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Incompressibility criteria for spun-normal surfaces

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We give a simple sufficient condition for a spun-normal surface in an ideal triangulation to be incompressible, namely that it is a vertex surface with non-empty boundary which has a quadrilateral in each tetrahedron. While this condition is far from being necessary, it is powerful enough to give two new results: the existence of alternating knots with non-integer boundary slopes, and a proof of the Slope Conjecture for a large class of 2-fusion knots.

While the condition and conclusion are purely topological, the proof uses the Culler-Shalen theory of essential surfaces arising from ideal points of the character variety, as reinterpreted by Thurston and Yoshida. The criterion itself comes from the work of Kabaya, which we place into the language of normal surface theory. This allows the criterion to be easily applied, and gives the framework for proving that the surface is incompressible.

We also explore which spun-normal surfaces arise from ideal points of the deformation variety. In particular, we give an example where no vertex or fundamental surface arises in this way.

Contents

1   Introduction

Let M be a compact oriented 3-manifold whose boundary is a torus. A properly embedded surface S in M is called *essential* if it is incompressible, boundary-incompressible, and not boundary-parallel. If S has boundary, this consists of pairwise-isotopic essential simple closed curves on the torus ∂M; the unoriented isotopy class of these curves is the *boundary slope* of S. Such slopes can be parameterized by the corresponding primitive homology class in $H_1(\partial M; \mathbb{Z})/(\pm 1)$; if a basis of $H_1(\partial M; \mathbb{Z})$ is fixed, slopes can also be recorded as elements of $\mathbb{Q} \cup \{\infty\}$.

Our focus here is on the set bs(M) of all boundary slopes of essential surfaces in M, which is finite by a fundamental result of Hatcher [Hat1]. This is an important invariant of M, for instance playing a key role in the study of exceptional Dehn filling. Building on Haken's fundamental contributions [Hak], Jaco and Sedgwick [JS] used normal surface theory to give a general algorithm for computing bs(M). As with most normal-surface algorithms, this method seems impractical even for modest-sized examples (however, some important progress has been made on this by [BRT]). For certain special cases, such as exteriors of Montesinos knots, fast algorithms do exist [HT, HO, Dun2], and additionally character-variety techniques can sometimes be used to find boundary slopes [CCGLS, Cul]. However, there remain quite small examples where bs(M) is unknown, e.g. for the exteriors of certain 9-crossing knots in S^3.

Here, we introduce a simple sufficient condition that ensures that a normal surface is essential. While our condition is far from being necessary, it is powerful enough to give two new results: the existence of alternating knots with non-integer boundary slopes, and a proof of the Slope Conjecture for all 2-fusion knots. Along with [BRT], these are the first results that come via applying directly normal surface algorithms, which have been much studied for their inherent interest in the past 50 years.

We work in the context of an ideal triangulation $\mathcal{T}$ of M and Thurston's corresponding theory of spun-normal surfaces (throughout, see Section 2 for definitions). In normal surface theory, *vertex surfaces* corresponding to the vertices of the projectivized space of normal surfaces play a key role. Our basic result is

1.1 Theorem   *Suppose S is a vertex spun-normal surface in $\mathcal{T}$ with non-trivial boundary. If S has a quadrilateral in every tetrahedron of $\mathcal{T}$, then S is essential.*

While this statement is purely topological, the proof uses the Culler-Shalen theory of essential surfaces arising from ideal points of the character variety [CS, CGLS], as reinterpreted by Thurston [Thu1] and Yoshida [Yos] in the context of the deformation variety defined by the hyperbolic gluing equations for $\mathcal{T}$. Theorem 1.1 is a strengthening of a result of Kabaya [Kab], who shows that, with the same hypotheses, that the boundary slope of S is in bs(M). Our contribution to Theorem 1.1 is restating Kabaya's work in the language of normal surface theory, allowing it to be easily applied, and showing that S is itself incompressible.

1.2 Alternating knots

Our main application of Theorem 1.1 concerns the boundary slopes of (the exteriors of) alternating knots in S^3. In the natural meridian-longitude basis for $H_1(\partial M)$, Hatcher and Oertel [HO] showed that the boundary slopes of alternating Montesinos knots were always even integers, generalizing what Hatcher and Thurston had found for 2-bridge knots [HT]. Hatcher and Oertel asked whether this was true for *all* alternating knots. We use Theorem 1.1 to settle this 20 year-old question:

1.3 Theorem *There are alternating knots with nonintegral boundary slopes. In particular, the knot 10_{79} has boundary slopes $10/3$ and $-10/3$.*

Many additional such examples are listed in Table 6.1.

1.4 Dehn filling

The technique of Kabaya that underlies Theorem 1.1 can be generalized to manifolds that arise from Dehn filling all but one boundary component of a more complicated manifold. Specifically, in the language of Section 7 we show:

1.5 Theorem *Let W be a compact oriented 3-manifold where ∂W consists of tori $T_0, T_1, \ldots, T_n$. Let S be a spun-normal surface in an ideal triangulation $\mathcal{T}$ of W, with nonempty boundary slope γ_k on each T_k. Suppose that S has a quadrilateral in every tetrahedra of $\mathcal{T}$, and is a vertex surface for the relative normal surface space corresponding to $(\cdot, \gamma_1, \ldots, \gamma_n)$. Then γ_0 is a boundary slope of $W(\cdot, \gamma_1, \ldots, \gamma_n)$.*

This broadens the applicability of Kabaya's approach since any given M arises in infinitely many ways by Dehn filling, and thus a fixed surface $S \subset M$ has many chances where Theorem 1.5 might apply.
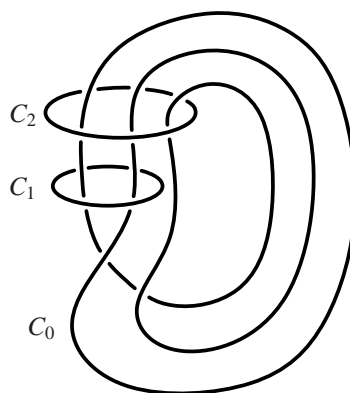
Figure 1.7: The link L.

1.6   The Slope Conjecture for 2-fusion knots

Our application of Theorem 1.5 involves constructing a boundary slope for every knot of a certain 2-parameter family. We use this to prove the Slope Conjecture of [Gar1] in the case of 2-fusion knots. This conjecture relates the degree of the Jones polynomial of a knot and its parallels to boundary slopes of essential surfaces in the knot complement. To state our result, consider the 3-component link L from Figure 1.7. For a pair of integers $(m_1, m_2) \in \mathbb{Z}^2$, let $L(m_1, m_2)$ denote the knot obtained by $(-1/m_1, -1/m_2)$ filling on the cusps C_1 and C_2 of L, leaving the cusp C_0 unfilled. The 2-parameter family of knots $L(m_1, m_2)$, together with the double-twist knots coming from filling 2 cusps of the Borromean link, is the set of all knots of fusion number at most 2; see [Gar2]. The family $L(m_1, m_2)$ has some well-known members: $L(2, 1) = L(-1, 2)$ is the $(-2, 3, 7)$ pretzel knot, $L(-2, 1)$ is the 5_2 knot, $L(-1, 3)$ is $k4_3$ which was the focus of [GL], and $L(m_1, 1)$ is the $(-2, 3, 3m_1 + 3)$ pretzel knot.
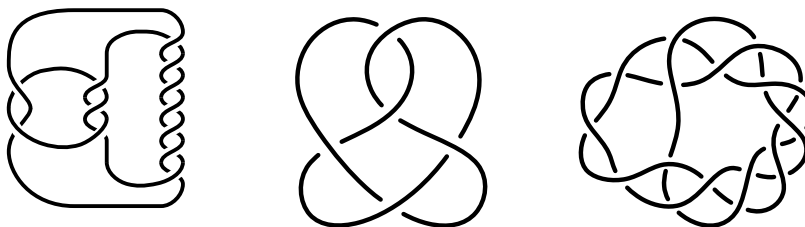

Figure 1.8: The $(-2, 3, 7)$ pretzel, the 5_2 knot, and the $k4_3$ knot.

Together with the results of [Gar2], the following confirms the Slope Conjecture for 2-fusion knots in one of three major cases:

1.9 Theorem *For $m_1 > 1, m_2 > 0$, one boundary slope of $L(m_1, m_2)$ is:*

$$3(1 + m_1) + 9m_2 + \frac{(m_1 - 1)^2}{m_1 + m_2 - 1}$$

It is mysterious how the Jones polynomial selects one (out of the many) boundary slopes of a knot, and it was fortunate that this slope happens to be one of the few accessible by the special method of Theorem 1.5 for the family $L(m_1, m_2)$ of 2-fusion knots. Indeed, we tried without success to apply our same method to confirm the Slope Conjecture for the rest of the 2-fusion knots. Note also that the results of [FKP] do not imply Theorem 1.9 as the former only produce integer boundary slopes.

1.10   Technical results

In addition to Theorems 1.1 and 1.5, we make progress on the question of which spun-normal surfaces in an ideal triangulation $\mathcal{T}$ arise from an ideal point of the deformation variety $D(\mathcal{T})$ (see Section 3 for more on the latter). In particular, given an ideal triangulation $\mathcal{T}$ of a manifold M with one torus boundary component, the goal is to determine all the boundary slopes that arise from ideal points of $D(\mathcal{T})$. Of course, one can find all such detected slopes by computing the A-polynomial, but this is often a very difficult computation, involving projecting an algebraic variety (i.e. eliminating variables).

For a fixed surface S, we give a relatively easy-to-check algebro-geometric condition (Lemma 4.15) which is both necessary and sufficient for S to come from an ideal point. However, there are often only finitely many ideal points but infinitely many spun-normal surfaces, and so Lemma 4.15 does not completely solve this problem. A natural hope is that the surfaces associated to ideal points would be vertex or fundamental surfaces, but we give a simple example in Section 9.2 where this is not the case.

1.11   Outline of contents

In Sections 2 and 3 we review the basics of spun-normal surfaces and deformation varieties. Then in Section 4, we study a class of algebraic varieties which includes these deformation varieties. We place Kabaya's motivating result into that context (Proposition 4.12) and also give a necessary and sufficient condition for there to be

an ideal point with certain data (Lemma 4.15). Section 5 is devoted to the proof of Theorem 1.1, and then Section 6 applies this result to give non-integral boundary slopes for alternating knots. Likewise, Section 7 proves Theorem 1.5 and Section 8 applies it to the Slope Conjecture for 2-fusion knots. Finally, Section 9 explores the effectiveness and limitations of the methods studied here.

1.12   Acknowledgments

2   Spun-normal surfaces

In this section, we sketch Thurston's theory of spun-normal surfaces in ideal triangulations. We follow Tillmann's exposition [Til1] which contains all the omitted details (see also [Kang, KR]). Let M be a compact oriented 3-manifold whose boundary is a nonempty union of tori. An ideal triangulation $\mathcal{T}$ of M is a Δ-complex (in the language of [Hat2]) made by identifying faces of 3-simplices in pairs so that $\mathcal{T} \setminus$ (vertices) is homeomorphic to int(M). Thus $\mathcal{T}$ is homeomorphic to M with each component of ∂M collapsed to a point.

A spun-normal surface S in $\mathcal{T}$ is one which intersects each tetrahedron in finitely many quads and infinitely many triangles marching out toward each vertex (see Figure 2.1(a)). While there are infinitely many pieces, in fact S is typically the interior of a properly embedded compact surface in M whose boundary has been "spun" infinitely many times around each component of ∂M. (The other possibility for S near a vertex is that it consists of infinitely many disjoint boundary-parallel tori.) Notice from Figure 2.1(a) that on any face of a tetrahedron, there is exactly one hexagon region and infinitely many four-sided regions. Thus to specify a spun-normal surface S, we need only record the number and type of quads in each tetrahedron of $\mathcal{T}$, since the need to glue hexagons to hexagons uniquely specifies how the local pictures of S must be glued together across adjoining tetrahedra. As there are three kinds of quads, if $\mathcal{T}$ has n tetrahedra then S
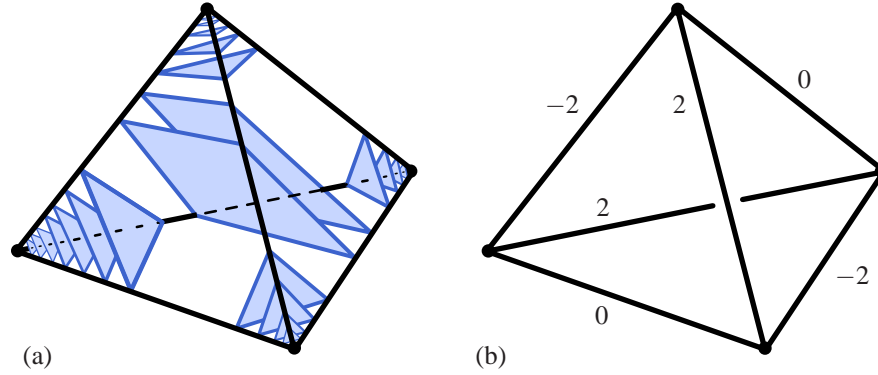
Figure 2.1: At left is the intersection of a spun-normal surface with a single tetrahedron, with infinitely many triangles in each corner. At right are the edge shifts of the hexagon regions as defined in Figure 2.2.

is uniquely specified by a vector in $\mathbb{Z}_+^{3n}$ called its Q-coordinates. This vector satisfies certain linear equations which we now describe as they will explain how ideal points of the deformation variety give rise to such surfaces.

For an edge of a tetrahedron, let s be the amount the adjacent hexagons are shifted relative to each other; the orientation convention is given in Figure 2.2, and Figure 2.1(b) shows the resulting shifts on all edges of a tetrahedron. It is not so hard to see that $v \in \mathbb{Z}_+^{3n}$ corresponds to a spun-normal surface if and only if

(a) There is at most one non-zero quad weight in any given tetrahedron.

(b) As we go once around an edge, the positions of the hexagons match up. That is, the sum of the shifts s must be 0.

The shifts are linear functions of the entries of v (see Figure 2.1), and so the conditions in (b) form a linear system of equations called the *Q-matching equations*.

As their Q-coordinates satisfy various linear equalities and inequalities, spun-normal surfaces fit into the following geometric picture. Let $C(\mathcal{T})$ be the intersection of $\mathbb{R}_+^{3n}$ with the subspace of solutions to the Q-matching equations. Thus $C(\mathcal{T})$ is a finite-sided convex cone. If we impose condition (a) as well, we get a set $F(\mathcal{T})$ which is a finite union of convex cones whose integral points are precisely the Q-coordinates of spun-normal surfaces. Within each convex cone of $F(\mathcal{T})$, vector addition of Q-coordinates corresponds to a natural geometric sum operation on the associated spun-normal surfaces.
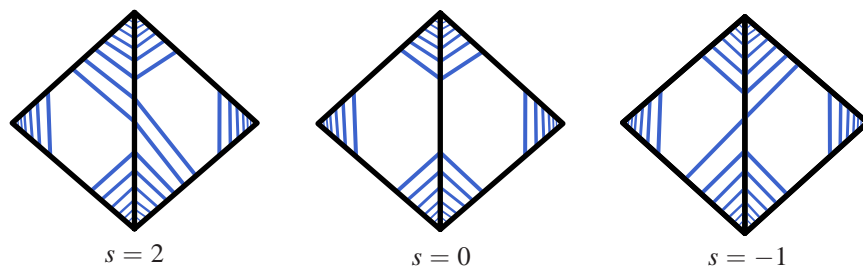
Figure 2.2: The shift parameter s of an edge describes the relative positions of the adjacent hexagons, as viewed from outside the simplex Δ. Here Δ is oriented by the orientation of M, and the induced orientation of $\partial\Delta$ distinguishes between $s > 0$ and $s < 0$. As each picture is invariant under rotation by π, the shift does *not* depend on an orientation of the edges themselves. The convention here agrees with [Til1].

It is natural to projectivize $F(\mathcal{T})$ by intersecting it with the affine subspace where the coordinates sum to 1. The resulting set $PF(\mathcal{T})$ is a finite union of compact polytopes. Since all the defining equations had integral coefficients, the vertices of these polytopes lie in $\mathbb{Q}_+^{3n}$. For such a vertex v, consider the smallest rational multiple of v which lies in $\mathbb{Z}_+^{3n}$; that vector gives a spun-normal surface, called a *vertex surface*. Vertex surfaces play a key role in normal surface theory generally and here in particular.

One major difference between spun-normal surface theory and the ordinary kind for non-ideal triangulations is that normalizing a given surface is much more subtle. This is because of the infinitely many intersections of a spun surface with the 1-skeleton of $\mathcal{T}$. However, building on ideas of Thurston, Walsh has shown that essential surfaces which are not fibers or semi-fibers can be spun-normalized, using characteristic submanifold theory [Wal]. Despite this, some key algorithmic questions remain unanswered for spun-normal surfaces. For instance, when M has one boundary component, do all the strict boundary slopes arise from *vertex* spun-normal surfaces which are also essential? For an ordinary triangulation of M (which will typically have more tetrahedra than an ideal one), the answer is yes [JS, Theorem 5.3].

2.3   Ends of spun-normal surfaces

We now describe how a spun-normal surface gives rise to a properly embedded surface in M, closely following Sections 1.9-1.12 of [Til1]. For notational simplicity, we assume that M has a single boundary component. Let $\mathfrak{v}$ be the vertex of $\mathcal{T}$, and consider a small neighborhood $N_\mathfrak{v}$ of $\mathfrak{v}$ bounded by a normal torus $B_\mathfrak{v}$ consisting of

one normal triangle in each corner of every tetrahedron in $\mathcal{T}$. We can assume that $B_{\mathfrak{v}}$ and S are in general position and that $N_{\mathfrak{v}}$ meets only normal triangles of S.

We put a canonical orientation on the curves of $S \cap B_{\mathfrak{v}}$ as follows. First, triangulate $N_{\mathfrak{v}}$ by taking the cone to $\mathfrak{v}$ of the triangulation of $B_{\mathfrak{v}}$. If $\mathfrak{n}$ is a normal triangle of S meeting $N_{\mathfrak{v}}$, its interior meets exactly one tetrahedron $\Delta^3_{\mathfrak{v}}$ in $N_{\mathfrak{v}}$. We orient $\mathfrak{n}$ by
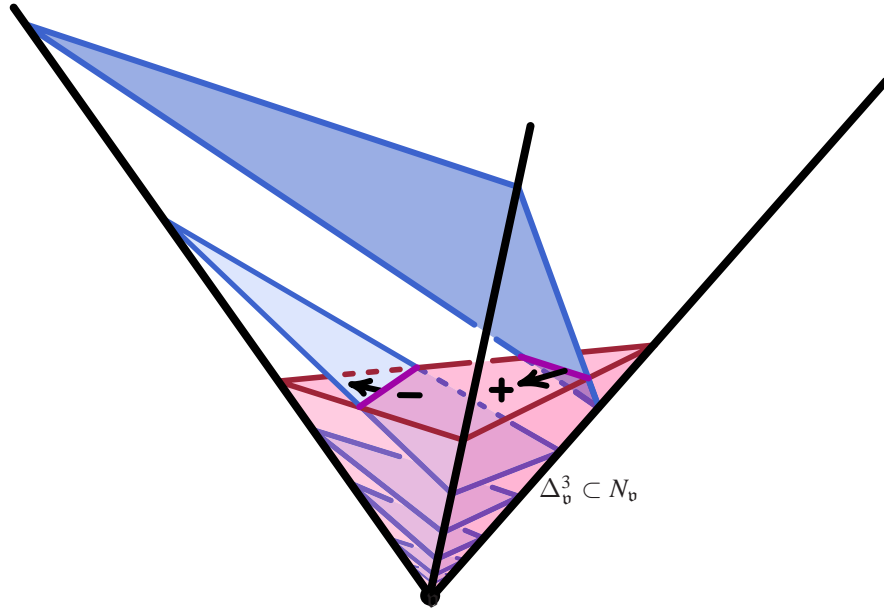
Figure 2.4: Orienting $S \cap B_{\mathfrak{v}}$. Notice that S meets the triangle of $B_{\mathfrak{v}}$ in at most two of the three possible types of normal arcs.

assigning $+1$ to the component of $\Delta^3_{\mathfrak{v}} \setminus \mathfrak{n}$ which contains $\mathfrak{v}$. This induces a consistent transverse orientation for each component of $S \cap B_{\mathfrak{v}}$ as shown in Figure 2.4.

By Lemma 1.31 of [Til1], we can also do a normal isotopy of S so that all the components of $S \cap B_{\mathfrak{v}}$ are nonseparating in the torus $B_{\mathfrak{v}}$. If the components of $S \cap B_{\mathfrak{v}}$ don't all have the same orientation, apply the proof of Lemma 1.31 of [Til1] to an annulus between two adjacent components with opposite orientations to reduce the size of $S \cap B_{\mathfrak{v}}$. Thus we can assume that all components of $S \cap B_{\mathfrak{v}}$ have the same orientation. It then follows from Lemma 1.35 of [Til1] that $S \cap N_{\mathfrak{v}}$ consists of parallel half-open annuli spiraling out toward $\mathfrak{v}$.

We now identify $\mathcal{T} \setminus \operatorname{int}(N_{\mathfrak{v}})$ with M. Then $S' = S \cap M$ is a properly embedded surface in M. Since we understand $S \cap N_{\mathfrak{v}}$, it's easy to see that the isotopy type of S'

$$z' = \frac{1}{1-z}$$

$$z'' = \frac{z-1}{z}$$

Figure 3.1: The relationship between the shape parameters of the edges of an oriented tetrahedron in $\mathbb{H}^3$. Our convention here agrees with [CDW] and [Til1, Section 2.2].
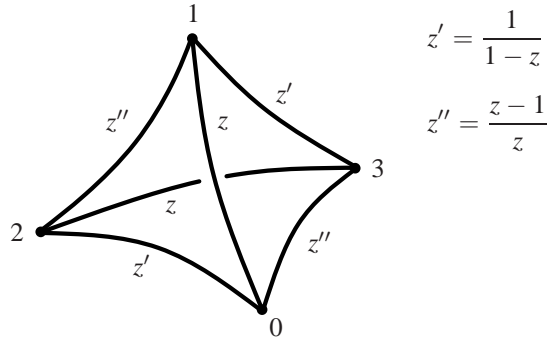
is independent of the choice of such $N_{\mathfrak{v}}$. (Here isotopies of S' are allowed to move $\partial S'$ within ∂M; the isotopy class of S' with $\partial S'$ fixed typically does depend on the choice of $N_{\mathfrak{v}}$.) Thus, it makes sense to talk about the number of boundary components of S and their slope.

3   Deformation varieties

As in Section 2, let $\mathcal{T}$ be an ideal triangulation of a compact oriented 3-manifold M with boundary a union of tori. Thurston [Thu2] introduced the deformation variety $D(\mathcal{T})$ parameterizing (incomplete) hyperbolic structures on $\mathrm{int}(M)$ where each tetrahedron in $\mathcal{T}$ has the shape of some honest ideal tetrahedron in $\mathbb{H}^3$. The deformation variety plays a key role in understanding hyperbolic Dehn filling [Thu2, NZ], and is closely related to the $\mathrm{PSL}_2\mathbb{C}$-character variety of $\pi_1(M)$. Via the latter picture, ideal points of $D(\mathcal{T})$ often give rise to essential surfaces in M, and spun-normal surfaces are the natural way to understand this process. In this section, we sketch the needed properties of $D(\mathcal{T})$ from the point of view of [Dun3, Til2] which contain the omitted details.

Suppose Δ is a non-degenerate ideal tetrahedron in $\mathbb{H}^3$, which has an intrinsic orientation (i.e. an ordering of its vertices). Each edge of Δ has a *shape parameter*, defined as follows. We apply an orientation preserving isometry of $\mathbb{H}^3$ so that the vertices of Δ are $(0, 1, \infty, z)$ and so this ordering induces the orientation of Δ. The shape parameter of the edge $(0, \infty)$ is then z, which lies in $\mathbb{C} \setminus \{0, 1\}$. Opposite edges have the same parameter, and any parameter determines all the others, as described in Figure 3.1, or

as encoded in

(3.2) $$z'(1-z) = 1 \quad \text{and} \quad z\,z'z'' = -1.$$

Returning to our ideal triangulation $\mathcal{T}$, suppose it has n tetrahedra. An assignment of hyperbolic shapes to all the tetrahedra is given by a point in $(\mathbb{C}^3)^n$ which satisfies n copies of the equations (3.2). The *deformation variety* $D(\mathcal{T})$, also called the *gluing equation variety*, is the subvariety of possible shapes where we require in addition that the *edge equations* are satisfied: for each edge the product of the shape parameters of the tetrahedra around it is 1. This requirement says that the hyperbolic structures on the individual tetrahedra glue up along the edge.

Because $D(\mathcal{T}) \subset \mathbb{C}^{3n}$ satisfies the conditions coming from (3.2), at a point of $D(\mathcal{T})$ no shape parameter takes on a degenerate value of $\{0, 1, \infty\}$. Consequently, a point of $D(\mathcal{T})$ gives rise to a developing map from the universal cover $\widetilde{M}$ to $\mathbb{H}^3$ which takes each tetrahedron of $\widetilde{\mathcal{T}}$ to one of the appropriate shape (see Lemma 3.5 below). This developing map is equivariant with respect to a corresponding holonomy representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$. In fact, there is a regular map

(3.3) $\quad D(\mathcal{T}) \to \overline{X}(M) \quad$ where $\overline{X}(M)$ is the $\mathrm{PSL}_2\mathbb{C}$-character variety of $\pi_1(M)$.

This map need not be onto, see e.g. the last part of Section 10 of [Dun3]. However, if M is hyperbolic and no edge of $\mathcal{T}$ is homotopically peripheral, then the image is nonempty. In particular, it contains a 1-dimensional irreducible component containing the discrete faithful representation $\pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ coming from the unique oriented complete hyperbolic structure on M.

3.4 Remark In Lemma 2.2 of [Til2], the existence of (3.3) is predicated on the edges of $\mathcal{T}$ being homotopically non-peripheral, whereas this condition is not mentioned in [Dun3]. Indeed it is not necessary to restrict $\mathcal{T}$, but as [Dun3] is terse on this point, we give a proof here of:

3.5 Lemma *For any triangulation $\mathcal{T}$, a point in $D(\mathcal{T})$ gives rise to a developing map $\widetilde{M} \to \mathbb{H}^3$, and hence a holonomy representation $\pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$.*

In fact, the proof will show that if $D(\mathcal{T})$ is nonempty, then a posteriori every edge in $\mathcal{T}$ is homotopically non-peripheral, meshing with Lemma 2.2 of [Til2]. A more detailed proof of a generalization of Lemma 3.5 is given in [ST].

Proof Let $N = M \setminus \partial M$ which we identify with the underlying space of $\mathcal{T}$ minus the vertices. Looking that the universal cover of N, we seek a map

$$d\colon \widetilde{N} \to \mathbb{H}^3$$

which takes each ideal simplex in N to an ideal simplex of $\mathbb{H}^3$ with the assigned shape (in particular, we are not yet trying to define the map at infinity). Let $N^\bullet$ be N minus the 1-skeleton of $\mathcal{T}$, which deformation retracts to the dual 1-skeleton of $\mathcal{T}$. In particular, $\pi_1(N^\bullet)$ is free, and universal cover U of $N^\bullet$ consists of tetrahedra with their 1-skeletons deleted, arranged so the dual 1-skeleton is an infinite tree. Thus it is trivial to inductively define a map

$$\widetilde{d}\colon U \to \mathbb{H}^3$$

which takes what's left of each tetrahedron in U to a correctly shaped ideal tetrahedron in $\mathbb{H}^3$ with its edges deleted. Let $\widetilde{N}^\bullet$ be $\widetilde{N}$ minus the lifted 1-skeleton of $\mathcal{T}$. Then we have covers $U \to \widetilde{N}^\bullet \to N^\bullet$. The cover $\widetilde{N}^\bullet \to N^\bullet$ corresponds to the normal subgroup Γ of $\pi_1(N^\bullet)$ generated by the boundaries of the dual 2-cells of $\mathcal{T}$, one corresponding to each edge. The condition that the shape parameters have product 1 for each edge mean that $\widetilde{d}$ is invariant under the deck transformation corresponding to the boundary of a dual 2-cell; hence $\widetilde{d}$ descends to a map d of $\widetilde{N}^\bullet = U/\Gamma$ to $\mathbb{H}^3$. The same edge condition also means that d extends over the deleted 1-skeleton to the desired map $d\colon \widetilde{N} \to \mathbb{H}^3$. (This is perhaps easier to understand if one only wants the corresponding representation: the holonomy representation $\pi_1(N^\bullet) \to \mathrm{PSL}_2\mathbb{C}$ for $\widetilde{d}$ clearly has the boundary of each dual 2-cell in its kernel, and thus factors through to a representation of $\pi_1(N)$.)

Now that we have $d\colon \widetilde{N} \to \mathbb{H}^3$ in hand, it is not hard to extend it to a continuous map from the end-compactification $\overline{N}$ of $\widetilde{N}$ to $\overline{\overline{\mathbb{H}}} = \mathbb{H}^3 \cup S^2_\infty$. This gives a pseudo-developing map in the sense of [Dun1, Section 2.5], and a posteriori certifies that the edges of $\mathcal{T}$ are homotopically non-peripheral, since they go to infinite geodesics under d which have two distinct limit points in S^2_∞. □

3.6 Ideal points and spun-normal surfaces

We now describe the connection between $D(\mathcal{T})$ and essential surfaces in M, which has its genesis in the work of Culler and Shalen on the character variety [CS, CGLS]. When M has one boundary component, a geometric component of $D(\mathcal{T})$ has complex dimension one, and it is common that all irreducible components of $D(\mathcal{T})$ are also curves. Thus for simplicity we focus on an irreducible curve $D \subset D(\mathcal{T})$; for the full story of ideal points as points in Bergman's logarithmic limit set, see [Til2].

As D is an affine algebraic variety, it is not compact. Let $\widetilde{D}$ be a smooth projective model for D, which in particular is a compact Riemann surface together with a rational map $f\colon \widetilde{D} \to D$ which is generically 1-1. An *ideal point* of D is a point of $\widetilde{D}$ where f

is not defined. For each edge of a tetrahedron in $\mathcal{T}$, the corresponding shape parameter z gives an everywhere-defined regular function $z\colon \widetilde{D} \to \mathbb{P}^1(\mathbb{C})$. From (3.2), it is easy to see that at an ideal point, the three shape parameters of a given tetrahedron are either $(z, z', z'') = (0, 1, \infty)$ (or some cyclic permutation thereof), or all take on values in $\mathbb{C} \setminus \{0, 1\}$.

We next describe how to define from an ideal point ξ of D a spun-normal surface $S(\xi)$. For each tetrahedron Δ of $\mathcal{T}$, we label each edge by the order of zero of the corresponding shape parameter at ξ (poles count as negative order zeros). For instance, in Figure 3.1, if z has a zero of order 2 at ξ, then the formulae for z' and z'' mean that the edges of Δ are labeled as shown in Figure 2.1(b). In general, the labeling associated to ξ similarly arises as the edge shifts of a unique spun-normal picture in Δ. In the case just mentioned, this is shown in Figure 2.1(a); in general, if n is the largest order of zero of the shape parameters, then $S(\xi) \cap \Delta$ has n quads which are disjoint from the edges whose shape parameters are 1 at ξ. That these local descriptions of $S(\xi)$ actually give a spun-normal surface can be seen as follows. Focus on an edge of $\mathcal{T}$, and let $z_1, \ldots, z_k$ be the shape parameters of the tetrahedra around it. Now on $D(\mathcal{T})$ and hence on $\widetilde{D}$ we have $\prod z_i = 1$, and taking orders of zeros turns this into the Q-matching equation for that edge, namely that the sum of the shifts is 0.

Before addressing the question of when $S(\xi)$ is essential, we mention that there is a closely related construction of Yoshida [Yos] which also associates a surface to an ideal point of D; see Segerman [Seg1] for the exact relationship between these two surfaces.

3.7 Ideal points and essential surfaces

Culler and Shalen showed how to associate to an ideal point of the character variety $\overline{X}(M)$ an essential surface via a non-trivial action on a tree [CS]. However, not every ideal point ξ of $D(\mathcal{T})$ gives rise to an essential surface, as sometimes ideal points of $D(\mathcal{T})$ map to ordinary points of $\overline{X}(M)$. We now describe how when $S(\xi)$ has non-empty boundary (in the sense of Section 2.3) then it does come from an ideal point of $\overline{X}(M)$.

As this is the key condition, we first sketch how to determine whether $S(\xi)$ has nonempty boundary along a component T of ∂M or instead consists of infinitely many boundary-parallel tori; for details see [Til2, Section 4] and [Til1, Sections 1 and 3]. For an element $\gamma \in \pi_1(\partial T)$, here is how to calculate the intersection number between γ and $\partial S(\xi)$. For a point $D(\mathcal{T})$, the holonomy in the sense of [Thu1] and [NZ] is given by

$$h(\gamma) = z_1 z_2 \cdots z_k \quad \text{for certain shape parameters } z_i.$$

View the components of $\partial S(\xi)$ on T as all oriented in the same direction, which direction being determined by how $S(\xi)$ is spinning out toward the boundary (see [Til1, Section 3.1]). Then the algebraic intersection number of γ and $\partial S(\xi)$ is the order of zero of $h(\gamma)$ at ξ. In particular, by taking a basis for $\pi_1(T)$, it is easy to check whether $S(\xi)$ has boundary and, if so, what the slope is.

We now turn to the question of when $S(\xi)$ can be reduced to an essential surface, in the following sense: a surface S is said to *reduce* to S' if there is a sequence of compressions, boundary compressions, elimination of trivial 2-spheres, and elimination of boundary-parallel components which turns S into S'. We then say that S' is a *reduction* of S. It will be convenient later to consider more broadly spun-normal surfaces S whose Q-coordinates are a rational multiple of those of $S(\xi)$; we call such S *associated* to ξ.

3.8 Theorem *Let ξ be an ideal point of a curve $D \subset D(\mathcal{T})$. Suppose a two-sided spun-normal surface S associated to ξ has non-empty boundary with slope α on a component T of ∂M. Then any reduction of S has nonempty boundary along T with slope α. In particular, S can be reduced to a non-empty essential surface in M which also has boundary slope α.*

Proof This will follow easily from [Til2, Section 6], but to this end we note that we have defined "spun-normal" slightly differently than [Til2]. In particular, what we call spun-normal with non-empty boundary he calls simply spun-normal. Moreover, in [Til2] the surface $S(\xi)$ is made two-sided simply by doubling its Q-coordinates if it's not; we adopt this convention for this proof.

First, we reduce from an arbitrary S associated to ξ to $S(\xi)$ itself. Let S_0 be the spun-normal surface corresponding to the primitive lattice point on the ray $\mathbb{R}_+ \cdot S$, i.e. $S_0 = (1/g)S$ where g is the gcd of the coordinates of S. If S_0 is two-sided, then both S and $S(\xi)$ are simply a disjoint union of parallel copies of S_0, and thus we can focus on $S(\xi)$ instead. Should S_0 have a one-sided component, then as S and $S(\xi)$ are two-sided, they are both integer multiples of $2 \cdot S_0$, and again we can focus on $S(\xi)$.

We now relate $S(\xi)$ to the Bass-Serre tree associated to an ideal point of the $\mathrm{PSL}_2\mathbb{C}$-character variety $\overline{X}(M)$. Following Section 5.3 of [Til2], we use $\mathfrak{T}_N$ to denote the simplicial tree dual to the spun-normal surface $S(\xi)$. (Unlike [Til2], we require $S(\xi)$ to have infinitely many triangles in every corner of every tetrahedron; hence the dual tree to $S(\xi)$ is $\mathfrak{T}_N$ rather than the $\mathfrak{T}_S$ of Section 5.2 of [Til2].) Let $N = \mathcal{T} \setminus \mathcal{T}^0 \cong M \setminus \partial M$, and let $p \colon \widetilde{N} \to N$ be the universal covering map. (Note: our N is called M in [Til2].) There is an equivariant map $f \colon \widetilde{N} \to \mathfrak{T}_N$ where the preimage of the midpoints of the edges in $\mathfrak{T}_N$ is precisely $p^{-1}(S(\xi))$.

Now fix a simple closed curve $\beta \in \pi_1(T)$ which intersects α exactly once. As discussed above, we can orient β so that the holonomy $h(\beta)$ has a pole at ξ. By Proposition 6.10 of [Til2], there is an associated ideal point ξ' of a curve in $\overline{X}(M)$ so that there is a $\pi_1(M)$-equivariant map from $\mathfrak{T}_N$ to the simplicial tree $T_{\xi'}$ associated to ξ'. In particular, since $h(\beta)$ has a pole, the action of β on $T_{\xi'}$ is by a fixed-point free loxodromic transformation. Because of the map $\mathfrak{T}_N \to T_{\xi'}$, it follows that β also acts on $\mathfrak{T}_N$ by a loxodromic.

As in Section 2.3, we identify M with a suitable subset of N, and henceforth abuse notation by denoting $S \cap M$ by S. By restricting the domain, we get that S is dual to the equivariant map $f : \widetilde{M} \to \mathfrak{T}_N$. Now if S' is a reduction of S, we can modify f so that that S' is still dual to $\mathfrak{T}_N$. If $T \cap S'$ were empty, it follows that $\pi_1(T)$ acts on $\mathfrak{T}_N$ with a global fixed point. Thus since β acts on $\mathfrak{T}_N$ as a loxodromic, we have that S' has nonempty boundary along T, as claimed. $\qquad\square$

4 Ideal points of varieties of gluing equation type

In this section, we consider a class of complex algebraic varieties that arise from the deformation varieties of the last section by focusing on a single shape parameter for each tetrahedron. Such varieties were first considered by Thurston [Thu2] and Neumann-Zagier [NZ].

We start with a subgroup $\Lambda \subset \mathbb{Z}^{2n+1}$, which we call a *lattice* even when its rank is not maximal. Let $\mathbb{C}^{*\bullet} = \mathbb{C} \setminus \{0, 1\}$, and consider the variety $V(\Lambda) \subset (\mathbb{C}^{*\bullet})^n$ of points satisfying

$$(4.1) \qquad z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} (1 - z_1)^{b_1} (1 - z_2)^{b_2} \cdots (1 - z_n)^{b_n} = (-1)^c$$

for all $(a_1, \ldots, a_n, b_1, \ldots, b_n, c) \in \Lambda$. Since z_i and $(1 - z_i)$ are never 0 for $z_i \in \mathbb{C}^{*\bullet}$, these equations always make sense even when some a_i or b_i is negative. Henceforth, we assume that $\mathrm{rank}(\Lambda) \leq n - 1$, and call such a $V(\Lambda)$ a *variety of gluing equation type*. In the final application, the variety $V(\Lambda)$ will be a complex curve and hence $\mathrm{rank}(\Lambda) = n - 1$.

4.2 Remark Replacing the lattice Λ with an arbitrary subset Ω of $\mathbb{Z}^{2n+1}$ does not broaden this class of examples, since $V(\Omega) = V\big(\mathrm{span}\,\langle\Omega\rangle\big)$. Conversely, when testing whether a point is in $V(\Lambda)$, it suffices to consider only the finitely many equations coming from a given $\mathbb{Z}$-basis for Λ. More precisely, let $M(\Lambda)$ be a matrix whose r

rows are a basis for Λ, and write it as

(4.3)
$$M(\Lambda) = \left(\begin{array}{c|c|c} A & B & c \end{array} \right)$$

where A and B are $r \times n$ matrices, and c is an $r \times 1$ column vector. Then $V(\Lambda)$ can be described by (4.1) for all rows $(a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ of the matrix $M(\Lambda)$.

4.4 Example As in Section 3, suppose $\mathcal{T}$ is an ideal triangulation of a manifold M, and consider its deformation variety $D(\mathcal{T}) \subset \mathbb{C}^{3n}$, where n is the number of tetrahedra in $\mathcal{T}$. If we fix a preferred edge in each tetrahedron, then its shape parameter z_i determines z_i' and z_i'' as noted in Figure 3.1; using these expressions for z_i' and z_i'' turns each edge equation into one of the form (4.1). Thus projecting away the other coordinates gives an injection $D(\mathcal{T}) \hookrightarrow (\mathbb{C}^{*\bullet})^n$, and the image variety V is given by $V(\Lambda)$ for some Λ. The number of edges of $\mathcal{T}$ is equal to n, but if $D(\mathcal{T})$ is non-empty then we argue that the rank of Λ is $n - k$, where k is the number of components of ∂M.

First, the matrix $M(\Lambda)$ minus its last column has rank $r = n - k$; this is Proposition 2.3 of [NZ] when M is hyperbolic, and Theorem 4.1 and remark following it in [Neu] for the general case. Thus Λ has a basis where r of the vectors have nonzero a or b components, and the rest have only the c component being nonzero. Since $D(\mathcal{T})$ is assumed nonempty, all of the latter must correspond to the equation $1 = 1$ rather than $1 = -1$ and hence may be omitted. Thus V is defined by a lattice Λ of rank r. As $r \leq n - 1$, the projection V of $D(\mathcal{T})$ is indeed a variety of gluing equation type.

4.5 Remark F. Rodriguez Villegas pointed out to us that $V(\Lambda)$ is the intersection of a toric variety with an affine subspace. Precisely, if $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ then it is isomorphic to the subspace of $(\mathbb{C}^*)^{2n+1}$ cut out by

(4.6) $z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} w_1^{b_1} w_2^{b_2} \cdots w_n^{b_n} u^c = 1$ for all $(a_1, \ldots, a_n, b_1, \ldots, b_n, c) \in \Lambda$.

together with $u = -1$ and $z_i + w_i = 1$ for $1 \leq i \leq n$. This seems potentially very useful, though we do not exploit it here.

4.7   Ideal points

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^\bullet = \mathbb{C} \setminus \{1\}$. Now in $(\mathbb{C}^\bullet)^n$, consider the closure $\overline{V}(\Lambda)$ of $V(\Lambda)$ in (equivalently) either the Zariski or the analytic (naive) topology. Points of $\overline{V} \setminus V$ will be called *ideal points*. In the context of Example 4.4 and Section 3.6, these

are images of ideal points ξ of $D(\mathcal{T})$ where the preferred shape parameters are either 0 or nondegenerate at ξ. By choosing the shape parameters appropriately, any ideal point of $D(\mathcal{T})$ gives an ideal point of the corresponding $V(\Lambda)$. The individual ideal points of $D(\mathcal{T})$ can be found by analyzing the local structure, typically highly singular, of the ideal points of the $V(\Lambda)$.

So returning to the context of a general $V = V(\Lambda)$, we seek to understand the local structure of $\overline{V}$ near an ideal point p. In particular, we need to find a holomorphic map from the open unit disc $D \subset \mathbb{C}$ of the form

$$(4.8) \qquad f \colon (D, 0) \to (\overline{V}, p) \quad \text{where } f(D \setminus \{0\}) \subset V.$$

Taking t as the parameter on D, we have

$$(4.9) \qquad z_i = t^{d_i} u_i(t)$$

where $d_i \geq 0$ and u_i are holomorphic functions on D with $u_i(0) \neq 0$ for all i, and $u_i(0) \neq 1$ when $d_i = 0$. As always, each u_i can be represented by a convergent power series in $\mathbb{C}[[t]]$.

The lattice Λ constrains the possibilities for $d = (d_1, d_2, \ldots, d_n)$ as follows. Consider the equations coming from a matrix $M(\Lambda)$ as in (4.3), and substitute (4.9) into (4.1). If we send $t \to 0$, it follows that d is in $\ker(A)$. This motivates:

4.10 Definition A *degeneration vector* is a nonzero element $d \in \ker(A) \cap (\mathbb{Z}_{\geq 0})^n$. It is *genuine* if it arises as in (4.9) for some ideal point of $V(\Lambda)$.

4.11 Remark If V comes from $D(\mathcal{T})$ as discussed in Example 4.4, then degeneration vectors correspond precisely to the Q-coordinates of certain spun-normal surfaces as follows. In a tetrahedron with a preferred shape parameter z, we say the *preferred quad* is the one with shift $+1$ along the preferred edge; equivalently, the preferred quad of the tetrahedron labeled as in Figure 3.1 is shown in Figure 2.1(a). Now, in the notation of Section 2, consider the face C' of $C(\mathcal{T})$ where all non-preferred quads have weight zero. The relationship described in Section 3.6 between edge equations and Q-matching equations shows that if we focus on the subspace of preferred quads, the Q-matching equations are simply given by the A part of the $M(\Lambda)$ matrix. Thus degeneration vectors are precisely the integer points of C', and each corresponds to a spun-normal surface. So when d is genuine, it is the Q-coordinates of a spun-normal surface $S(d)$ associated to an ideal point ξ of $D(\mathcal{T})$. (Technical aside: we have not insisted that f in (4.8) is generically $1 - 1$, thus d may be an integer multiple of the vector of the orders of zero of the z at the corresponding ideal point ξ. Hence, $S(d)$ may be some integer multiple of $S(\xi)$.)

Thus the key question for us here is when a given degeneration vector is genuine. The following is the main technical tool from [Kab], and underlies our Theorems 1.1 and 1.5:

4.12 Proposition   *Suppose a degeneration vector d is totally positive, i.e. each $d_i > 0$. If A has rank $n - 1$, then d is genuine.*

We include a detailed proof of this in our current framework, as part of a more general discussion of which degeneration vectors are genuine.

4.13   Genuine degeneration vectors

Fix a degeneration vector d which we wish to test for being genuine. For convenience, we reorder our variables so that $d_i = 0$ for precisely $i \geq k > 1$. Taking our lead from the substitution in (4.9), and arbitrarily folding u_1 into t, we consider

$$\pi \colon \mathbb{C}^n \to \mathbb{C}^n \quad \text{given by} \quad (t, u_2, \ldots, u_n) \mapsto (t^{d_1}, t^{d_2} u_2, \ldots, t^{d_n} u_n)$$

We set $W(\Lambda, d)$ to be the preimage of V under π, regarded as a subvariety of

$$U = \pi^{-1}\big((\mathbb{C}^{* \bullet})^n\big) = (\mathbb{C}^*)^n \setminus \{t^{d_1} = 1, t^{d_i} u_i = 1\}$$

Equivalently, using (4.9), we see $W(\Lambda, d)$ is the subset of U cut out by

$$(4.14) \qquad u_2^{a_2} \cdots u_n^{a_n} (1 - t^{d_1})^{b_1} (1 - t^{d_2} u_2)^{b_2} \cdots (1 - t^{d_n} u_n)^{b_n} = (-1)^c$$

for $(a, b, c) \in \Lambda$. To examine whether d is genuine, we need to allow t to be zero. So consider

$$\overline{U} = \mathbb{C} \times (\mathbb{C}^*)^{n-1} \setminus \{t^{d_1} = 1, t^{d_i} u_i = 1\}$$

and let $\overline{W}(\Lambda, d)$ be the closure of $W(\Lambda, d)$ in $\overline{U}$. Defining

$$\overline{W}_0(\Lambda, d) = \overline{W}(\Lambda, d) \cap \{t = 0\},$$

we have a simple test for when d is genuine:

4.15 Lemma   *If d is genuine, then $\overline{W}_0(\Lambda, d)$ is nonempty. Almost conversely, if $\overline{W}_0(\Lambda, d)$ is nonempty then a positive integer multiple of d is genuine.*

The reader whose focus is on Theorems 1.1 and 1.5 may skip the proof of Lemma 4.15, as the proof of Proposition 4.12 does not depend on it.

Proof First suppose that d is genuine. Consider the analytic functions $u_i(t)$ in (4.9); by replacing t with $t\bigl(u_1(t)\bigr)^{-1/d_1}$, which is analytic near $t = 0$, we may assume $u_1(t)$ is the constant function 1. Now, for small $t \neq 0$ the function

$$t \mapsto \bigl(t, u_2(t), u_3(t), \dots, u_n(t)\bigr)$$

has image contained in $W(\Lambda, d)$. Thus by continuity, the point $\bigl(0, u_2(0), \dots, u_n(0)\bigr)$ is in $\overline{W}_0(\Lambda, d)$, as needed.

Now suppose instead p is a point of $\overline{W}_0(\Lambda, d)$. Dropping Λ and d from the notation, we argue it is enough to show

4.16 Claim *There is an irreducible curve $C \subset \overline{W}$ containing p on which t is nonconstant.*

If the claim holds, let $\widetilde{C}$ be a smooth projective model for C, with $f \colon \widetilde{C} \to C$ the corresponding rational map. If we take s to be a holomorphic parameter on $\widetilde{C}$ which is 0 at some preimage of p, then $\pi \circ f \circ s$ shows that $m \cdot d$ is a genuine degeneration vector, where $m > 0$ is the order of zero of the t-coordinate of f at $s = 0$.

To prove the claim, let Y be an irreducible component of $\overline{W}$ containing p. Since $\overline{W}$ was defined by taking the closure of W in $\overline{U}$, it follows that t is nonconstant on Y. If $j = \dim Y > 1$, we will construct an irreducible subvariety Y' of dimension $j-1$ which contains p and on which t is nonconstant. Repeating this inductively will produce the needed curve C.

As Y is irreducible, and $Y_0 = Y \cap \{t = 0\}$ is a nonempty proper algebraic subset, it follows that $\dim Y_0 = j - 1$. There are coefficients $\alpha_i \in \mathbb{C}$ so that the polynomial

$$g = \alpha_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n$$

is nonconstant on *every* irreducible component of Y_0, and where $g(p) = 0$. (If we temporarily view p as the origin of our coordinate system, then any linear functional whose kernel fails to contain the linear envelope of any component of Y_0 works for g.) Now set $Y' = Y \cap \{g = 0\}$, which contains p and has dimension $j - 1$ as g is nonconstant on Y. Moreover $Y' \cap \{t = 0\} = Y_0 \cap \{g = 0\}$ has dimension $j - 2$ as g is nonconstant on every component of Y_0. Thus an irreducible component of Y' containing p has dimension $j - 1$ and t is nonconstant on it, as needed.   □

4.17   The first-order system

Suppose that $\beta = (0, \beta_2, \beta_3, \ldots, \beta_n)$ is a point of $\overline{W}(\Lambda, d)$. Substituting $t = 0$ into (4.14) we get that β satisfies

$$(4.18) \qquad \beta_2^{a_2} \cdots \beta_n^{a_n} (1 - \beta_k)^{b_k} \cdots (1 - \beta_n)^{b_n} = (-1)^c \quad \text{for all } (a; b; c) \in \Lambda.$$

We call the union of all such equations, together with $t = 0$, the *first-order system*, and denote the corresponding subset of $\{0\} \times (\mathbb{C}^*)^{k-2} \times (\mathbb{C}^{*\bullet})^{n-k+1}$ by $W_0(\Lambda, d)$. Notice that $W_0(\Lambda, d)$ contains $\overline{W}_0(\Lambda, d)$, but is not a priori equal to it, as the latter may contain points which are not in the closure of $W(\Lambda, d)$. As the former is easier to work with in practice, we show

4.19 Lemma   *Suppose $W_0(\Lambda, d)$ is nonempty and has dimension 0. Then some multiple of d is genuine.*

As we discuss later, in small examples this condition is easy to check using Gröbner bases. As with Lemma 4.15, on which it depends, it is not actually used to prove Proposition 4.12.

Proof   Consider the subvariety $\widetilde{W}$ of $\overline{U}$ cut out by the equations (4.14) coming from the r rows of a fixed matrix $M(\Lambda)$ defining our original variety $V(\Lambda)$. Then $\widetilde{W}$ contains both $W_0(\Lambda, d)$ and $\overline{W}(\Lambda, d)$. Let p be a point of $W_0(\Lambda, d)$, and Y an irreducible component of $\widetilde{W}$ containing p. As $\widetilde{W}$ is defined by $r \leq n - 1$ equations and $\dim \overline{U} = n$, the variety Y must have dimension at least 1. As $Y \cap \{t = 0\}$ is contained in the finite set $W_0(\Lambda, d)$, it follows that all but finitely many points of Y are in $W(\Lambda, d)$. Hence $p \in \overline{W}(\Lambda, d)$, and Lemma 4.15 implies that a multiple of d is genuine. $\qquad\square$

We now have the needed framework to show Proposition 4.12.

Proof of Proposition 4.12   Let d be a totally positive degeneration vector. By hypothesis, the submatrix A of $M(\Lambda)$ has rank $n - 1$, and so in particular $M(\Lambda)$ has $n - 1$ rows. We reorder the variables so that the matrix A' gotten by deleting the first column of A also has rank $n - 1$.

To show d is genuine, we start by examining the solutions $W_0(\Lambda, d)$ to the first-order equations. As all $d_i > 0$, these equations are simply $t = 0$ and

$$(4.20) \qquad \beta_2^{a_2} \beta_3^{a_3} \cdots \beta_n^{a_n} = (-1)^c \quad \text{for all } (a; b; c) \in \Lambda.$$

where we require each $\beta_i \in \mathbb{C}^*$. Note that any solution in $\mathbb{C}^{n-1}$ to the linear equations

$$(4.21) \qquad a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = c\pi i \quad \text{for all } (a; b; c) \in \Lambda$$

gives rise to one of (4.20) via the map $\mathbb{C}^{n-1} \to (\mathbb{C}^*)^{n-1}$ which exponentiates each coordinate. Since $\operatorname{rank}(A') = n - 1$, the equations (4.21) have a solution and hence so do (4.20).

We will use the inverse function theorem to show that d is genuine. To set this up, let $\widetilde{W}$ be the subvariety of $\mathbb{C}^n$ with coordinates $(t, u_2, u_3, \ldots, u_n)$ cut out by the $n - 1$ equations (4.14) coming from rows of the matrix $M(\Lambda)$. Fix a point $\beta \in W_0(\Lambda, d) \subset \widetilde{W}$, and let J be the $(n - 1) \times n$ Jacobian matrix of these equations at β. Let J' be the submatrix of J gotten by deleting the first column (which corresponds to $\partial/\partial t$). If J' has rank $n - 1$, then the inverse function theorem implies that $\widetilde{W}$ is a smooth curve at β. Moreover, this curve is transverse to $\{t = 0\}$ since $\operatorname{rank}(J') = n - 1$ forces any nonzero element of $\ker(J) = T_p\widetilde{W}$ to have nonzero first component.

Thus it remains to calculate J'. As all $d_i > 0$, taking $\partial/\partial u_i$ of (4.14) at β gives $a_i(-1)^c/\beta_i$. Thus the columns of J' are nonzero multiples of those of A', and hence $\operatorname{rank}(J') = \operatorname{rank}(A') = n - 1$ as needed. Thus d is genuine. $\qquad\square$

4.22 Examples

Both hypotheses of Proposition 4.12 are independently necessary, even for the weaker conclusion that the first-order equations have a solution. Here are two examples with $V(\Lambda) \neq 0$ which illustrate this.

First, for $n = 2$ consider the span Λ of $(0, 1;\ 1, -1;\ 1)$; here, $V(\Lambda)$ is given by a single equation

$$(4.23) \qquad \frac{z_2(1 - z_1)}{1 - z_2} = -1$$

which defines the nonempty plane conic $z_1 z_2 = 1$. For the degeneration vector $d = (1, 0)$, the first-order system is

$$\frac{\beta_2}{1 - \beta_2} = -1,$$

which is equivalent to $0 = -1$ and hence has no solutions. So d is not genuine, even though $A = (0, 1)$ has maximal rank. This shows the total positivity of d is necessary for Proposition 4.12.

Second, again for $n = 2$, consider the span Λ of $(0, 0;\ 1, 1;\ -1)$. Then $V(\Lambda)$ is again a nonempty plane conic, and is given by

$$(4.24) \qquad\qquad (1 - z_1)(1 - z_2) = -1.$$

Here, any d is a degeneration vector since $A = (0, 0)$, so take $d = (1, 2)$. Then the first-order system is simply $1 = -1$ which has no solutions. So d is not genuine, even though d is totally positive. This shows that the condition that $\mathrm{rank}(A)$ is maximal is also necessary for Proposition 4.12.

5 Proof of Theorem 1.1

In this section, we prove

1.1 Theorem *Let $\mathcal{T}$ be an ideal triangulation of a compact oriented 3-manifold M with ∂M a torus. Suppose S is a vertex spun-normal surface in $\mathcal{T}$ with non-trivial boundary. If S has a quad in every tetrahedron of $\mathcal{T}$, then S is essential.*

The requirement that ∂M is a single torus, rather than several, is simply for notational convenience; the proof works whenever S has at least one non-trivial boundary component.

We first rephrase Theorem 1.1 in the form in which we will prove it. Throughout this section, let $\mathcal{T}$ be an ideal triangulation as in Theorem 1.1. Recall from Section 2 if $\mathcal{T}$ has n tetrahedra, then the Q-coordinates of a spun-normal surface are given by a vector in $\mathbb{R}^{3n}$ that lives in the linear subspace $L(\mathcal{T})$ of solutions to the Q-matching equations. Specifically, each spun-normal surface gives an integer vector in the convex cone $C(\mathcal{T}) = L(\mathcal{T}) \cap \mathbb{R}_+^{3n}$.

Suppose we fix a preferred type of quad in each tetrahedron; such a choice will be denoted by Q. Let $\mathbb{R}_Q^n \subset \mathbb{R}^{3n}$ be the corresponding subspace where all non-preferred quads have weight 0. Define $L(\mathcal{T}, Q) = L(\mathcal{T}) \cap \mathbb{R}_Q^n$ and $C(\mathcal{T}, Q) = C(\mathcal{T}) \cap \mathbb{R}_Q^n$. We will show the following:

5.1 Theorem *Suppose S is a spun-normal surface with non-empty boundary which has a quad in every tetrahedron. Let Q be the corresponding quad type. If $\dim L(\mathcal{T}, Q) = 1$, then S is essential.*

Proof of Theorem 1.1 from Theorem 5.1 Let S be a vertex spun-normal surface which has a quad in every tetrahedron; we need to show that $\dim L(\mathcal{T}, Q) = 1$. Since $C(\mathcal{T}) = L(\mathcal{T}) \cap \mathbb{R}_+^{3n}$, a face (of any dimension) of $C(\mathcal{T})$ corresponds to setting some subset of the coordinates to 0. Thus since S is a vertex solution, there are coordinates u_i so that $C(\mathcal{T}) \cap \{u_1 = \cdots = u_k = 0\}$ is the ray $\mathbb{R}_+ \cdot S$. Let Q be the unique quad type compatible with S. As S has nonzero weight on every quad in Q, we must have

$$(5.2) \qquad \mathbb{R}_+ \cdot S = C(\mathcal{T}) \cap \{u_i = 0 \mid u_i \notin Q\} = C(\mathcal{T}) \cap \mathbb{R}_Q^n = C(\mathcal{T}, Q)$$

Next we argue that $C(\mathcal{T}, Q) = L(\mathcal{T}, Q) \cap \mathbb{R}_+^{3n}$ has the same dimension as $L(\mathcal{T}, Q)$ itself. This follows since all $\mathbb{R}_Q^n$-coordinates of S are positive, and thus all nearby points to S in $L(\mathcal{T}, Q)$ are also in $C(\mathcal{T}, Q)$. Thus $\dim C(\mathcal{T}, Q) = \dim L(\mathcal{T}, Q)$. As $\dim C(\mathcal{T}, Q) = 1$ by (5.2), the fact that $\dim C(\mathcal{T}, Q) = \dim L(\mathcal{T}, Q)$ shows that the hypotheses of Theorem 1.1 imply those of Theorem 5.1. (In fact, the hypotheses of the two theorems are equivalent.) $\qquad\square$

We break the proof of Theorem 5.1 into two lemmas.

5.3 Lemma *Suppose S is a spun-normal surface with a quad in every tetrahedron. Suppose that $\dim L(\mathcal{T}, Q) = 1$ for the quad type Q determined by S. Then there is an ideal point ξ of $D(\mathcal{T})$ so that S is associated to ξ.*

5.4 Lemma *Suppose S is a connected, two-sided, spun-normal surface with a quad in every tetrahedron. Suppose that $\dim L(\mathcal{T}, Q) = 1$ for the quad type Q determined by S. If every reduction of S has nonempty boundary, then S is essential.*

We establish these lemmas below after first deriving the theorem from them.

Proof of Theorem 5.1 First, we reduce to the case that S is two-sided and connected. Let S_0 be the spun-normal surface corresponding to the primitive lattice point on the ray $\mathbb{R}_+ \cdot S$, i.e. $S_0 = (1/g)S$ where g is the gcd of the coordinates of S. The surface S_0 must be connected, since if not it would be the sum of two surfaces in $C(\mathcal{T}, Q)$ which is just $\mathbb{R}_+ \cdot S$ since $\dim L(\mathcal{T}, Q) = 1$. Now, the surface S is essential if and only if S_0 is, so we shift focus to S_0. If S_0 is one-sided, then by definition S_0 is essential if and only if $2 \cdot S_0$ is, and we focus on the latter (which is still connected). Thus we have reduced to the case that S is connected and two-sided.

Now by Lemma 5.3, there is an ideal point ξ of $D(\mathcal{T})$ so that S is associated to ξ. By Theorem 3.8, the surface S can be reduced to a nonempty essential surface S' with nonempty boundary. By Lemma 5.4, the surface $S = S'$ and S is essential, as required. $\qquad\square$

Proof of Lemma 5.3 In each tetrahedron Δ of $\mathcal{T}$, focus on the edge which has shift $+1$ with respect to the quad that is in S. By Example 4.4, if we focus solely on the corresponding shape parameters, this expresses the deformation variety $D(\mathcal{T})$ as a variety $V(\Lambda)$ of gluing equation type. Moreover, as discussed in Remark 4.11, the degeneration vectors d of $V(\Lambda)$ correspond precisely to the spun-normal surfaces in $C(\mathcal{T}, Q)$. Indeed, the Q-matching equations cutting out $L(\mathcal{T}, Q)$ from $\mathbb{R}_Q^n$ are equivalent to those given by the A submatrix of $M(\Lambda)$.

Let d be the degeneration vector corresponding to the surface S. Since $\dim L(\mathcal{T}, Q) = 1$, by the connection above we must have $\operatorname{rank}(A) = n - 1$. Thus by Proposition 4.12, the degeneration vector d is genuine, and so by Remark 4.11 the surface S is associated to some ideal point ξ of $D(\mathcal{T})$, as needed. $\qquad\square$

Before proving Lemma 5.4, we sketch the basic idea, which was suggested to us by Saul Schleimer and Eric Sedgwick. If S compresses, do so once to yield a surface S' which is disjoint from S. Now normalize S' to S''; while this may result in additional compressions, the surface S'' is nonempty by hypothesis. The original normal surface S acts as a barrier during the normalization of S' [Rub], and so S'' is disjoint from S. Thus the quads in S'' are compatible with those of S. Now as $\dim L(\mathcal{T}, Q) = 1$, we must have that $S = S''$ and so the initial compression was trivial and hence S is essential.

If S was an ordinary (non-spun) normal surface, this sketch would essentially be a complete proof. Unfortunately, the spun-normal case introduces some additional technicalities, particularly as we are not assuming that M is hyperbolic, and hence we can't appeal directly to [Wal] to ensure that S' can be normalized at all.

Proof of Lemma 5.4 As in Section 2.3, we pick a neighborhood $N_{\mathfrak{v}}$ of the vertex $\mathfrak{v}$ of $\mathcal{T}$ so that S meets the torus $B_{\mathfrak{v}} = \partial N_{\mathfrak{v}}$ in nonseparating curves with consistent canonical orientations. We now identify M with $\mathcal{T} \setminus \operatorname{int}(N_{\mathfrak{v}})$. Except for the very end of the proof, we will focus on $S \cap M$ and so denote it simply by S.

If S is not essential as the lemma claims, there are three possibilities:

(a) S has a genuine compressing disc D.

(b) S is incompressible but has a genuine ∂-compression D.

(c) S is boundary parallel.

Case (c) is ruled out since S can be reduced to an essential surface. In case (b), consider the arc $\alpha = D \cap \partial M$. If the end points of α are on the same component of ∂S, then
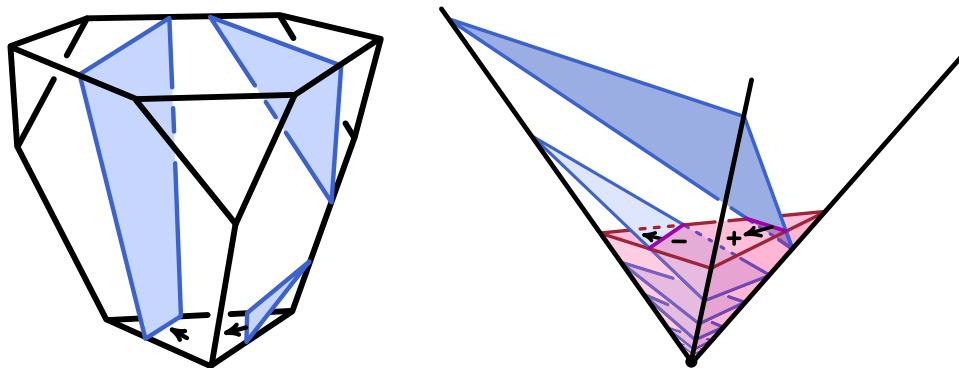
Figure 5.5: Some normal discs in a truncated tetrahedron, with a choice of orientation for the arcs meeting ∂M. Comparing with the copy of Figure 2.4 at right, we see the quad can be spun but not the lower triangle.

the incompressibility of S forces D to be a trivial ∂-compression. When instead α joins two components of ∂M, incompressibility means that the connected surface S is an annulus. But then compressing S along D gives a disc S' whose boundary is inessential in ∂M. This contradicts that that every reduction of S yields a surface with nonempty boundary.

Thus it remains to rule out (a). Now let D be a compressing disc for S. Compress S along D and slightly isotope the result to yield a surface S_1 disjoint from S. Now further compress and otherwise reduce S_1 in the complement of S to give a surface S_2 which is disjoint from S and essential in its complement. By hypothesis, S_2 has non-empty boundary. If S_2 is not connected, replace it by any connected component with non-empty boundary.

Now M has a cell structure $\overline{\mathcal{T}}$ coming from $\mathcal{T}$ consisting of truncated tetrahedra, and note that S is normal with respect to $\overline{\mathcal{T}}$. Our goal is to normalize S_2 in $\overline{\mathcal{T}}$ and then spin the result into a spun-normal surface. However, not every normal surface in $(M, \overline{\mathcal{T}})$ can be spun. The boundary curves need an orientation which satisfies the condition in Section 1.12 of [Til1], and that orientation must be compatible with the "tilt" of the normal discs (see Figure 5.5). To finesse this issue, we isotope S_2 in the complement of S so that ∂S_2 consists of normal curves, each of which lies just to the positive side of a parallel curve in ∂S.

Now normalize S_2 with respect to $\overline{\mathcal{T}}$ to yield a surface S_3 (see e.g. [Mat, Ch. 3]). As mentioned above, this normalization takes place in the complement of S. A concise
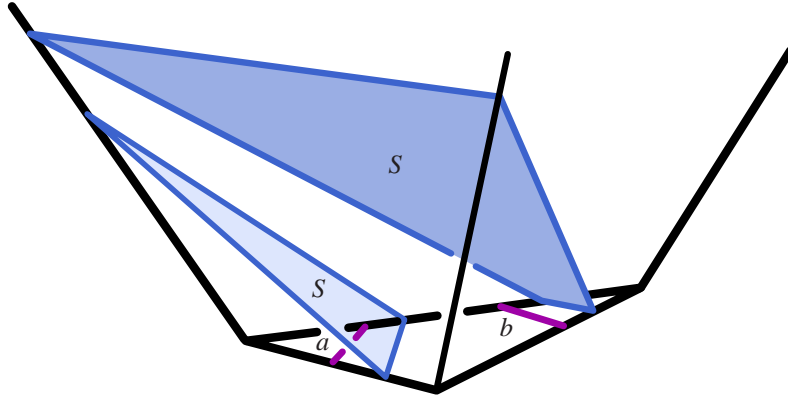
Figure 5.6: There are at most two kinds of normal arcs in ∂S_3, labeled here a and b. From the their position relative to the surface S, any normal disc of S_3 adjacent to ∂S_3 must be parallel to those in S.

way of seeing this is to cut M open along S to yield M' with a cell structure $\overline{T}'$. If we normalize S_2 in M' with respect to $\overline{T}'$, the result is necessarily normal with respect to $\overline{T}$. Moreover the final surface is still disjoint from the two copies of S in $\partial M'$ since normalizing never increases the number of intersections of the surface with an edge.

The normalization process may result in compressions or other reductions to the surface. However, since S_2 is essential in M', it follows that S_3 has the same topology as S_2. (If M' is irreducible, then S_2 and S_3 are of course isotopic.) Focus on a component of $\partial M \cap M'$, which is an annulus A. The components of ∂S_2 in A are all normally isotopic, and moreover intersect any 2-dimensional face of $\overline{T}'$ at most once (see the right half of Figure 5.5). Thus the first ∂-compression that occurs while normalizing S_2 must join two distinct boundary components, reducing the total number of boundary components. As S_2 and S_3 have the same number of boundary components, there can be no ∂-compressions during normalization and so ∂S_2 and ∂S_3 are setwise the same.

Focus now on one normal arc α in ∂S_3. By construction, it lies just to the positive side of a normal arc α' of ∂S. If $\mathfrak{n}$ is a normal disc of S_3 with α as an edge, from Figure 5.6 we see that $\mathfrak{n}$ must be parallel to the normal disc of S along α'. Hence, we can build a spun-normal surface S' from S_3 which is disjoint from S by attaching half-open annuli in $N_\mathfrak{v}$ which are combinatorially parallel to those of $N_\mathfrak{v} \cap S$.

Now S and S' are disjoint spun-normal surfaces in T and hence they have compatible quad types. Thus S' is in $L(T, Q)$. We know that both S and S' are nonempty, two-sided, connected, and not vertex linking tori. Hence as $L(T, Q)$ is one dimensional,

the Q-coordinates of S and S' must be the same. Hence they are normally isotopic and so they have the same topology. This contradicts that we started with a genuine compression of S, ruling out (a). Hence S is essential. $\qquad\square$

6 Slopes of alternating knots

In this section, we prove Theorem 1.3 by showing that the alternating knot $10_{79} = 10a78$ has nonintegral boundary slopes, namely $10/3$ and $-10/3$. Additional nonintegral slopes of alternating knots are given in Table 6.1. Let M denote the complement of 10_{79}; as M is amphichiral, we simply show that $10/3$ is a boundary slope.

To apply Theorem 1.1, we need to specify an ideal triangulation $\mathcal{T}$ with a spun-normal surface S and check:

(a) The ideal triangulation $\mathcal{T}$ is homeomorphic to the complement of 10_{79} and the peripheral basis that comes with $\mathcal{T}$ is the standard homological one.

(b) The surface S is a vertex surface with a quad in every tetrahedron. In the reformulation of Theorem 5.1, the former is equivalent to $\dim L(\mathcal{T}, Q) = 1$, where Q is the quad type determined by S.

(c) The boundary slope of S is $10/3$, which can be done as described in Section 3.7.

The triangulation $\mathcal{T}$ we use has 14 tetrahedra and is given by the file "10_79-certificate.tri" available at [DG]. The surface S has the same quad type in each tetrahedron, namely the one disjoint from the edges 01 and 23 in Figure 3.1, which also corresponds to the shape degeneration $z \to 0$. The number of quads is given by

$$S = (2, 3, 3, 3, 2, 5, 2, 1, 4, 1, 3, 1, 3, 3) \in \mathbb{R}_Q^{14}$$

Now (a) above is easily checked using SnapPy [CDW]. The information needed for (b-c) comes directly from the A part of the matrix $M(\Lambda)$ describing the gluing equations for $\mathcal{T}$ together with the corresponding part of the cusp equations. Explicitly, using SnapPy within Sage [SAGE] the following suffices to confirm Theorem 1.3:

```
sage: from snappy import *
sage: M = Manifold("10_79-certificate.tri")
sage: N = Manifold("10_79")
sage: M.is_isometric_to(N, return_isometries=True)[1]
0 -> 0
[1 0]
[0 1]
Extends to link
```

10a8:	$-20/3$	11a275:	$-20/3$	12a120:	$-52/3$
10a78:	$-10/3, 10/3$	11a281:	$-28/3$	12a125:	$-10/3, -2/3, 2/3$
10a95:	$4/3$	11a284:	$-2/3$	12a126:	$-2/3$
11a17:	$-2/3$	11a296:	$34/3$	12a127:	$-22/3$
11a19:	$-2/3$	11a299:	$-4/3$	12a132:	$-2/3$
11a25:	$-2/3$	11a300:	$-28/3, 34/3$	12a134:	$2/3$
11a38:	$10/3$	11a301:	$-22/3, 34/3$	12a154:	$-20/3$
11a49:	$-28/3$	11a313:	$-20/3$	12a155:	$4/3$
11a102:	$-16/3$	11a314:	$-2/3$	12a162:	$-20/3$
11a113:	$2/3$	11a320:	$-46/3, -22/3$	12a177:	$10/3, 22/3$
11a125:	$-34/3$	11a321:	$20/3$	12a186:	$34/3$
11a127:	$-40/3$	11a323:	$26/3$	12a188:	$2/3$
11a129:	$40/3$	11a326:	$22/3, 28/3$	12a211:	$2/3$
11a130:	$-34/3$	11a329:	$-4/3$	12a222:	$-23/2$
11a136:	$-34/3$	11a345:	$-10/3$	12a223:	$-10/3$
11a147:	$-34/3$	11a349:	$34/3$	12a224:	$-27/2$
11a151:	$34/3$	12a45:	$26/3$	12a233:	$-23/2$
11a152:	$-40/3$	12a46:	$7/2, 22/3$	12a264:	$-40/3$
11a156:	$-4/3$	12a52:	$-52/5$	12a267:	$-28/3$
11a157:	$40/3$	12a53:	$-32/3$	12a276:	$-20/3$
11a158:	$28/3$	12a57:	$16/3$	12a284:	$4/3$
11a162:	$34/3$	12a59:	$-8/3$	12a292:	$22/3$
11a164:	$34/3$	12a63:	$-8/3$	12a293:	$-14/3$
11a168:	$-10/3$	12a65:	$26/3$	12a294:	$34/3$
11a169:	$22/3$	12a70:	$34/3, 46/3$	12a296:	$-8/3$
11a171:	$-34/3$	12a72:	$34/3, 46/3$	12a301:	$-22/3$
11a217:	$40/3$	12a88:	$-52/3, -8/3, 68/5$	12a309:	$22/3$
11a218:	$2/3$	12a89:	$34/3$	12a311:	$-76/3$
11a227:	$-2/3$	12a91:	$-58/3, -8/3$	12a315:	$-4/5, 4/3$
11a233:	$28/3$	12a93:	$-48/5$	12a316:	$36/5, 28/3$
11a239:	$22/3$	12a94:	$-32/3$	12a317:	$10/3$
11a244:	$-2/3$	12a100:	$7/2$	12a318:	$46/5, 34/3$
11a249:	$-20/3$	12a101:	$78/5$	12a319:	$-13/2$
11a251:	$-4/3$	12a102:	$-32/3, 28/5$	12a320:	$-52/3, -4/3$
11a253:	$-4/3$	12a105:	$-8/3$	12a321:	$-40/3, -26/3, -8/3$
11a255:	$34/3$	12a107:	$-32/3, 28/5$	12a334:	$16/3$
11a256:	$-40/3, -20/3$	12a108:	$-52/3$	12a337:	$22/3$
11a272:	$-10/3$	12a109:	$7/2$	12a339:	$-40/3, -16/3$
11a273:	$22/3$	12a111:	$-58/3, -8/3$	12a340:	$-64/3$
11a274:	$-28/3, 34/3$	12a115:	$78/5$	12a344:	$-52/3, -27/2$

Table 6.1: Some nonintegral boundary slopes of alternating knots, numbered as in [HT]; the first three are $10_{80}, 10_{79},$ and 10_{106} in the standard table [Rol]. These were proven to exist by Theorem 1.1 using triangulations with 14–23 tetrahedra.

```
sage: data = M.gluing_equations(form="rect")
sage: gluing_data, cusp_data = data[:-2], data[-2:]
sage: A = matrix( [e[0] for e in gluing_data] )
sage: B = matrix( [e[1] for e in gluing_data] )
sage: c = matrix( [ [e[2]] for e in gluing_data] )
sage: cusp_holonomy_A_part = matrix( [e[0] for e in cusp_data] )
sage: L = A.right_kernel(); L
Free module of degree 14 and rank 1 over Integer Ring
Echelon basis matrix:
[2 3 3 3 2 5 2 1 4 1 3 1 3 3]
sage: S = L.basis()[0]
sage: cusp_holonomy_A_part * S
(-3, 10)
```

7 Dehn filling

We turn to the case of a 3-manifold W where ∂W consists of several tori $T_0, T_1, \ldots, T_b$. For $k > 0$, we pick a slope γ_k on T_k. If we fix an ideal triangulation $\mathcal{T}$ of W, we can consider all spun-normal surfaces S whose boundary slope on T_k is either γ_k or $\emptyset$. Equivalently, we consider surfaces S where the geometric intersection of γ_k with $S \cap T_k$ is 0. By the discussion in Section 3.7, for each k this requirement imposes an additional linear condition on the cone $C(\mathcal{T})$ of spun-normal surfaces. We call the resulting subcone $C(\mathcal{T}, \{\gamma_k\})$ the *relative normal surface space corresponding to* $(\,\cdot\,, \gamma_1, \ldots, \gamma_b)$. This section is devoted to:

1.5 Theorem *Let W be a compact oriented 3-manifold where ∂W consists of tori $T_0, T_1, \ldots, T_b$. Let S be a spun-normal surface in an ideal triangulation $\mathcal{T}$ of W, with nonempty boundary slope γ_k on each T_k. Suppose that S has a quadrilateral in every tetrahedra of $\mathcal{T}$, and is a vertex surface of $C(\mathcal{T}, \{\gamma_1, \ldots, \gamma_b\})$. Then γ_0 is a boundary slope of $W(\cdot, \gamma_1, \ldots, \gamma_b)$.*

Proof We consider the relative gluing equation variety $D(\mathcal{T}, \{\gamma_k\})$ obtained from adding the b conditions that the holonomy $h(\gamma_k)$ of each γ_k is 1. For the Dehn filled manifold $M = W(\,\cdot\,, \gamma_1, \ldots, \gamma_b)$, the relative variety $D(\mathcal{T}, \{\gamma_k\})$ is closely related to the character variety $\overline{X}(M)$. However, while every point in $D(\mathcal{T}, \{\gamma_k\})$ gives a representation $\rho \colon \pi_1(W) \to \mathrm{PSL}_2\mathbb{C}$, these representations do not all factor through $\pi_1(M)$; the condition $h(\gamma_k) = 1$ only gives that $\rho(\gamma_k)$ is trivial or parabolic. However, $\rho(\gamma_k)$ can only be nontrivial if $h(\alpha) = 1$ for every element $\alpha \in \pi_1(T_k)$.

As in Section 4.11, we take the preferred shape parameter z in a tetrahedron to be the one where the quad of S has shift $+1$. Then following Section 4.4, we consider the variety $V = V(\Lambda)$ arising from $D(\mathcal{T}, \{\gamma_k\})$ by focusing on the preferred shape parameters. If $\mathcal{T}$ has n tetrahedra, then the rank of Λ is at most $n-1$ since there are $n - b - 1$ equations coming from $D(\mathcal{T})$ (by Section 4.4) and also one equation for each condition $h(\gamma_k) = 1$ (by Section 3.7). Thus V is indeed a variety of the kind studied in Section 4. Just as in Section 4.11, the degeneration vectors for V are precisely the spun-normal surfaces in the relative space $C(\mathcal{T}, \{\gamma_k\})$. Thus we can apply Proposition 4.12 to see that the surface S is associated to an ideal point p of V. Let $f\colon (D, 0) \to (\overline{V}, p)$ be an associated holomorphic map. For each $k \geq 0$, pick a curve α_k on T_k which meets γ_k in one point. Then as γ_k is the boundary slope of S, the function $h(\alpha_k) \circ f$ has a nontrivial pole or zero at 0. In particular, we can restrict the domain D of f so that $h(\alpha_k) \neq 1$ on $f(D \setminus \{0\})$. Then every point in $f(D \setminus \{0\})$ gives rise to a representation of $\pi_1(M)$. Thus we have found an ideal point ξ of $\overline{X}(M)$ where $\mathrm{tr}\,(\rho(\alpha_0))$ has a pole and $\mathrm{tr}\,(\rho(\gamma_0)) = \pm 2$. The essential surface associated to ξ has boundary slope γ_0, as needed. $\qquad\qquad\square$

8   The 2-fusion link

Let W be the complement of the link in Figure 1.7. The manifold W has a hyperbolic structure obtained by gluing two regular ideal octahedra. We consider a certain ideal triangulation $\mathcal{T}$ of W with 8 tetrahedra described in the file "2fusion-certificate.tri" available at [DG]. As in Section 6, we look at surfaces with the same quad type in each tetrahedron, the one which corresponds to the shape degeneration $z \to 0$, and use Q to denote this choice of quads.

One finds that the first part of the matrix $M(\Lambda) = (A|B|c)$ is

$$
A = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 1 & 1 & -1 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 2 & 1 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

which has rank 5. Three vectors which span $\ker A = L(\mathcal{T}, Q)$ are

$$
\begin{aligned}
S_1 &= \big(0, 1, 0, 1, 1, 0, 0, 0\big) \\
S_2 &= \big(0, 2, 0, 0, 0, 1, 0, 2\big) \\
S_3 &= \big(1, 0, 1, 0, 0, 0, 1, 0\big)
\end{aligned}
$$

Thus on $L(\mathcal{T}, Q) = \{a_1 S_1 + a_2 S_2 + a_3 S_3\}$, the condition defining $C(\mathcal{T}, Q)$ that the original variables satisfy $u_k \geq 0$ translates into having each $a_k \geq 0$. Hence $C(\mathcal{T}, Q)$ is simply the positive orthant in $L(\mathcal{T}, Q)$ with respect to the basis $\{S_1, S_2, S_3\}$. So we can identify the projective solution space $P(\mathcal{T}, Q)$ with the triangle spanned by the vertex surfaces S_k.

Now with the peripheral basis curves ordered $(\mu_0, \lambda_0, \mu_1, \lambda_1, \mu_2, \lambda_2)$, the A part of the matrix specifying the cusp equations is

$$
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

and hence the boundary slopes of each of our vertex surfaces is

(8.1)

		T_0	T_1	T_2
∂S_1	:	$2\mu_0 + \lambda_0$	$\emptyset$	μ_2
∂S_2	:	$4\mu_0 + 2\lambda_0$	$-2\mu_1 + 2\lambda_1$	$\emptyset$
∂S_3	:	$-\mu_0$	λ_1	$-\lambda_2$

We will show

8.2 Proposition *The surface $S = a_1 S_1 + a_2 S_2 + a_3 S_3$ for $a_k \in \mathbb{N}$ has non-empty boundary slopes $\gamma_0, \gamma_1, \gamma_2$ on each boundary torus T_k, and γ_0 is a boundary slope of $M = W(\cdot, \gamma_1, \gamma_2)$.*

Proof Since all $a_k > 0$, it is clear from (8.1) that ∂S has nontrivial coefficients along each λ_k and so has non-empty boundary slope γ_k on each T_k. Consider the boundary slope map from the convex hull of the S_k to the space $\mathbb{R}^4 = H_1(T_1; \mathbb{R}) \oplus H_1(T_2; \mathbb{R})$. From (8.1), it is clear this is injective, even if we projectivize the image. Thus, the relative normal surface space $C(\mathcal{T}, \{\gamma_1, \gamma_2\})$ is just the ray generated by S, and so S is a vertex surface for $C(\mathcal{T}, \{\gamma_1, \gamma_2\})$. Hence we can apply Theorem 1.5 to see that γ_0 is a boundary slope of M. $\square$

We now prove Theorem 1.9 by considering the surface

$$
S = 2(m_1 - 1)S_1 + m_2 S_2 + 2(m_1 - 1)m_2 S_3
$$

for some $m_1 > 1$ and $m_2 > 0$. This surface has boundary slopes as follows, written as elements of $\mathbb{Q}$:

$$
\gamma_0 = -\frac{(m_1 - 3)m_2 - 2m_1 + 2}{m_1 + m_2 - 1}, \quad \gamma_1 = -\frac{1}{m_1}, \quad \gamma_2 = -\frac{1}{m_2}
$$

Thus by Proposition 8.2, the slope γ_0 above is a boundary slope for

$$M = W\big(\,\cdot\,, -1/m_1, -1/m_2\big)$$

Now, the manifold M is the exterior of the knot $L(m_1, m_2)$ from Theorem 1.9, but the peripheral basis $\{\mu_0, \lambda_0\}$ is the one that comes from W, and so λ_0 is *not* the homological longitude λ_0' for M. As the components C_1 and C_2 are unlinked, we can adjust for this via

$$\lambda_0' = \big(-m_1 \cdot \mathrm{lk}(C_0, C_1)^2 - m_2 \cdot \mathrm{lk}(C_0, C_2)^2\big)\,\mu_0 + \lambda_0 = (-4m_1 - 9m_2)\mu_0 + \lambda_0$$

and thus find that in the usual homological basis

$$\gamma_0' = (4m_1 + 9m_2) + \gamma_0 = 3(m_1 + 1) + 9m_2 + \frac{(m_1 - 1)^2}{(m_1 + m_2 - 1)}$$

is a boundary slope of $L(m_1, m_2)$, proving Theorem 1.9.

9 Which spun-normal surfaces come from ideal points?

Given an ideal triangulation $\mathcal{T}$ of a 3-manifold M with one torus boundary component, we would like to determine all the boundary slopes that arise from ideal points of $D(\mathcal{T})$. Of course, one can find all such detected slopes by computing the A-polynomial, but this is often a very difficult computation, involving projecting an algebraic variety (i.e. eliminating variables). While Culler has a clever new numerical method for such computations [Cul], there are still 9 crossing knots whose A-polynomials have not been computed.

For a spun-normal surface S with nonempty boundary, we have an effectively checkable condition (Lemma 4.15) which is necessary and sufficient for S be associated to an ideal point of $D(\mathcal{T})$. However, since there are typically infinitely many spun-normal surfaces, this does not allow for the computation of all such detected slopes unless we can restrict to a finite set of surfaces. From the point of view of normal surface theory, two natural finite subsets are:

(a)   The vertex surfaces introduced in Section 2.

(b)   The larger set of fundamental surfaces, which are the integer points in $C(\mathcal{T})$ which are not proper sums of other such points.

However, we show below that neither of these subsets suffices. In fact, there is a geometric triangulation $\mathcal{T}$ of the complement of the knot 6_3 where *none* of the 22 fundamental surfaces is associated with an ideal point of $D(\mathcal{T})$!

Independent of this issue, we've also seen three conditions which ensure that a surface S is associated to an ideal point of $D(\mathcal{T})$:

(a) Kabaya's original criterion, Proposition 4.12, which was used in proving Theorem 1.1. This requires that S is a vertex surface and has a quad in every tetrahedron.

(b) Lemma 4.19 applies when the first-order system $W_0(\Lambda, d)$ has dimension 0, where d is the degeneration vector corresponding to S.

(c) Lemma 4.15 applies when $\overline{W}_0(\Lambda, d)$ is nonempty.

Here, each condition implies the next, and (c) is necessary as well as sufficient. Condition (b) is easier to check than (c) as it only needs the dimension of a variety, which is one of the easiest tasks for Gröbner bases. In contrast, (c) requires eliminating a variable, albeit one that appears only in fairly simple equations, and thus (c) is still much easier than finding the A-polynomial using Gröbner bases. For manifolds with less than 20 tetrahedra, tests (a) and (b) are usually quite feasible for any given surface. However, a naive Gröbner basis approach to applying (c) sometimes failed even for manifolds with less than 10 tetrahedra.

9.1 Experimental Data

There are 173 Montesinos knots with < 11 crossings. As we know the boundary slopes for these [HO, Dun2], we tested the three methods above on each of them, using triangulations with between 2 and 15 tetrahedra. These knots have an average of 6.1 boundary slopes, but the method (a) yields an average of only 1.2 slopes, or about 20% of the total. When (b) is applied to all vertex surfaces, it finds an average of 3.8 slopes, or about 64% of the actual number of slopes. The third test (c) was not practical on enough of these vertex surfaces to give any real data.

When the manifolds were ordered by the size of their triangulations, the number of slopes found by (a) decreased (in absolute and relative terms) as the number of tetrahedra increased. A more marked variant of this pattern was observed in punctured torus bundles; when the triangulations were small there was an average of 1.0 slope found with (a), but when there were 15 tetrahedra the average had dropped to below $<$ 0.1.

Also for punctured torus bundles, method (b) always found exactly two slopes. Henry Segerman pointed out to us that these are the two surfaces corresponding to the edges of the Farey strip in [FH]. This can be deduced from [Seg2] where the solutions to

the tilde equations of Section 8 are closely related to our $W(\Lambda, d)$. Another interesting observation of Segerman is the following simple way that (b) that can fail. If the detected surface S has a tube of quads encircling an edge, then all the edge parameters around it are 1 at the ideal point. Thus the equation (4.18) for that edge is simply $1 = 1$ and so the dimension of $W_0(\Lambda, d)$ will be at least one if it is not empty, and hence (b) will not apply. Of course, such an S has an obvious compression from the tube of quads (which is typically a genuine compression), but for the examples in [Seg2] the spun-normal surfaces associated to ideal points frequently do have such tubes.

9.2 The knot 6_3

We illustrate some of the subtleties of these questions with the complement M of the hyperbolic knot 6_3 in S^3. Using that this is the two-bridge knot $K(5/13)$, one finds that the boundary slopes are: $-6, -2, 0, 2, 6$. (The symmetry here comes from the fact that M is amphichiral.)

From the A-polynomial, we see that the character variety $\overline{X}(M)$ has a single irreducible component (excluding the component of reducible representations). All boundary slopes are strongly detected on $\overline{X}(M)$, with the exception of 0 which com1es from a fibration of M over the circle. We focus on the boundary slope 2, which is associated to a unique ideal point of $\overline{X}(M)$.

From now on, let $\mathcal{T}$ be the canonical triangulation of M as saved in "6_3-canon.tri" available at [DG]. It has 6 tetrahedra, which come in three different shapes.

- Tets 0 and 2 have the same shape, which is an isosceles triangle.
- Tets 1 and 3 have the same shape, which has no symmetries.
- Tets 4 and 5 have the same shape, which is the mirror image of those of tets 1 and 3.

All of this is compatible with the fact that the isometry group of M is the dihedral group with eight elements. It turns out that there are 16 spun-normal vertex surfaces, all of which have non-trivial boundary slopes, and also 4 other fundamental surfaces.

Four vertex surfaces have slope 2, all of which are compatible with a single quad-type

$$Q = [Q03, Q13, Q13, Q03, Q13, Q13]$$

and have weights

$$S_8 = [0, 1, 2, 0, 1, 1] \qquad S_{10} = [2, 1, 0, 0, 1, 1]$$
$$S_9 = [0, 0, 2, 1, 1, 1] \qquad S_{11} = [2, 0, 0, 1, 1, 1]$$

While each of these vertex surfaces has exactly one boundary component, they differ in the direction the surface is spun out to the boundary. The surfaces S_8 and S_9 are spun one way, and S_{10} and S_{11} are spun the other. Additionally, there are two other fundamental surfaces in this component of $F(\mathcal{T})$:

$$S_a = (1/2)\big(S_8 + S_{10}\big) = [1, 1, 1, 0, 1, 1]$$
$$S_b = (1/2)\big(S_9 + S_{11}\big) = [1, 0, 1, 1, 1, 1]$$

Oddity the first: The surfaces S_8 and S_{10} are compatible and each has nonempty boundary, but $S_8 + S_{10}$ is actually a closed surface. In fact, it's the double of S_a which has genus 2 and is just the boundary torus plus a tube linking the edge e_3. This is in stark contrast with the non-spun case, where compatible normal surfaces with boundary always sum to a surface with nonempty boundary. (This is because the two surfaces lie on a common branched surface.) Thus this is a potential problem for proving that all boundary slopes can be determined solely by looking at the spun-normal vertex surfaces.

Oddity the second: None of the 22 fundamental surfaces arise from an ideal point of the gluing equation variety $D(\mathcal{T})$. For instance, for the surfaces with slope 2, chose the preferred edge parameters so that $z_i \to 0$ corresponds to the quad in Q. Then the gluing equations include $z_1 = z_3$ and $z_4 = z_5$; the former is not compatible with any of the above fundamental surfaces. Instead, after some work it turns out that Lemma 4.15 shows that the surfaces

(9.3)
$$S = S_8 + S_b = S_9 + S_a = [1, 1, 3, 1, 2, 2]$$
$$S' = S_{10} + S_b = S_{11} + S_a = [3, 1, 1, 1, 2, 2]$$

are associated to the two ideal points of $D(\mathcal{T})$ which detect the slope $+2$. (These two ideal points map to the same ideal point of $\overline{X}(M)$, and differ in the direction the associated surfaces are spun out toward the boundary.)

A posteriori, the failure of the fundamental surfaces to appear at ideal points of $D(\mathcal{T})$ is not so surprising given the large symmetry group G of $\mathcal{T}$. The four vertex surfaces above are the vertices of a tetrahedron Δ in the projectivized space $PF(\mathcal{T})$ of embedded spun-normal surfaces. The subgroup H of G which preserves Δ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and acts transitively on the vertices of Δ by orientation preserving symmetries. Now $D(\mathcal{T})$ has two ideal points with slope 2 and there is a unique non-trivial element g of H which fixes both; this g acts by $S_8 \leftrightarrow S_9$ and $S_{10} \leftrightarrow S_{11}$ (i.e. interchanges the pairs of surfaces that spin in the same direction). Thus a surface associated with either ideal point must lie on the line segment joining $(1/2)(S_8 + S_9)$ to $(1/2)(S_{10} + S_{11})$ and hence can not be a fundamental surface. However, since we also know that g

interchanges S_a and S_b, we can see that the surfaces S and S' in (9.3) will indeed be fixed by g.

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