10	14	17	24	27	30	28	28	20	22	6	11	0	4	-2	1	-1	0	0	0	0
-7	0	-2	-1	-5	-6	-12	-14	-22	-24	-27	-29	-23	-20	-15	-6	-3	-1	3	1	0	1	0	-1	0	0
-5	0	0	1	1	3	4	9	10	14	15	16	11	11	-2	4	-7	-2	-7	2	0	2	2	2	2	0
-3	0	0	0	0	-1	0	-3	-3	-4	-5	-5	-3	2	2	5	3	7	0	-3	-6	-5	-5	-4	-4	-1
-1	0	0	0	0	0	0	0	1	0	1	1	0	-3	0	-3	-1	-1	3	6	8	6	5	6	3	2
1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	-2	-2	-5	-4	-4	-3	-3	-2	-1
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	0	1	0

(C.3)

Table 19. LMOV invariants $N_{3,i,j}$ for the 6_2 knot

$i \backslash j$	-4	-2	0	2	4
-3	0	1	-1	1	0
-1	-1	0	-2	0	-1
1	1	0	2	0	1
3	0	-1	1	-1	0

(C.4)

Table 20. LMOV invariants $N_{1,i,j}$ for the 6_3 knot

$i \backslash j$	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11
-6	0	-1	1	0	-1	0	1	-1	0	0	0	0
-4	1	1	-1	1	3	-3	2	0	0	0	0	0
-2	-2	0	-1	-3	-1	1	-2	1	1	0	1	0
0	1	1	1	3	-1	3	-3	1	-3	-1	-1	-1
2	0	-1	0	-1	-1	2	-1	1	3	1	0	2
4	0	0	0	0	0	-2	3	-3	-1	1	-1	-1
6	0	0	0	0	1	-1	0	1	0	-1	1	0

(C.5)

Table 21. LMOV invariants $N_{2,i,j}$ for the 6_3 knot

i \ j	−24	−22	−20	−18	−16	−14	−12	−10	−8	−6	−4	−2	0	2	4	6	8	10	12	14	16	18	20	22	24
−9	0	1	−1	0	0	1	0	−1	1	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−7	−1	−1	0	1	−2	−3	−1	1	−1	−1	0	−3	3	−2	0	0	0	0	0	0	0	0	0	0	0
−5	2	1	2	0	6	6	4	3	2	4	7	−4	1	1	1	−1	0	0	0	0	0	0	0	0	0
−3	−1	−3	−1	−4	−7	−10	−8	−7	−8	−8	−6	5	−6	5	−4	2	0	−2	−1	0	−1	0	0	0	0
−1	0	2	1	4	5	9	9	8	9	8	−2	8	−8	3	1	2	3	6	5	3	3	1	1	0	0
1	0	0	−1	−1	−3	−3	−5	−6	−3	−2	−1	−3	8	−8	2	−8	−9	−8	−9	−9	−5	−4	−1	−2	0
3	0	0	0	0	1	0	1	2	0	−2	4	−5	6	−5	6	8	8	7	8	10	7	4	1	3	1
5	0	0	0	0	0	0	0	0	0	1	−1	−1	−1	4	−7	−4	−2	−3	−4	−6	−6	0	−2	−1	−2
7	0	0	0	0	0	0	0	0	0	0	2	−3	3	0	1	1	−1	1	3	2	−1	0	1	1	0
9	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	0	−1	1	0	−1	0	0	1	−1	0	0

(C.6)

Table 22. LMOV invariants $N_{3,i,j}$ for the 6_3 knot

i \ j	−4	−2	0	2	4
3	−1	1	−1	1	−1
5	0	0	−1	0	0
7	1	0	2	0	1
9	0	−1	0	−1	0

(C.7)

Table 23. LMOV invariants $N_{1,i,j}$ for the 7_3 knot

i \ j	−5	−3	−1	1	3	5	7	9	11	13	15	17	19
6	0	1	−2	2	0	0	0	1	−1	1	0	0	0
8	−1	1	0	−1	−1	1	−1	0	1	−1	0	0	0
10	1	−2	2	0	1	0	1	−1	0	0	0	0	0
12	0	−2	0	−5	−3	−4	−4	−3	−2	−2	−2	0	−1
14	1	2	4	6	8	7	10	6	6	5	4	1	2
16	−1	−1	−4	−4	−6	−7	−7	−5	−5	−5	−2	−2	−1
18	0	1	0	2	1	3	1	2	1	2	0	1	0

(C.8)

Table 24. LMOV invariants $N_{2,i,j}$ for the 7_3 knot

i \ j	−4	−2	0	2	4
−9	0	1	−1	1	0
−7	−1	1	−1	1	−1
−5	0	−1	0	−1	0
−3	1	−1	2	−1	1

(C.9)

Table 25. LMOV invariants $N_{1,i,j}$ for the 7_5 knot

i \ j	−19	−17	−15	−13	−11	−9	−7	−5	−3	−1	1	3	5
−18	0	−1	1	−1	−1	0	0	−1	0	−1	1	−1	0
−16	1	1	−1	3	2	−1	2	1	1	−1	2	−1	1
−14	−2	1	−1	−3	0	0	−2	0	−1	1	1	0	0
−12	1	−1	1	1	0	1	1	1	−2	4	−4	2	−1
−10	0	0	0	−1	−2	0	−5	−1	−3	0	−5	3	−2
−8	0	0	0	2	0	2	5	0	3	3	2	−1	2
−6	0	0	0	−1	1	−2	−1	0	0	−4	3	−2	0

(C.10)

Table 26. LMOV invariants $N_{2,i,j}$ for the 7_5 knot

$i \setminus j$	-8	-6	-4	-2	0	2	4	6	8
-13	0	0	0	-1	0	-1	0	0	0
-11	0	1	1	2	2	2	1	1	0
-9	-1	-1	-2	-2	-3	-2	-2	-1	-1
-7	1	0	1	1	1	1	1	0	1

(C.11)

Table 27. LMOV invariants $N_{1,i,j}$ for the 10_{124} knot

$i \setminus j$	-31	-29	-27	-25	-23	-21	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13
-26	0	0	0	0	-1	0	-2	-1	-2	-1	-3	-1	-3	-1	-2	-1	-1	0	0	0	0	0	0
-24	0	0	1	2	4	6	9	11	13	15	17	18	18	17	15	12	8	5	2	1	0	0	0
-22	0	-2	-3	-9	-12	-22	-26	-39	-41	-55	-55	-65	-60	-63	-51	-46	-31	-23	-12	-7	-2	-1	0
-20	1	4	7	15	24	35	49	64	77	92	103	109	112	107	96	81	63	44	29	16	8	3	1
-18	-2	-3	-9	-13	-25	-32	-50	-58	-79	-85	-103	-102	-111	-100	-97	-77	-66	-45	-33	-18	-11	-4	-2
-16	1	2	4	7	12	17	24	31	38	45	50	53	54	52	47	41	33	25	17	11	6	3	1
-14	0	-1	0	-2	-2	-4	-4	-8	-6	-11	-9	-12	-10	-12	-8	-10	-6	-6	-3	-3	-1	-1	0

(C.12)

Table 28. LMOV invariants $N_{2,i,j}$ for the 10_{124} knot

$i \setminus j$	-4	-2	0	2	4
-7	0	1	0	1	0
-5	-1	-1	-1	-1	-1
-3	1	0	1	0	1

(C.13)

Table 29. LMOV invariants $N_{1,i,j}$ for the 10_{132} knot

$i \setminus j$	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7
-14	0	-1	0	-1	-1	-2	-1	-3	-1	-2	0	-1	0	0
-12	1	2	2	4	5	7	8	8	7	6	4	2	1	0
-10	-2	-2	-5	-6	-10	-11	-15	-11	-14	-9	-8	-3	-3	0
-8	1	2	4	5	9	10	11	11	11	8	6	4	2	1
-6	0	-1	-1	-3	-3	-5	-4	-6	-3	-4	-2	-2	0	-1
-4	0	0	0	1	0	1	1	1	0	1	-1	0	0	-1
-2	0	0	0	0	0	0	0	0	0	1	0	0	1	1
0	0	0	0	0	0	0	0	0	0	-1	1	0	-1	0

(C.14)

Table 30. LMOV invariants $N_{2,i,j}$ for the 10_{132} knot

$i \setminus j$	-8	-6	-4	-2	0	2	4	6	8
7	-1	0	-1	-1	0	-1	-1	0	-1
9	1	1	2	1	2	1	2	1	1
11	0	-1	-1	-1	-1	-1	-1	-1	0
13	0	0	0	1	-1	1	0	0	0

(C.15)

Table 31. LMOV invariants $N_{1,i,j}$ for the 10_{139} knot

$i \setminus j$	−13	−11	−9	−7	−5	−3	−1	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
14	0	1	1	2	2	4	4	7	5	8	8	8	6	9	4	7	3	3	2	2	0	1	0
16	−1	−3	−5	−8	−11	−16	−21	−25	−29	−33	−35	−34	−33	−32	−27	−23	−18	−13	−10	−6	−4	−2	−1
18	2	4	9	13	21	27	38	43	54	57	63	59	63	53	52	38	35	22	19	10	8	3	2
20	−1	−3	−7	−11	−18	−24	−32	−41	−46	−52	−56	−54	−55	−51	−43	−38	−30	−21	−16	−11	−5	−4	−1
22	0	1	2	5	7	11	14	19	21	25	24	27	24	26	18	20	13	12	6	6	2	2	0
24	0	0	0	−1	−1	−2	−3	−4	−5	−5	−5	−6	−6	−5	−4	−5	−4	−2	−2	−1	−1	0	0
26	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	1	1	−1	1	0	0	0	0

$$\text{(C.16)}$$

Table 32. LMOV invariants $N_{2,i,j}$ for the 10_{139} knot

References

[AGSF12]   Awata, H., Gukov, S., Sulkowski, P., Fuji, H.: Volume conjecture: refined and categorified. Adv.Theor.Math.Phys., **16**, 1669–1777 (2012). arXiv:1203.2182

[AKV02]   Aganagic, M., Klemm, A., Vafa, C.: Disk instantons, mirror symmetry and the duality web. Z. Naturforsch. A **57**, 1–28 (2002). arXiv:hep-th/0105045

[AM98]   Aiston, A.K., Morton, H.R.: Idempotents of Hecke algebras of type *A*. J. Knot Theory Ramifications **7**(4), 463–487 (1998)

[AV00]   Aganagic, M., Vafa, C.: Mirror symmetry, D-branes and counting holomorphic discs (2000), arXiv:hep-th/0012041

[AV12]   Aganagic, M., Vafa, C.: Large *N* Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots (2012). arXiv:1204.4709

[BE12]   Borot, G., Eynard, B.: All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials (2012). arXiv:1205.2261

[BEM12]   Brini, A., Eynard, B., Mariño, M.: Torus knots and mirror symmetry. Ann. Henri Poincaré **13**(8), 1873–1910 (2012)

[Beu10]   Beukers, F.: Algebraic *A*-hypergeometric functions. Invent. Math. **180**(3), 589–610 (2010)

[BH89]   Beukers, F., Heckman, G.: Monodromy for the hypergeometric function $_nF_{n-1}$. Invent. Math. **95**(2), 325–354 (1989)

[CCG⁺94]   Cooper, D., Culler, M., Gillet, H., Long, D.D., Shalen, P.B.: Plane curves associated to character varieties of 3-manifolds. Invent. Math. **118**(1), 47–84 (1994)

[CDGS14]   Chung, H.-J., Dimofte, T., Gukov, S., Sulkowski, P.: 3d-3d Correspondence Revisited. JHEP **1604**, 140 (2016)

[CKK14]   Choi, J., Katz, S., Klemm, A.: The refined BPS index from stable pair invariants. Commun.Math.Phys., **328**, 903–954 (2014). arXiv:1210.4403

[DF09]   Dijkgraaf, R., Fuji, H.: The volume conjecture and topological strings. Fortschr. Phys. **57**(9), 825–856 (2009)

[DFM11]   Dijkgraaf, R., Fuji, H., Manabe, M.: The volume conjecture, perturbative knot invariants, and recursion relations for topological strings. Nuclear Phys. B **849**(1), 166–211 (2011)

[DGG14]   Dimofte, T., Gaiotto, D., Gukov, S.: Gauge theories labelled by three-manifolds. Comm. Math. Phys. **325**(2), 367–419 (2014)

[DGR06]   Dunfield, N.M., Gukov, S., Rasmussen, J.: The superpolynomial for knot homologies. Experiment. Math. **15**(2), 129–159 (2006)

[DSV13]   Diaconescu, D.E., Shende, V., Vafa, C.: Large N duality, lagrangian cycles, and algebraic knots. Commun.Math.Phys., **319**, 813–863 (2013). arXiv:1111.6533

[FGS13]   Fuji, Hiroyuki, Gukov, Sergei, Sulkowski, Piotr: Super-A-polynomial for knots and BPS states. Nucl. Phys. B, **867**, 506 (2013). arXiv:1205.1515

[FGSS13]   Fuji, H., Gukov, S., Sulkowski, P., Stosic, M.: 3d analogs of Argyres-Douglas theories and knot homologies. JHEP, **01**, 175 (2013). arXiv:1209.1416

[FS15]   Fuji, H., Sulkowski, P.: Super-A-polynomial. Proceedings of Symposia in Pure Mathematics, **90**, 277 (2015). arXiv:1303.3709

[Gar04]   Garoufalidis, S.: On the characteristic and deformation varieties of a knot. Geom. Topol. Monogr. 7:291–304 (2004). arXiv:math/0306230

[Gar11a]   Garoufalidis, Stavros: The degree of a q-holonomic sequence is a quadratic quasi-polynomial. Electron. J. Combin., **18**(2):Paper 4, 23 (2011)

[Gar11b]   Garoufalidis, S.: What is a sequence of Nilsson type?. In: Interactions between hyperbolic geometry, quantum topology and number theory, volume **541** of Contemp. Math., pp. 145–157. Amer. Math. Soc., Providence, RI (2011)

[GGS13]   Gorsky, E., Gukov, S., Stosic, M.: Quadruply-graded colored homology of knots (2013). arXiv:1304.3481

[GJKS15]   Gu, J., Hans, J., Albrecht, K., Masoud S.: Knot invariants from topological recursion on augmentation varieties. Commun. Math. Phys. **336**(2), 987–1051 (2015). arXiv:1401.5095

[GL05]   Garoufalidis, S., Lê, Thang, T.Q.: The colored Jones function is q-holonomic. Geom. Topol., **9**, 1253–1293 (2005) (electronic)

[GLL]   Garoufalidis, S., Lauda, A.D., Lê, T.T.Q.: The colored HOMFLY-PT polynomial is q-holonomic. Preprint (2016)

[GS11]   Gukov, S., Stosic, M.: Homological algebra of knots and BPS states. Geom. Topol. Monographs **18**, 309–367 (2012)

[GS12]   Gukov, S., Sulkowski, P.: A-polynomial, B-model, and Quantization. JHEP, **1202**, 070 (2012), arXiv:1108.0002

[Guk05]   Gukov, S.: Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial. Comm. Math. Phys. **255**(3), 577–627 (2005)

[Hik04]   Hikami, K.: Difference equation of the colored Jones polynomial for torus knot. Internat. J. Math., **15**, 959–965 (2004). arXiv:math/0403224

[HKP13]   Huang, M.X., Klemm, A., Poretschkin, M.: Refined stable pair invariants for E-, M- and $[p, q]$-strings. JHEP, **1311**, 112 (2013). arXiv:1308.0619

[IMMM12]   Itoyama, H., Mironov, A., Morozov, A., Morozov, An.: HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations. JHEP **1207**, 131 (2012)

[JKS14]   Jockers, H., Klemm, A., Soroush, M.: Torus Knots and the Topological Vertex. Lett.Math.Phys., **104**, 953–989 (2014). arXiv:1212.0321

[Kon]   Maxim Kontsevich. Private communication

[KR14]   Kassel, C., Reutenauer, C.: Algebraicity of the zeta function associated to a matrix over a free group algebra. Algebra Number Theory **8**(2), 497–511 (2014)

[KSV06]   Kontsevich, M., Schwarz, A., Vologodsky, V.: Integrality of instanton numbers and p-adic B-model. Phys. Lett. B **637**(1–2), 97–101 (2006)

[LM01]   Labastida, J.M.F., Mariño, M.: Polynomial invariants for torus knots and topological strings. Comm. Math. Phys. **217**(2), 423–449 (2001)

[LM02]   José, M.F.: Labastida and Marcos Mariño. A New Point of View in the Theory of Knot and Link Invariants. J. Knot Theory Ramifications **11**(2), 173–197 (2002)

[LMV00]   Labastida, J., Mariño, M.F., Marcos, V.C: Knots, links and branes at large N. J. High Energy Phys., (11):Paper 7, 42 (2000)

[MS15]   Mulase, M., Sulkowski, P.: Spectral curves and the Schrdinger equations for the Eynard-Orantin recursion. Adv. Theor. Math. Phys., **19**, 955–1015 (2015). arXiv:1210.3006

[Ng08]   Ng, L.: Framed knot contact homology. Duke Math. J. **141**(2), 365–406 (2008)

[Ng11]   Ng, L.: Combinatorial knot contact homology and transverse knots. Adv. Math. **227**(6), 2189–2219 (2011)

[NRZ13]   Nawata, S., Ramadevi, P.Z.: Colored HOMFLY polynomials from Chern-Simons theory. J.Knot Theor., **22**, 1350078 (2013). arXiv:1302.5144

[NRZS12]   Nawata, S., Ramadevi, P., Zodinmawia, S.X.: Super-A-polynomials for twist knots. J. High Energy Phys., (11):157, front matter + 38 (2012)

[OV00]   Ooguri, H., Vafa, C.: Knot invariants and topological strings. Nucl. Phys. B **577**, 419–438 (2000). arXiv:hep-th/9912123

[RS01]   Ramadevi, P., Sarkar, T.: On link invariants and topological string amplitudes. Nucl. Phys. **B 600**, 487–511 (2001). arXiv:hep-th/0009188

[SV09]   Schwarz, A., Vologodsky, V.: Integrality theorems in the theory of topological strings. Nuclear Phys. B **821**(3), 506–534 (2009)

[VdV]   Roland Van der V.: The degree of the colored HOMFLY polynomial (2010). arXiv:1501.00123 Preprint

[Vol07]   Vologodsky, V.: Integrality of instanton numbers (2007). arXiv:0707.4617

[Wed]   Wedrich, P.: q-holonomic formulas for colored homfly polynomials of 2-bridge links (2014). arXiv:1410.3769 **(Preprint)**

[Zei06]   Zeidler, E.: Quantum field theory. I. Basics in mathematics and physics. Springer-Verlag, Berlin (2006). A bridge between mathematicians and physicists

[ZR12]   Zodinmawia, R.P.: Reformulated invariants for non-torus knots and links (2012). arXiv:1209.1346

[ZR13]   Zodinmawia, R.P.: SU(N) quantum Racah coefficients and non-torus links. Nucl. Phys., B 870,
           205–242 (2013). arXiv:1107.3918