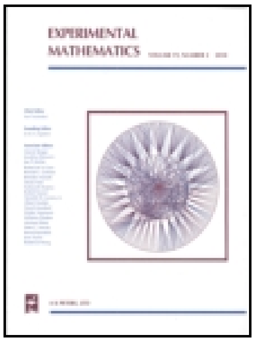

Experimental Mathematics

ISSN: 1058-6458 (Print) 1944-950X (Online) Journal homepage: https://www.tandfonline.com/loi/uexm20

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To cite this article: Stavros Garoufalidis, Eric Sabo & Shane Scott (2016) Exact Computation of the *n*-Loop Invariants of Knots, Experimental Mathematics, 25:2, 125-129, DOI: 10.1080/10586458.2015.1048012

To link to this article: https://doi.org/10.1080/10586458.2015.1048012

Published online: 02 Dec 2015.

EXPERIMENTAL MATHEMATICS
2016, VOL. 25, NO. 2, 125–129
http://dx.doi.org/10.1080/10586458.2015.1048012

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Exact Computation of the *n*-Loop Invariants of Knots

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ABSTRACT
The loop invariants of Dimofte–Garoufalidis is a formal power series with arithmetically interesting coefficients that conjecturally appears in the asymptotics of the Kashaev invariant of a knot to all orders in 1/*N*. We develop methods implemented in `SnapPy` that compute the first six coefficients of the formal power series of a knot. We give examples that illustrate our method and its results.

KEYWORDS
knots; Jones polynomia; Kashaev invariant; volume conjecture; volume; hyperbolic geometry; ideal triangulations; shapes; Neumann–Zagier data; 1-loop; *n*-loop; formal Gaussian integration; Feynman diagrams

2000 AMS SUBJECT CLASSIFICATION
Primary 57N10; Secondary 57M25

1. Introduction

1.1. *The volume conjecture to all orders in 1/N*

The best known quantum invariant of a knot in 3-space is the Jones polynomial [Jones 87]. The Kashaev invariant $\langle K\rangle_N$ of a knot K (for $N = 1, 2, \ldots$) [Kashaev 95] coincides with the evaluation of the Jones polynomial of a knot and its parallels at complex roots of unity [Murakami and Murakami 01]. The volume conjecture of Kashaev [Kashaev 97] states that for a hyperbolic knot K,

$$\lim_{N\to\infty} \frac{1}{N} \log |\langle K\rangle_N| = \frac{\mathrm{Vol}(K)}{2\pi},$$

where $\mathrm{Vol}(K)$ is the hyperbolic volume of K. An extension of the volume conjecture to all orders in $1/N$ was proposed independently by Gukov and the first author [Gukov 05, Garoufalidis 08]. Namely, for every hyperbolic knot K, there exists a formal power series $\phi_K(\hbar) \in \mathbb{C}[[\hbar]]$ such that

$$\langle K\rangle_N \sim N^{3/2} e^{C_K N} \phi_K(2\pi i/N), \tag{1-1}$$

where C_K is the complexified volume of K divided by $2\pi i$,

$$\begin{aligned} \phi_K(\hbar) &= \tau_K^{-\frac{1}{2}}\ \phi^+_{K,1}(\hbar), && \text{(1-2a)}\\ \phi^+_K(\hbar) &\in 1 + \hbar F_K[[\hbar]], && \text{(1-2b)}\\ \tau_K &\in F_K, && \text{(1-2c)} \end{aligned}$$

and F_K is the trace field of K.

1.2. *Ideal triangulations, shapes, and the loop invariants*

The left-hand side of equation (1-1) is concretely defined given a planar projection or an ideal triangulation of a knot and is typically given by a finite state-sum where the summand is a ratio of quantum factorials. Examples of state-sum formulas for the Kashaev invariant of the 4_1, 5_2, and 6_1 knots are given in [Kashaev 97, (2.2)–(2.4)].

On the other hand, the power series $\phi_K(\hbar)$ that conjecturally appears in the right-hand side is not an explicit function of the knot. Numerical computations of the Kashaev invariant were performed by Zagier and G., and using numerical interpolation and a variety of guessing methods, it was possible to recognize the first few coefficients of the power series ϕ_K for several knots [Garoufalidis and Zagier 13].

The main result of Dimofte-G. [Dimofte and Garoufalidis 13] was the construction of a power series $\phi_\gamma(\hbar)$ that depends on an ideal triangulation of a knot complement. For a detailed discussion on ideal triangulations and their gluing equations, see [Thurston 77, Neumann and Zagier 85] and also [Dimofte and Garoufalidis 13, Section 1.2]. Explicitly, an ideal triangulation $\mathcal{T}$ with N tetrahedra gives rise to a vector $z = (z_1, \ldots, z_N)$ of shapes that satisfy the Neumann–Zagier equations

$$\prod_{j=1}^{N} z_j^{\mathbf{A}_{ij}} (z''_j)^{\mathbf{B}_{ij}} = (-1)^{\nu_i}$$

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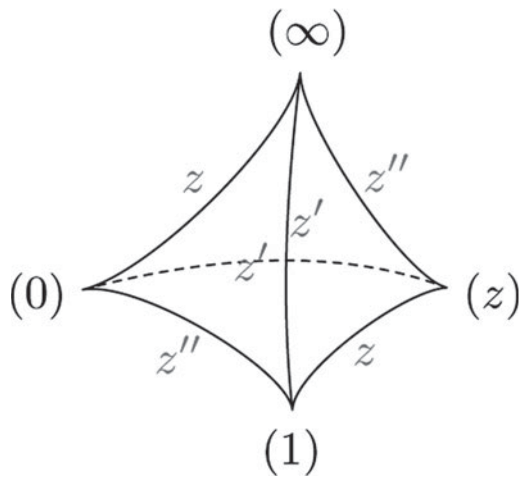

Figure 1. An ideal tetrahedron and its shape assignment.

for $i = 1, \ldots, N$ where $z''_j = 1 - 1/z_j$. These equations are obtained as follows. Fix an edge of the ideal triangulation $\mathcal{T}$, and set the product of the shape parameters of all tetrahedra that go around the edge e equal to 1. If z is the shape of a tetrahedron that contains the fixed edge, then its contribution to the above product is z or z' or z'' according to the convention of Figure 1. Finally, replace $z' = -(zz'')^{-1}$ using the relation $zz'z'' = -1$. This gives rise to the equations (1-2), one for each edge of $\mathcal{T}$. Likewise, there is an equation of the same type for each peripheral curve.

In the above equations, we have removed one edge equation and replaced it with the meridian cusp equation. Neumann–Zagier [Neumann and Zagier 85] prove that $(\mathbf{A}|\mathbf{B})$ is the upper part of a symplectic matrix. It follows that $(\mathbf{A}|\mathbf{B})$ has rank N and $\mathbf{A}\mathbf{B}^T$ is symmetric, where $\mathbf{B}^T$ is the transpose of $\mathbf{B}$. We will assume that $\mathbf{B}$ has nonvanishing determinant. Furthermore, we will assume that our triangulation $\mathcal{T}$ is such that there exists a solution to the gluing equations in $(\mathbb{C} \setminus \{0, 1\})^N$ that recovers the complete hyperbolic structure of the hyperbolic knot K. In that case, z is a vector of algebraic numbers and $\mathbb{Q}(z_1, \ldots, z_N)$ is a number field (the shape field) which coincides with the invariant trace field and with the trace field of the knot [Neumann and Reid 92, Theorems 2.2 and 2.4]. Finally, one can choose a flattening (f, f''), which is an integer solution of the linear equation $\mathbf{A}f + \mathbf{B}f'' = \nu$. This determines a Neumann–Zagier datum $\gamma = (A, B, \nu, f, f', z)$, which in turn defines the power series $\phi_\gamma(\hbar)$. Of course, different ideal triangulations give rise to different Neumann–Zagier data, hence to potentially different formal power series $\phi_\gamma(\hbar)$. On the other hand, the left-hand side of equation (1-1) depends only on the hyperbolic knot K. Although the topological invariance of $\phi_\gamma(\hbar)$ is not known, from the computational point of view, this gives an excellent consistency check of correctness of the code.

Equations (1-2a)–(1-2c) are manifest by the definition of $\phi_\gamma(\hbar)$. In [Dimofte and Garoufalidis 13], it was shown that τ_γ is a topological invariant, defined up to a sign. We may call τ_γ the 1-loop invariant. If we write

$$\phi^+_\gamma(\hbar) = \exp\left(\sum_{n=2}^{\infty} S_{\gamma,n}\hbar^{n-1}\right),$$

then $S_{\gamma,n}$ are the n-loop invariants of the γ. In [Dimofte and Garoufalidis 13], it was conjectured that $S_{\gamma,2}$ is well-defined up to addition of an integer multiple of 1/24, and that $S_{\gamma,n}$ are topological invariants for $n \geq 3$.

The definition of $\phi^+_\gamma(\hbar)$ is given explicitly by formal Gaussian integration. It follows that $S_{\gamma,n}$ is a weighted sum of a finite set of Feynman diagrams with Feynman loop number at most n. The Feynman rules were explained in detail in [Dimofte and Garoufalidis 13, Sections 1.6–1.8], and the contributing Feynman diagrams for $n = 2$ and $n = 3$ were explicitly drawn. For $n > 3$, the number of Feynman diagrams gets large and drawings-by-hand is not advisable.

For the benefit of the reader, we recall the Feynman rules from [Dimofte and Garoufalidis 13, Sections 1.6–1.8]. By connected *Feynman diagram* G, we mean a connected multigraph, possibly with loops and multiple edges. If G is a Feynman diagram, its *Feynman loop number* $L(G)$ is given by

$$L(G) = |V_1(G)| + |V_2(G)| + b_1(G)$$

where $|V_k(G)|$ is the number of k-valent vertices of G and $b_1(G)$ is the first betti number (also known as the number of holes) of G. It is easy to see that a connected Feynman diagram with loop number at most n has at most $2n - 2$ vertices and at most n holes. Hence, there are finitely many Feynman diagrams of loop number at most n.

Fix a Neumann–Zagier datum $\gamma = (\mathbf{A}, \mathbf{B}, \nu, f, f'', z)$ which we assume is non-degenerate, that is the propagator (defined below) makes sense. In each Feynman diagram G, the edges represent an $N \times N$ propagator

$$\Pi = \hbar\left(-\mathbf{B}^{-1}\mathbf{A} + \mathrm{diag}(1/(1-z))\right)^{-1}$$

while each k-vertex comes with an N-vector of factors $\Gamma^{(k)}_i$,

$$\Gamma^{(k)}_i = (-1)^k \sum_{p=\alpha_k}^{\alpha_k+n-L(D)} \frac{\hbar^{p-1}(-1)^p B_p}{p!} \mathrm{Li}_{2-p-k}(z_i^{-1}) + \begin{cases} -\frac{1}{2}(\mathbf{B}^{-1}\nu)_i & k = 1 \\ 0 & k \geq 2 \end{cases},$$

where $\alpha_k = 1$ (resp., 0) if $k = 1, 2$ (resp., $k \geq 3$). Here B_k is the k-th Bernoulli number ($B_1 = -1/2$, $B_2 = 1/6$) and $\mathrm{Li}_s(z) = \sum_{m=1}^{\infty} z^m/m^s \in \mathbb{Q}(z)$ is the s-polylogarithm function for s a nonpositive integer. The diagram G is then evaluated by contracting the *vertex factors* $\Gamma_i^{(k)}$ with propagators, multiplying by a standard *symmetry factor*, and taking the $\hbar^{n-1}$ part of the answer. In the end, $S_{\gamma,n}$ is the sum of evaluated diagrams, plus an additional *vacuum* contribution

$$\Gamma^{(0)} = \frac{B_n}{n!}\sum_{i=1}^{N} \mathrm{Li}_{2-n}(z_i^{-1}) + \begin{cases} \frac{1}{8} f \cdot \mathbf{B}^{-1}\mathbf{A} f & n=2 \\ 0 & n \geq 3 \end{cases}.$$

1.3. *Our code*

Our goal is to give an exact computation for the n-loop invariants for $n = 1, \ldots, 6$ of a Neumann–Zagier datum of a `SnapPy` triangulation. Our method is implemented in `SnapPy`. We accomplished this in three steps.

(a) We wrote a Python method `generate_feynman_diagrams.py` that generates all Feynman diagrams that contribute to the n-loop invariant. The Feynman diagrams were generated by first generating trees, and then adding to them multiple edges or loops. The number of such diagrams is shown in Table 1. Observe that if G is a multigraph with corresponding simple graph $S(G)$, then $S(G)$ has at most $2n - 2$ vertices and at most n holes, and $L(G)$ can be obtained from $S(G)$ by adding at most $n - L(S(G)) + |V_1(S(G))| + |V_2(S(G))|$ edges. Thus, all Feynman diagrams with Feynman loop number at most n can thus be generated by first generating all trees with at most $2n - 2$ vertices, then iteratively adding edges between pairs of vertices. Every edge added also adds an additional hole. If multigraph G has more than $n - |V_1(G)| - |V_2(G)|$ holes, it cannot be the subgraph of a Feynman diagram with Feynman loop number at most n.

(b) We wrote a Python class `NeumannZagierDatum` which gives the
Neumann–Zagier matrices and the exact value of the shape parameters that recover the geometric representation of an ideal triangulation. The exact computation of the shape parameters was done using the Ptolemy module [Garoufalidis et al. 15a, Culler et al. 09], and the numerical computation is already implemented in `SnapPy`.

(c) We wrote Python classes `nloop_exact.py` and `nloop_num.py` which give a Neumann–Zagier datum γ and a natural number $n = 1, \ldots, 6$ computes $S_{\gamma,n}$ exactly (as an element of the trace field) or numerically to arbitrary precision.

Table 1. The number g_n of graphs that contribute to the n-loop invariant for $n = 2, \ldots, 6$.

n	2	3	4	5	6
g_n	6	40	331	3700	53,758

To verify the correctness of our code, we computed the n-loop invariants for $n = 1, \ldots, 5$ for different triangulations of each of a fixed knot, such as 5_2, $(-2, 3, 7)$ pretzel, 6_1, and 6_2. In all cases, the results agreed (up to a sign when $n = 1$ and up to addition of $1/24$ times an integer when $n = 2$). This illustrates both the topological invariance of the n-loop invariants and the correctness of our code.

1.4. *Usage*

The essence of our code lies in two Python classes `NeumannZagierDatum` and `nloop`. The former takes as input a manifold and generates the Neumann–Zagier datum $\gamma = (A, B, \nu, f, f', z)$, and the latter takes as input Neumann–Zagier datum, an integer n, and a list of Feynman diagrams and returns the n-th loop invariant $S_{\gamma,n}$.

The `NeumannZagierDatum` class has three optional arguments `engine`, `verbose`, and `file_name`, which are set to `None`, `False`, and `None`, respectively, by default. The `engine` variable is passed as an option into the Ptolemy module and controls the method in which solutions to the Ptolemy variety are found. The preferred value for this variable for manifolds in `CensusKnots` is `engine="magma"`, which refers to the `Sage` interface to the `Magma` Computational Algebra System [Bosma et al. 97]. If Magma is not available, `engine="None"` will attempt to compute solutions of the Ptolemy variety using `Sage`. Solutions for manifolds in `HTLinkExteriors` and `LinkExteriors` have been precomputed and are available with the Ptolemy module using `engine="retrieve"`, [Garoufalidis et al. 15a]. This option requires an internet connection, but will automatically switch to recomputing locally if the download is unsuccessful. The output of the Ptolemy module including the `retrieve` option are suppressed with `verbose=False` and are displayed with `verbose=True`.

To utilize the `NeumannZagierDatum` class use a terminal to navigate to the directory containing `nloop_exact.py`, available at [Garoufalidis et al. 15b], and load `Sage`. Once loaded, the class must first be initiated via

```
sage: attach('nloop_exact.py')
sage: M = Manifold('6_1')
sage: D = NeumannZagierDatum(M, engine="retrieve").
```

To generate the Neumann–Zagier datum, use

```
sage: D.generate_nz_data().
```

This will assign a Python list [A, B, ν, f, f', z, embedding] consisting of the

Neumann–Zagier datum plus the embedding of the something in the something to the class variable `nz`. If the optional argument `file_name` is used, this variable will be saved as a `Sage` object file (*.sobj) in the current directory. To view the data, simply use

```
sage: D.nz.
```

The shape equations z and field embedding may be computed separately via

```
asage: D.exact_shapes_via_ptolemy_
lifted()
```

and

```
sage: D.compute_ptolemy_field_and_
embedding(),
```

respectively.

Once the Neumann–Zagier datum has been computed, one may use it to compute the n-loop invariants $S_{\gamma,n}$. First, load the Feynman diagrams you wish to use and choose an invariant you wish to calculate,

```
sage: n = 2
sage: diagrams = load('6diagrams.
sobj')
sage: E = nloop(D.nz, n, diagrams).
```

Here, we have chosen to calculate $S_{\gamma,2}$ using Feynman diagrams up to six loops. Note that the manifold `M` is not directly used when initiating the `nloop` class, as all the information about the manifold we need is encoded in the Neumann–Zagier datum `D.nz`. To compute the invariant use

```
sage: E.one_loop()
```

if $n = 1$ or

```
sage: E.nloop_invariant()
```

otherwise. To do this using a precomputed Neumann–Zagier datum `Sage` object file instead of defining `D` as above use

```
sage: nz = load('nz_exact_6_1.sobj')
sage: E = nloop(nz, n, diagrams).
```

The entire process described above has been streamlined into two automated functions for convenience. For example, to start with a specified manifold `M` and diagrams list, compute the Neumann–Zagier datum, and then compute the n-loop invariant, simply use

```
sage: nloop_from_manifold(M, n,
diagrams, engine="retrieve").
```

The `NeumannZagierDatum` optional arguments described above may be entered here as seen in the example. On the other hand, to start with a precomputed Neumann–Zagier datum `Sage` object file (loaded as `nz`) and a diagrams list, then compute the n-loop invariant, simply use

```
sage: nloop_from_nzdatum(nz, n,
diagrams, engine="retrieve").
```

Also available at [Garoufalidis et al. 15b] is an almost identical version of our code, `nloop_num.py`, which produces numerical results to arbitrary precision instead of exact computations. The usage for this file is the same.

1.5. *Sample computations*

The results of our computations are available from [Garoufalidis et al. 15b], along with the code and data files.

To illustrate our method, consider the $6_1 = K4_1$ knot with trace field $F_{6_1} = \mathbb{Q}(x)$, where $x = -1.50410836415074\ldots + i1.22685163774658\ldots$ is a root of

$$x^4 + 2x^3 + x^2 - 3x + 1 = 0.$$

F_{6_1} is a number field of type [0, 2] with discriminant 257, a prime number. It follows that the Bloch group $\mathcal{B}(F_{6_1})$ is a finitely generated abelian group of rank 2 [Suslin 90, Zickert 09]. The default `SnapPy` triangulation for $K4_1$ uses four ideal tetrahedra with shapes

$$z = \left(\frac{3}{2}x^3 + \frac{7}{2}x^2 + 3x - \frac{5}{2},\ 2x^3 + 5x^2 + 5x - 3,\ -\frac{1}{2}x^3 - \frac{3}{2}x^2 - x + \frac{3}{2},\ \frac{1}{2}x^3 + \frac{3}{2}x^2 + 2x + \frac{1}{2}\right).$$

A Neumann–Zagier datum $\gamma = (A, B, \nu, f, f', z)$ is given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\nu = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad f = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad f'' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The n-loop invariants for $n = 1, \ldots, 6$ are given by:

$$\tau = -\frac{7}{2}x^3 - \frac{17}{2}x^2 - \frac{17}{2}x + 6$$

$$S_2 = \frac{46490}{198147}x^3 + \frac{231209}{396294}x^2 + \frac{473191}{792588}x - \frac{62777}{264196}$$

$$S_3 = \frac{570416}{16974593}x^3 + \frac{2833463}{33949186}x^2 + \frac{1122215}{16974593}x - \frac{1386486}{16974593}$$

$$S_4 = -\frac{2255130587026}{50451970187565}x^3 - \frac{91695358340911}{807231523001040}x^2$$

$$-\frac{85651263871967}{807231523001040}x+\frac{1596902056811}{20180788075026}$$

$$S_5=-\frac{37040877003091}{1728820845093894}x^3-\frac{330280282463219}{6915283380375576}x^2$$
$$-\frac{53499149965837}{1728820845093894}x+\frac{72838757049049}{1152547230062596}$$

$$S_6=\frac{1449319256564305241317}{17984434859623040256945}x^3$$
$$+\frac{2359284241023023907679 9}{11510038310158745764444 8}x^2$$
$$+\frac{1105674328328997547081 87}{5755019155079372882222 40}x$$
$$-\frac{2000849458562016874831 9}{1438754788769843220555 60}$$

Acknowledgments

The authors wish to thank the computer support group of the School of Mathematics and especially Justin Filoseta and Lew Lefton. We also wish to thank the anonymous referee for a careful reading of our article and for suggested improvements in the presentation.

Funding

S.G. was supported in part by NSF.

References

[Bosma et al. 97] Wieb Bosma, John Cannon, and Catherine Playoust. “The Magma Algebra System. I. The User Language.” *J. Symbol. Comput.* 24:3–4 (1997), 235–265. Computational algebra and number theory (London, 1993).

[Culler et al. 09] Marc Culler, Nathan M. Dunfield, and Jeffery R. Weeks. “SnapPy.” Available at http://www.math.uic.edu/t3m/SnapPy, 2009.

[Dimofte and Garoufalidis 13] Tudor Dimofte and Stavros Garoufalidis. “The Quantum Content of the Gluing Equations.” *Geom. Topol.* 17:3 (2013), 1253–1315.

[Garoufalidis 08] Stavros Garoufalidis. “Chern-Simons Theory, Analytic Continuation and Arithmetic.” *Acta Math. Vietnam.* 33:3 (2008), 335–362.

[Garoufalidis et al. 15a] Stavros Garoufalidis, Matthias Goerner, and Christian Zickert. “*The Ptolemy Field of 3-Manifold Representations*.” *Algebr. Geom. Topol.* 15:1 (2015), 371–397.

[Garoufalidis et al. 15b] Stavros Garoufalidis, Eric Sabo, and Shane Scott. Available at http://www.math.gatech.edu/~stavros/publications/nloop.data/index.html/, 2015.

[Gukov 05] Sergei Gukov. “Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial.” *Commun. Math. Phys.* 255:3 (2005), 577–627.

[Garoufalidis and Zagier 13] Stavros Garoufalidis and Don Zagier. “Asymptotics of Quantum Knot Invariants.” Preprint, 2013.

[Jones 87] V. F. R. Jones. “Hecke Algebra Representations of Braid Groups and Link Polynomials.” *Ann. Math. (2)* 126:2 (1987), 335–388.

[Kashaev 95] Rinat M. Kashaev. “A Link Invariant from Quantum Dilogarithm.” *Mod. Phys. Lett. A* 10:19 (1995), 1409–1418.

[Kashaev 97] R. M. Kashaev. “The Hyperbolic Volume of Knots from the Quantum Dilogarithm.” *Lett. Math. Phys.* 39:3 (1997), 269–275.

[Murakami and Murakami 01] Hitoshi Murakami and Jun Murakami. “The Colored Jones Polynomials and the Simplicial Volume of a Knot.” *Acta Math.* 186:1 (2001), 85–104.

[Neumann and Reid 92] Walter D. Neumann and Alan W. Reid. “Arithmetic of Hyperbolic Manifolds.” In *Topology ’90*, Columbus, OH, 1990, Ohio State Univ. Math. Res. Inst. Publ., vol. 1, pp. 273–310. Berlin: de Gruyter, 1992.

[Neumann and Zagier 85] Walter D. Neumann and Don Zagier. “Volumes of Hyperbolic Three-Manifolds.” *Topology* 24:3 (1985), 307–332.

[Suslin 90] Andrei Suslin. “K_3 of a Field, and the Bloch Group.” *Trudy Mat. Inst. Steklov.* 183 (1990), 180–199, 229. Translated in Proc. Steklov Inst. Math. 1991, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).

[Thurston 77] William Thurston. *The Geometry and Topology of 3-Manifolds*. Berlin: Universitext, Springer-Verlag, 1977. Lecture notes, Princeton.

[Zickert 09] Christian K. Zickert. “The Extended Bloch Group and Algebraic K-Theory.” Preprint, arXiv:0910.4005, 2009.