# ASYMPTOTICS OF $q$-DIFFERENCE EQUATIONS

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Abstract. In this paper we develop an asymptotic analysis for formal and actual solutions of $q$-difference equations, under a regularity assumption, namely the non-collision and non-vanishing of the eigenvalues. In particular, evaluations of regular solutions of regular $q$-difference equations have an exponential growth rate which can be computed from the $q$-difference equation.

The motivation for the paper comes from a problem in Quantum Topology, the Hyperbolic Volume Conjecture, which states that a sequence on Laurent polynomials (the so-called colored Jones function of a knot), appropriately evaluated, becomes a sequence of complex numbers that grows exponentially. Moreover, the exponential growth rate is proportional to the volume of the knot complement.

The connection of the Hyperbolic Volume Conjecture with the paper comes from the fact that the colored Jones function of a knot is a solution of a $q$-difference equation, as was proven by TTQ. Le and the author.

Contents



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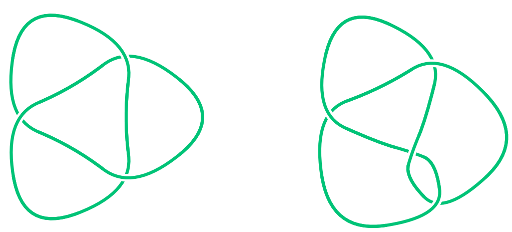
## 1. Introduction

1.1. **The goal.** The goal of the paper is to intiate an approach to the Hyperbolic Volume Conjecture, via asymptotics of solutions of difference equations with a small parameter. The Generalized Volume Conjecture links (conjecturally) the (colored) Jones polynomial of a knot to hyperbolic geometry of its complement.

Since the colored Jones polynomial is a specific solution to a linear $q$-difference equation, it follows that the generalized volume conjecture is the WKB limit of a specific solution of a linear difference equation with a small parameter.

Motivated by this, we study WKB asymptotics of formal and actual solutions of difference equations with a small parameter, under certain regularity asymptions.

1.2. **The colored Jones function.** A knot in 3-space is a smooth embedding of a circle, considered up to isotopy. Two of the simplest knots, the Trefoil ($3_1$) and the Figure Eight ($4_1$) are shown here:



By the very definition, knots are flexible objects defined up to isotopy, which allows the embedding to move in a smooth and arbitrary way as long as it does not cross itself. In algebraic topology, a common way of studying knots (and more generally, spaces) is to associate computable numerical invariants (such as Euler characteristic, or Homology). Invariants are useful in deciding whether two knots are not the same. It is a much harder problem to construct computable invariants that separate knots.

The invariant that we will consider in this paper is the Jones polynomial of a knot; [J], which is a Laurent polynomial with integer coefficients, associated to each knot. The quantum nature of the Jones polynomial is apparent both in the original definition of Jones (using Temperley-Lieb algebras) and in the reformulation, due to Witten, in terms of the expectation value of a Quantum Field Theory; see [J, Wt].

The combinatorics associated to a planar projection of a knot show that the Jones polynomial is a computable invariant. However, it is hard to see from this point of view the relation between the Jones polynomial and Geometry. In Quantum Field Theory, one often reproduces Geometry by moving carefully chosen parameters of the theory to an appropriate limit.

In our case, we will introduce a new parameter, a natural number which roughly speaking corresponds to taking a connected $n$-parallel of a knot. The resulting invariant is no longer a Laurent polynomial, but rather a sequence of Laurent polynomials.

The *colored Jones function* of a knot $K$ in 3-space is a sequence of Laurent polynomials

$$J_K : \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}].$$

The first term in the above sequence, $J_K(1)$ is the Jones polynomial of $K$; see [GL1].

1.3. **The Hyperbolic Volume Conjecture.** Although knots are flexible objects, Thurston had the idea that their complements have a unique decomposition in pieces of unique "crystaline" shape. The shapes in question are the 8 different geometries in dimension 3, and the idea in question was termed the "Geometrization Conjecture". The most common of the 8 geometries is Hyperbolic Geometry, that is the existence of a complete, finite volume, constant curvature $-1$ Riemannian metric on knot complements. Thurston proved that unless the knot is torus or a satellite, then it carries a unique such metric; see [Th].

The *Hyperbolic Volume Conjecture* (HVC, in short) connects two very different views of knot: namely Quantum Field Theory and Riemannian Geometry. The HVC states for every hyperbolic knot $K$

$$\lim_{n\to\infty} \frac{\log |J_K(n)(e^{\frac{2\pi i}{n}})|}{n} = \frac{1}{2\pi} \operatorname{vol}(S^3 - K).$$

where $\operatorname{vol}(S^3 - K)$ is the *volume* of a complete hyperbolic metric in the knot complement $S^3 - K$. The conjecture was formulated in this form by Murakami-Murakami [MM] following an earlier version due to Kashaev, [K]. More generally, Gukov (see [Gu]) formulated a Generalized Hyperbolic Volume Conjecture that identifies the limit

$$\lim_{n\to\infty} \frac{\log |J_K(n)(e^{\frac{2\pi i\alpha}{n}})|}{n}$$

of a hyperbolic knot with known hyperbolic invariants (such as the volume of cone manifolds obtained by hyperbolic Dehn filling), for $\alpha \in (0,1] - \mathbb{Q}$ or $\alpha = 1$. Actually, the GHVC is stated for *complex* numbers $\alpha$. For simplicity, we will study asympotics for real $\alpha \in [0,1]$.

At present, it is not known whether the limit in the HVC exists, let alone that it can be computed. Explicit finite multisum formulas for the colored Jones function of a knot exist; see for example [GL1]. From these formulas alone, it is difficult to study the above limit. In a sense, the question is to understand the sequence of Laurent polynomials that appears in the HVC. If the sequence is in some sense random, then it is hard to expect that the limit exists, or that it can be computed.

Since the first term of this sequence is the Jones polynomial, and since we know little about the possible values of the Jones polynomial, one would expect that there is even less to be said about the colored Jones function.

1.4. ***q*-difference equations.** Luckily, the colored Jones function behaves in a better way than its first term, namely the Jones polynomial. This can be quantified by recent work of TTQ Le and the first author, who proved that the colored Jones function of a knot satisfies a $q$-difference equation.

In other words, for every knot $K$ there exist rational functions $b_1(u,v), \dots, b_d(u,v) \in \mathbb{Q}(u,v)$ (which of course depend on $K$) such that for all $n \in \mathbb{N}$ we have:

$$\sum_{j=0}^{d} b_j(q^n, q) J_K(n+j) = 0.$$

This opens the possibility of studying the $q$-difference equation rather than one of its solutions, namely the colored Jones function. Although the $q$-difference equation is not unique, it was shown by the first author in [Ga1] that one can choose a unique $q$-difference equation, which is a knot invariant. Moreover, it was conjectured in [Ga1] that the characteristic polynomial of this $q$-difference equation determines the characters of $\mathsf{SL}_2(\mathbb{C})$ representations of the knot complement, viewed from the boundary.

As was explained by the first author on several occasions, asymptotics of solutions of $q$-difference equations would have consequences on the HVC.

In this introductory article we review the history of asymptotics of solutions of $q$-difference equations.

1.5. **Asymptotics of differential equations with a parameter.** Excellent references for differential equations with a parameter are Olver's and Wasau's books; [O] and [Wa]. In 1837, Liouville and Green independently studied systematically existence of formal (i.e., perturbative) and actual solutions for second order differential equations with a parameter; see [Gr, L]. Second order equations are very important for classical and quantum physics.

In 1908 Birkhoff had the insight to introduce and study *arbitrary* order differential equation with a parameter (see [B1]):

$$y^{(n)} + \rho a_{n-1}(x,\rho) y^{(n-1)} + \dots + \rho^n a_0(x,\rho) y = 0 \tag{1}$$

where $y = y(x,\rho)$, $y^{(n)}$ means $n$-th derivative with respect to $x$ (assumed to be restricted to a real interval), and $\rho$ is a large complex parameter, and where the coefficient $a_j(x,\rho)$ are complex $C^\infty$ functions with an expansion

$$a_j(x,\rho) = \sum_{s=0}^{\infty} a_{j,s}(x)\rho^{-s}$$

Birkhoff's working assumption was that the eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ of the characteristic equation

$$\lambda^n + a_{n-1,0}(x)\lambda^{n-1} + \dots + a_{0,0}(x) = 0$$

were distinct but not necessarily nowhere vanishing.

In 1926, three theoretical physicists, Wentzel-Krammer-Brillouin studied the second order differential equation (1) under the assumption that its eigenvalues do not collide, and developed connection formulas linking solutions in the exponential region with those in the oscillatory region. Their method is often referred to as the WKB method.

1.6. **Asymptotics of difference equations.** As a motivation for our results, let us recall some fundamental results of Birkhoff and Trjitzinsky from 1930 on difference equations without a parameter; see [B2] and [BT].

A *difference equation* for a discrete function $f : \mathbb{N} \to \mathbb{C}$ has the form:

$$\sum_{j=0}^{d} a_j(n) f(n+j) = 0 \tag{2}$$

where $a_j : \mathbb{N} \longrightarrow \mathbb{C}$ are discrete functions so that $a_0(n)a_d(n) \neq 0$ for all $n \in \mathbb{N}$. We will assume the existence of asymptotic expansions of $a_j(n)$ around $n \to \infty$ for all $j = 1, \dots, d$:

$$a_j(n) \sim_{n\to\infty} n^{d_j/\omega}(a_{j,0} + a_{j,1}n^{-1/\omega} + a_{j,2}n^{-2/\omega} + \dots)$$

where $\omega \in \mathbb{N}$. This certainly holds for $\omega = 1$ if $a_j$ are rational functions of $n$, as is often the case in combinatorial problems.

Due to the nowhere vanishing of $a_d \cdot a_0$, it follows that the set of solutions of (2) is a vector space of dimension $d$.

There are two main problems of difference equations:

- Existence of *formal series solutions* $\tilde{\psi}_1, \dots, \tilde{\psi}_d$ to (2).
- Existence of a basis $\{\psi_1, \dots, \psi_d\}$ of solutions so that $\psi_k(n)$ is asymptotic to $\tilde{\psi}_k(n)$ for large $n$.

In [B2], Birkhoff solved the existence of formal solutions in complete generality (that is, without any assumptions on the eigenvalues of the characteristic equation). In a sequel paper [BT], Birkhoff-Trjitzinsky solved the second problem in complete generality.

Among other things, the formal solutions of Birkhoff lead to the development of differential Galois theory, see [vPS].

Decades later, the results of Birkhoff and Trjitzinsky on difference equations have found applications to enumerative combinatorics and numerical analysis; see for example Wimp and Zeilberger in [Wi, WZ] and references therein. It is not surprising that difference equations are used in numerical analysis, since difference equations are numerical schemes of approximating differential equations. In enumerative combinatorics and complexity theory, difference equations appear in recursive computation. For example, the number $f(n)$ of involutions of $\{1, 2, \dots, n\}$ (that is, permutations which are a product of 1 and 2-cycles) is given by

$$f(n+2) = f(n+1) + (n+1)f(n)$$

with $f(1) = 1$, $f(2) = 2$. Using the results of Birkhoff-Trjitzinsky and the fact that $f(n)$ is monotone, it follows that

$$f(n) \sim_{n\to\infty} K n^{n/2} e^{-n/2+n^{1/2}} \left(1 + \frac{c_1}{n^{1/2}} + \frac{c_2}{n} + \frac{c_3}{n^{3/2}} + \dots\right)$$

for nonzero constants $c_i$ and some $K > 0$. Actually, the $c_k$ can be computed recursively from the difference equation; see [WZ, p.169].

1.7. **Asymptotics of difference equations with a parameter.** By some historical accident, asymptotics of solutions of difference equations with a parameter was not discussed a century ago. The first paper that discusses second order difference equation with a parameter appears to be due to Deift-McLaughlin (see [DM]) which was generalized by Costin-Costin to arbitrary order difference equations, [CC].

The purpose in this paper is to show that for regular $q$-difference equations, a regular solution has a well-defined and computable exponential growth rate in terms of a relative entropy of the characteristic polynomial of the $q$-difference equation; see Theorem 1 below.

This subject is classical and has been reinvented over the past hundred years by several groups, often unaware of each others results. In a sense, the problem of formal solutions of $q$-difference equations is a problem in differential Galois theory; [vPS], and a problem in numerical analysis; see for example [CC].

Our results are hardly new and are contained or can be obtained by minor modifications from results of Costin-Costin or from work of Birkhoff and collaborators, [B1, BT, CC].

Since the presentation in the above papers varies by time and taste, we have decided to give a self-contained account of the theory with complete proofs. Hopefully, this will benefit the researchers in Quantum Topology and in Analysis.

1.8. **Statement of the results.** In this paper, we will describe asymptotics of solutions of $q$-difference equations.

A *$q$-difference equation* for a sequence $(f(1), f(2), f(3), \dots)$ of smooth functions of $q$ has the form:

$$\sum_{j=0}^{d} b_j(q^k, q) f(k+j, q) = 0 \tag{3}$$

where $b_j(v, u)$ are smooth functions and $f(k, q) = f(k)(q)$.

Before we proceed further, let us remark that $q$ is a *variable* in (3) and not a complex number of absolute value less (or more) than 1. In the usual analytic theory of $q$-difference equations, $q$ is a complex number inside or outside the unit disk.

Moreover, in the GHVC, we need to compute the $n$th term $f(n, q)$ in the above $q$-difference equation, and then evaluate it at $q_n = e^{2\pi i\alpha/n}$, for $\alpha$ fixed. In other words, in the GHVC, $q_n$ is a complex number that varies with $n$ in such a way that it stays in the unit circle and approaches 1 as $n \to \infty$.

With this in mind, $\epsilon$-difference equations (defined below) are obtained from $q$-difference equations by the substitution $q = e^{2\pi i\epsilon}$ where $\epsilon$ is a small nonnegative real number, that plays the role of Planck's constant.

The *characteristic polynomial* of the $q$-difference equation (3) is

$$P(v, \lambda) = \sum_{j=0}^{d} b_j(v, 1)\lambda^j$$

**Definition 1.1.** We will say that (3) is *regular* if

$$\mathrm{Dsc}_\lambda P(v, \lambda) \cdot b_0(v, 1) \cdot b_d(v, 1) \neq 0$$

for all $v \in S^1$, where $\mathrm{Dsc}_\lambda P(v, \lambda)$ is the *discriminant* of $P(v, \lambda)$, which is a polynomial in the coefficients of $P(v, \lambda)$.

Let $\lambda_1(v), \dots, \lambda_d(v)$ denote the roots of the characteristic polynomial, which we call the *eigenvalues* of (3). It turns out that (3) is regular iff the eigenvalues $\lambda_1(v), \dots, \lambda_d(v)$ never collide and never vanish, for every $v \in S^1$. Moreover, it follows by the implicit function theorem that the roots are smooth functions of $v \in S^1$.

Since we are interested in asymptotics of solutions of $q$-difference equations which, as we shall see, are governed by the magnitude of the eigenvalues, we need to partition the circle according to the magnitudes of the eigenvalues.

Let $S^1 = \cup_{p \in \mathcal{P}} I_p$ denote a partition of $S^1$ into a finite union of closed arcs (with nonoverlapping interiors), such that the magnitude of the eigenvalues does not change in each arc. In other words, for each $p \in \mathcal{P}$, there is a permutation $\sigma_p$ of the set $\{1, \dots, d\}$ such that

$$|\lambda_{\sigma_p(1)}(v)| \geq |\lambda_{\sigma_p(2)}(v)| \geq \dots \geq |\lambda_{\sigma_p(d)}(v)| \qquad \text{for all} \qquad v \in I_p.$$

The following definition introduces a locally fundamental set of solutions of $q$-difference equations.

**Definition 1.2.** Fix a partition of $I$ as above. A set $\{\psi_1, \dots, \psi_d\}$ is a *locally fundamental set of solutions* of (3) iff for every solution $\psi$ for every $p \in \mathcal{P}$ and for every $m = 1, \dots, d$ there exist smooth functions $c_m^p$ such that

$$\psi(k,q) = c_1^p(q)\psi_{\sigma_p(1)}(k,q) + \dots + c_d^p(q)\psi_{\sigma_p(d)}(k,q) \qquad \text{for all} \quad (k,q): \quad q^k \in I_p. \tag{4}$$

**Theorem 1.** *Assume that* (3) *is regular. Then, there exists a locally fundamental set of solutions* $\{\psi_1, \dots, \psi_d\}$ *such that*

- *For every* $m = 1, \dots, d$ *and* $(k,q)$ *such that* $q^k \in I_p$ *we have*

$$\psi_m\left(k, e^{\frac{2\pi i \alpha}{n}}\right) = \exp\left(\frac{n}{\alpha}\Phi_m\left(\frac{k\alpha}{n}, \frac{\alpha}{n}\right)\right).$$

- *for some smooth functions* $\Phi_m$ *with uniform (with respect to* $x \in I = [0,1]$*)* asymptotic expansion

$$\Phi_m(x,\epsilon) \sim_{\epsilon \to 0} \sum_{s=0}^{\infty} \phi_{m,s}(x)\epsilon^s$$

  *where* $\phi_{m,s} \in C^\infty(I)$ *for all* $s$
- *and leading term*

$$\phi_{m,0}(x) = \int_0^x \log(\lambda_m(e^{2\pi i t}))dt \tag{5}$$

  *where we have chosen a branch for the logarithm of* $\lambda_m$.

*Remark* 1.3. For every $j = 1, \dots, d$ the smooth functions $\phi_{j,s}$ for positive $s$ are uniquely determined from the coefficients $b_j(u,v)$ of (3) by a hierarchy of first-order differential equations along with specified initial conditions. On the other hand, the smooth functions $\Phi_m$ are not uniquely determined, since they are obtained by a smooth interpolation. Thus, the locally fundamental set of solutions is not uniquely determined from the $q$-difference equation, although its asymptotic behavior is.

It follows from Theorem 1 that each locally fundamental solution $\psi_m(n,q)$ of the $q$-difference equation (3) satisfies the GHVC in the sense that for every $\alpha \in [0,1]$ we have:

$$\lim_{n \to \infty} \frac{\log|\psi_m(n, e^{\frac{2\pi i \alpha}{n}})|}{n} = \int_0^1 \log|\lambda_m(e^{2\pi i \alpha t})|dt$$

Fix a solution $\psi$ of (3). Theorem 1 expresses $\psi$ as a linear combination of $\psi_m$'s in each arc $I_p$. For every $p \in P$, let

$$S_p = \{m \in \{1, \dots, d\} | \; c_m^p \neq 0\}. \tag{6}$$

Later (in Section 6.3) we will define the notion of a regular solution to a $q$-difference equation.

As a prototypical example, consider an $q$-difference equation that satisfies $|\lambda_1(v)| > |\lambda_j(v)|$ for all $j \neq 1$ and all $v \in S^1$. Then, any solution that satisfies $c_1(0) \neq 0$ (or more generally, $c_1$ has a nonvanishing derivative at 0) is regular.

*Remark* 1.4. It is possible that $S_p \neq S_{p+1}$. In other words, the restriction of a fixed solution $\psi$ to different intervals $I_p$ may be a linear combination of different $\psi_j$s. This is an important phenomenon, referred by the name of *Stokes phenomenon*; see [Wa].

Our next definition captures the growth rate of regular solutions to regular $q$-difference equations.

**Definition 1.5.** Fix a collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$. The *$S$-entropy*

$$\sigma_S : [0,1] \to \mathbb{R}$$

of the $q$-difference equation (3) is defined by

$$\sigma_S(\alpha) = \int_0^1 \log \chi_S(e^{2\pi i \alpha t}) dt.$$

where $\chi_S : [0,1] \to \mathbb{R}$ is defined by

$$\chi_S(v) = \max_{j \in S_p} |\lambda_{\sigma_p(j)}(v)| \qquad \text{if} \qquad v \in I_p.$$

The *entropy* of (3) is the set of functions

$$\{\sigma_S : [0,1] \to \mathbb{R} \quad | \quad S \subset \{1, \dots, d\}\}.$$

Notice that the entropy of a $q$-difference equation is not a real number, but rather a finite collection of functions.

The main result is the following

**Theorem 2.** *If $f$ is an $S$-regular solution of the regular $q$-difference equation* (3)*, then for every $\alpha \in [0,1]$ we have:*

$$\lim_{n \to \infty} \frac{\log |f(n)(e^{\frac{2\pi i \alpha}{n}})|}{n} = \sigma_S(\alpha)$$

Finally, let us define the $J$-entropy of a knot. In [Ga1] the first author showed that to every knot $K$ one can associate a canonical $q$-difference equation of degree $d$, and a specific solution of it, namely the colored Jones function of $K$.

The $q$-difference equation itself is an invariant of a knot, which (by definition) is determined by the colored Jones function of the knot. Thus, any invariant of the $q$-difference equation is also an invariant of a knot, which is determined by the colored Jones function of the knot.

**Definition 1.6.** The $J$-entropy of a knot is the entropy of its associated $q$-difference equation. We denote the $J$-entropy of a knot $K$ by

$$\{\sigma^J_{S,K} : [0,1] \to \mathbb{R} \quad | \quad S \subset \{1, \dots, d\}\}.$$

1.9. **What's next?** The paper was written in the spring of 2004. Since then, a number of papers that discuss the asymptotics of the colored Jones function have appeared; see [Ga2, Ga3, GL2, GL3].

1.10. **Acknowledgement.** The results of this paper were announced in the JAMI 2003 meeting in Johns Hopkins. The first author wishes to thank J. Morava for the invitation, and P. Deligne who suggested the asymptotic behavior of solutions of $q$-difference equations. The first author wishes to thank D. Boyd for sharing and explaining his unpublished work and also A. Riese, T. Morley, and D. Zeilberger.

## 2. $\epsilon$-DIFFERENCE EQUATIONS

2.1. **$\epsilon$-difference equations.** In this section, we will translate asymptotics of solutions of $q$-difference equation in terms of asymptotics of solutions of *$\epsilon$-difference equations*. The latter are defined as follows.

Fix a positive number $\epsilon_0$, a compact interval $I$ of $\mathbb{R}$ and a natural number $d$. We will consider functions $\phi : \Delta_{\epsilon_0, I} \to \mathbb{C}$ with domain

$$\Delta_{\epsilon_0, I} := \{(k\epsilon, \epsilon) | \quad k \in \mathbb{N}, \epsilon \in (0, \epsilon_0], \quad k\epsilon \in I\}. \tag{7}$$

Consider the $\epsilon$-difference equation for a function $\phi : \Delta_{\epsilon_0, I} \to \mathbb{C}$

$$\sum_{j=0}^d a_j(k\epsilon, \epsilon) \phi((k+j)\epsilon, \epsilon) = 0 \tag{8}$$

where $a_j \in C^\infty(I \times [0, \epsilon_0])$.

We will assume that for all $j = 0, \dots, d$, $a_j(x, \epsilon)$ has a uniformly (with respect to $x$) asymptotic expansion

$$a_j(x, \epsilon) \sim_{\epsilon \to 0} \sum_{s=0}^{\infty} a_{j,s}(x) \epsilon^s \tag{9}$$

where $a_{j,s} \in C^\infty(I)$.

As we mentioned before, $\epsilon$-difference equations are obtained from $q$-difference equations by the substitution $q = e^{2\pi i \epsilon}$ where $\epsilon$ is a small nonnegative real number, that plays the role of Planck's constant.

The *characteristic polynomial* of (8) is

$$P(x, \lambda) = \sum_{j=0}^{d} a_j(x, 0) \lambda^j$$

**Definition 2.1.** We will say that (8) *regular* if

$$\mathrm{Dsc}_\lambda P(x, \lambda) \cdot a_0(x, 0) \cdot a_d(x, 0) \neq 0$$

for all $x \in I$.

Let $\lambda_1(x), \dots, \lambda_d(x)$ denote the roots of the characteristic polynomial, which we call the *eigenvalues* of (8).

It turns out that (8) is regular iff the eigenvalues $\lambda_1(x), \dots, \lambda_d(x)$ never collide and never vanish, for every $x \in I$. Moreover, it follows by the implicit function theorem that the roots are smooth functions of $x \in I$.

Since we are interested in asymptotics of solutions of $\epsilon$-difference equations which, as we shall see, are governed by the magnitude of the eigenvalues, we need to partition the interval $I$ according to the magnitudes of the eigenvalues.

Let $I = \cup_{p \in \mathcal{P}} I_p$ denote a partition of $I$ into a finite union of closed intervals (with nonoverlapping interiors), such that the magnitude of the eigenvalues does not change in each interval. In other words, for each $p \in \mathcal{P}$, there is a permutation $\sigma_p$ of the set $\{1, \dots, d\}$ such that

$$|\lambda_{\sigma_p(1)}(x)| \geq |\lambda_{\sigma_p(2)}(x)| \geq \dots \geq |\lambda_{\sigma_p(d)}(x)| \qquad \text{for all} \qquad x \in I_p.$$

The following definition introduces a locally fundamental set of solutions of $\epsilon$-difference equations.

**Definition 2.2.** Fix a partition of $I$ as above. A set $\{\psi_1, \dots, \psi_d\}$ is a *locally fundamental set of solutions* of (8) iff for every solution $\psi : \Delta_{\epsilon, I} \to \mathbb{C}$, for every $p \in \mathcal{P}$ and for every $m = 1, \dots, d$ there exist smooth functions $c_m^p \in C^\infty[0, \epsilon]$ such that

$$\psi(k\epsilon, \epsilon) = c_1^p(\epsilon) \psi_{\sigma_p(1)}(k\epsilon, \epsilon) + \dots + c_d^p(\epsilon) \psi_{\sigma_p(d)}(k\epsilon, \epsilon) \qquad \text{for all} \qquad (k\epsilon, \epsilon) \in \Delta_{\epsilon, I}.$$

Here, the notation $c_m^p$ does not indicate the $p$th power of $c_m$.

The next theorem summarizes the results of Costin-Costin.

**Theorem 3.** *([CC]) Assume that (8) is regular. Then, there exists a positive $\epsilon' \leq \epsilon_0$ and a locally fundamental set of solutions $\{\psi_1, \dots, \psi_d\}$ of (8) such that*

- *For every $m = 1, \dots, d$ and $(k\epsilon, \epsilon) \in \Delta_{e', I}$ we have*

$$\psi_m(k\epsilon, \epsilon) = \exp\left(\epsilon^{-1} \Phi_m(k\epsilon, \epsilon)\right).$$

- *for some smooth functions $\Phi_m \in C^\infty(I \times [0, \epsilon'])$ with uniform (with respect to $x \in I$)* asymptotic expansion

$$\Phi_m(x, \epsilon) \sim_{\epsilon \to 0} \sum_{s=0}^{\infty} \phi_{m,s}(x) \epsilon^s \tag{10}$$

  *where $\phi_{m,s} \in C^\infty(I)$ for all $s$*
- *and leading term*

$$\exp(\phi'_{m,0}(x)) = \lambda_m(x). \tag{11}$$

Fix a solution $\psi$ of (8). Theorem 3 expresses $\psi$ as a linear combination of the $\psi_j$'s in each interval $I_p$. For every $p \in P$, let

$$S_p = \{m \in \{1, \dots, d\}|\ c_m^p \neq 0\}. \tag{12}$$

Later (in Section 6.2) we will define the notion of a regular solution to an $\epsilon$-difference equation.

As a prototypical example, consider an $\epsilon$-difference equation that satisfies $|\lambda_1(x)| > |\lambda_j(x)|$ for all $j \neq 1$ and all $x \in I = [a, b]$. Then, any solution that satisfies $c_1(a) \neq 0$ (or more generally, $c_1$ has a nonvanishing derivative at $a$) is regular.

Our next definition captures the growth rate of regular solutions to regular $\epsilon$-difference equations.

**Definition 2.3.** Fix a collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$. The $S$-*entropy*

$$\sigma_S : I \to \mathbb{R}$$

of the $\epsilon$-difference equation (8) is defined by

$$\sigma_S(x) = \int_0^x \log \chi_S(t) dt.$$

where $\chi_S : I \to \mathbb{R}$ is defined by

$$\chi_S(x) = \max_{j \in S_p} |\lambda_{\sigma_p(j)}(x)| \qquad \text{if} \qquad x \in I_p.$$

**Theorem 4.** *If $\psi$ is an $S$-regular solution to a regular $\epsilon$-difference equation, and $x \in I$, we have:*

$$\lim_{\epsilon \to 0^+} \epsilon \log |\psi(x, \epsilon)| = \sigma_S(x)$$

The next remarks concern the uniqueness of a set of locally fundamental solutions to (8).

*Remark* 2.4. For every $m = 1, \dots, d$ the smooth functions $\phi_{m,s}$ for positive $s$ are uniquely determined by (3) and the initial condition $\phi_{m,s}(0) = 0$. Indeed, applying Taylor series (with respect to $\epsilon$) in (8) and collecting terms, we get for example:

$$\begin{aligned} \phi'_{m,1}(x) &= -\frac{1/2 \phi''_{m,0}(x) \sum_{j=0}^d a_j(x, 0) j^2 \lambda_m^j(x) + \sum_{j=0}^d \partial_\epsilon a_j(x, 0) \lambda_m^j(x)}{\sum_{j=0}^d a_j(x, 0) j \lambda_m^j(x)} \\ &= -\left( \frac{1}{2} \lambda' \frac{P_{\lambda\lambda}}{P_\lambda} + \frac{P_\epsilon}{P_\lambda \lambda} \right) |_{\lambda = \lambda_m(x)} \end{aligned}$$

where $f_x(x, \lambda) = \partial/\partial_x f(x, \lambda)$ and $f_\lambda(x, \lambda) = \partial/\partial_\lambda f(x, \lambda)$.

Similarly, for $s \geq 1$ we have:

$$\phi'_{m,s}(x) = -\frac{H_s(x)}{\sum_{j=0}^d a_j(x, 0) j \lambda_m^j(x)}$$

where $H_s(x)$ is a function of derivatives of $a_j(x, 0)$ and $\phi_{m,t}$ for $t < s$. Notice that the denominator vanishes nowhere since the roots do not collide and do not vanish for every $x \in I$.

*Remark* 2.5. If the coefficients $a_j(x, \epsilon)$ of the regular $\epsilon$-difference equation (8) are analytic functions, then the functions $\phi_{m,s}$ of Theorem 3 are also analytic, for every $m$ and $s$. This follows by induction from the differential hierarchy which these functions satisfy, and from the fact that the eigenvalues are analytic functions. Even though $\phi_{m,s}$ is analytic for every $m$ and $s$, the series

$$\sum_{s=0}^\infty \phi_{m,s}(x) \epsilon^s$$

is in general divergent, and the functions $\Phi_{m,s}$ of Theorem 3 are not analytic.

*Remark* 2.6. Even though the functions $\phi_{m,s}$ are uniquely determined by the $\epsilon$-difference equation, the smooth functions $F_m$ (and thus the locally fundamental set of solutions $\psi_m$) are not uniquely determined by the $\epsilon$-difference equation. The problem is that smooth interpolation is not unique. Recently developed ideas of exponentially small corrections might construct a unique set of locally fundamental solutions when the coefficients of (8) are analytic functions. We will elaborate on this in a separate occasion.

2.2. **Converting $q$-difference equations to $\epsilon$-difference equations.** The translation of $q$-difference equations to $\epsilon$-difference equations is as follows. If $f$ satisfies the $q$-difference equation

$$\sum_{j=0}^{d} b_j(q^k, q) f(k+j, q) = 0$$

then set

$$q = e^{2\pi i\epsilon}, \quad b_j(e^{2\pi i x}, e^{2\pi i\epsilon}) = a_j(x, \epsilon)$$

and consider the $\epsilon$-difference equation for a function $\phi$ (with domain $\Delta_{\epsilon_0, I}$ for some $\epsilon_0 > 0$ and $I = [0, 2\pi]$):

$$\sum_{j=0}^{d} a_j(k\epsilon, \epsilon)\phi((k+j)\epsilon, \epsilon) = 0$$

The following lemma, although elementary, is the key to translating $q$-difference equations to $\epsilon$-difference equations.

**Lemma 2.7.** *For every $(k\epsilon, \epsilon) \in I \times [0, \epsilon_0]$ we have:*

$$\phi(k\epsilon, \epsilon) = f(k, e^{2\pi i\epsilon}). \tag{13}$$

*Consequently, for every $\alpha \in (0, 1]$, we have:*

$$\lim_{k\to\infty} \frac{1}{k} \log |f(k, e^{2\pi i\alpha/k})| = \alpha^{-1} \lim_{\epsilon\to 0} \epsilon \log |\phi(\alpha, \epsilon)|. \tag{14}$$

*Thus, Theorem 3 implies Theorem 1.*

*Proof.* Observe that $a_j(k\epsilon, \epsilon) = b_j(e^{2\pi ik\epsilon}, e^{2\pi i\epsilon})$, thus $(k, \epsilon) \to \phi(k\epsilon, \epsilon)$ satisfies the equation

$$\sum_{j=0}^{d} b_j(e^{2\pi ik\epsilon}, e^{2\pi i\epsilon})\phi((k+j)\epsilon, \epsilon) = 0$$

and so does $(k, \epsilon) \to f(k, e^{2\pi i\epsilon})$. Since solutions with matching initial conditions are unique, (13) follows.

Equation (14) follows from equation (13) by the substitution $\epsilon = \alpha/k$:

$$\frac{1}{k} \log |f(k, e^{2\pi i\alpha/k})| = \frac{1}{k} \log |\phi(k\epsilon, \alpha/k)| = \alpha^{-1}\epsilon \log |\phi(\alpha, \epsilon)|.$$

□

## 3. Some linear algebra

In this section we will review some linear algebra. It is obvious that the complex roots of a monic polynomial uniquely determine it. It is also known [GLR] that the eigenvalues of a companion matrix uniquely determine it, in case they are distinct.

Consider a *companion* $d$ by $d$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{d-1} \end{pmatrix}$$

The characteristic polynomial of $A$ is

$$\lambda^d + \sum_{j=0}^{d-1} a_j \lambda^j$$

with roots $\lambda_1, \dots, \lambda_d$. Let $M = (\lambda_j^{i-1})_{i,j}$ be the Vandermonde matrix, and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ be the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_d$.

**Lemma 3.1.** *If a companion matrix has distinct eigenvalues, then with the above notation we have:*

$$A = MDM^{-1}$$

*Proof.* Observe that $v_j = (1, \lambda_j, \dots, \lambda_j^{d-1})^T$ is an eigenvector of $A$ with eigenvalue $\lambda_j$. Thus, $M = (v_1, \dots, v_d)$ and $AM = MD$. The result follows. $\square$

Now, consider a companion matrix $A(u)$ whose entries in the bottom row are smooth functions in a variable $u$, with roots $\lambda_1(u), \dots, \lambda_d(u)$ which we assume are distinct for all $u$.

The next lemma is a key estimate for the norm of long products of slowly varying matrices. In the language of physics, $A(u)$ is the *transfer matrix* and $A(n) \dots A(2)A(1)$ is the *transition matrix*.

**Lemma 3.2.** *Assume that the eigenvalues $\lambda_1(u), \dots, \lambda_d(u)$ of $A(u)$ are distinct for all $u$ and*

$$\max_j \sup_u |\lambda_j(u)| \leq 1 + C\epsilon.$$

*Then for $m \leq n$, $n\epsilon \in I$, we have*

$$\|A(n)A(n-1)\dots A(m)\| \leq C'$$

*Proof.* By Lemma 3.1, we have:

$$A(u) = M(u)D(u)M(u)^{-1}$$

If $m \leq n$, it follows that

$$A(n)A(n-1)\dots A(m) = M(n)D(n)M(n)^{-1}M(n-1)D(n-1)M(n-1)^{-1}\dots M(m)D(m)M(m)^{-1},$$

which implies that

$$\begin{aligned} \|A(n)A(n-1)\dots A(m)\| &\leq \|M(n)\|\|M(m)^{-1}\| \cdot \|D(n)\| \dots \|D(m)\| \\ &\quad \|M(n)^{-1}M(n-1)\| \dots \|M(m+1)^{-1}M(m)\|. \end{aligned}$$

Now, we have

$$\begin{aligned} \|D(k)\| &\leq 1 + C\epsilon \qquad \text{for} \quad k = m, \dots, n \\ \|M(k)M(k-1)^{-1}\| &\leq 1 + C'\epsilon \qquad \text{by Lemma 3.3.} \end{aligned}$$

If $I = [a, b]$, using the fact that $n\epsilon, m\epsilon \in I$, we obtain:

$$\begin{aligned} \|A(n)A(n-1)\dots A(m)\| &\leq (1 + C\epsilon)^{n-m}(1 + C'\epsilon)^{n-m} \\ &\leq (1 + C''\epsilon)^{2(n-m)} \\ &\leq (1 + C''\epsilon)^{2\frac{b-a}{\epsilon}} \\ &\leq e^{2C''(b-a)}. \end{aligned}$$

$\square$

**Lemma 3.3.** *If $M = (x_j^{i-1})_{i,j}$ and $N = (y_j^{i-1})_{i,j}$ are Vandermonde matrices, such that $M$ is nonsingular, then*

$$(M^{-1}N)_{i,j} = \prod_{l \neq i} \frac{y_j - x_l}{x_i - x_l}.$$

## 4. Existence of formal solutions

In this section we will prove that (8) has a unique set of formal solutions. Let us define those first.

**Definition 4.1.** A *formal series* $\tilde{\psi}(x,\epsilon)$ is one of the form

$$\tilde{\psi}(x,\epsilon) = \exp\left(\epsilon^{-1}\sum_{s=0}^{\infty}\phi_s(x)\epsilon^s\right) \tag{15}$$

where $\phi_s \in C^\infty(I)$ are smooth functions for all $s$.

Note that $\epsilon \log \tilde{\psi}(x,\epsilon)$ lies in the ring $C^\infty(I)[[\epsilon]]$ of formal power series with coefficients smooth functions on $I$.

Note further that if $\tilde{\psi}(x,\epsilon)$ is a formal series, so is $\tilde{\psi}(x+j\epsilon,\epsilon)$ for every $j \in \mathbb{Z}$, where the latter may be defined using the Taylor series of $\phi_s(x+j\epsilon) = \sum_{t=0}^\infty \frac{1}{t!}\phi_s^{(t)}(x)j^t\epsilon^t$:

$$\begin{aligned}\tilde{\psi}(x+j\epsilon,\epsilon) &= \exp\left(\epsilon^{-1}\sum_{s=0}^{\infty}\left(\sum_{t=0}^{s}\frac{1}{t!}\phi_{s-t}^{(t)}(x)\right)\epsilon^s\right)\\ &= \tilde{\psi}(x,\epsilon)\exp\left(\phi_0'(x)j + \left(\phi_1'(x)j + \frac{\phi_0''(x)}{2!}j^2\right)\epsilon + \left(\phi_2'(x)j + \frac{\phi_1''(x)}{2!}j^2 + \frac{\phi_0'''(x)}{3!}j^3\right)\epsilon^2 + \dots\right)\end{aligned}$$

It follows that if $\tilde{\psi}(x,\epsilon)$ is a formal series, then the ratio $\tilde{\psi}(x+\epsilon,\epsilon)/\tilde{\psi}(x,\epsilon)$ lies in the ring $C^\infty(I)[[\epsilon]]$.

Using the language of *difference Galois theory* (see [vPS, p.4]) this implies that

**Lemma 4.2.** *$C^\infty(I)[[\epsilon]]$ is a finite difference ring, under the map $x \to x+\epsilon$.*

**Definition 4.3.** We say that a formal series $\tilde{\psi}$ of (15) is a *formal series solution* to (8) iff

$$\frac{1}{\tilde{\psi}(x,\epsilon)}\sum_{j=0}^{d} a_j(x,\epsilon)\tilde{\psi}(x+j\epsilon,\epsilon) = 0 \in C^\infty(I)[[\epsilon]]. \tag{16}$$

It is easy to see that if $\tilde{\psi}$ is a formal solution to (8), then the leading term $\phi_0$ satisfies the equation

$$\exp(\phi_0'(x)) = \lambda(x) \tag{17}$$

where $\lambda(x)$ is an eigenvalue of (8).

**Proposition 4.4.** *Assume that* (8) *is regular. Then,* (8) *has $d$ unique formal series solutions $\tilde{\psi}_1,\dots,\tilde{\psi}_d$ with leading terms corresponding to the eigenvalues of* (8).

*Proof.* First we need to show that (16) is indeed an equation in the power series ring $C^\infty(I)[[\epsilon]]$, i.e., that the terms involving negative powers of $\epsilon$ cancel.

Suppose that $\tilde{\psi}$ is given by (15). It follows from the calculation preceding Lemma 4.2 that for every $s \in \mathbb{N}$, we have:

$$\operatorname{coeff}\left(\epsilon^s, \frac{\tilde{\psi}(x+j\epsilon,\epsilon)}{\tilde{\psi}(x,\epsilon)}\right) = \begin{cases}\exp(j\phi_0'(x)) & \text{if} \quad s=0\\ j\exp(j\phi_0'(x))\phi_s'(x) + \text{terms}_s & \text{if} \quad s>0\end{cases} \tag{18}$$

where $\operatorname{coeff}(\epsilon^s, g(\epsilon))$ denotes the coefficient of $\epsilon^s$ in a power series $g(\epsilon)$, and where $\text{terms}_s$ is a polynomial in the derivatives of $\phi_t$ for $t<s$.

Expand the terms of Equation (16) into power series in $\epsilon$ using the above equation and (9), and collect terms of powers of $\epsilon$. It follows that (16) is equivalent to a hierarchy of first order differential equations:

$$\begin{aligned}\sum_{j=0}^{d} a_j(x,0)\exp(j\phi_0'(x)) &= 0\\ \sum_{j=0}^{d} a_j(x,0)j\exp(j\phi_0'(x))\phi_s'(x) + \text{Terms}_s &= 0\end{aligned}$$

where for positive $s$, $\mathrm{Terms}_s$ is a polynomial in the derivatives of $\phi_t$ and $a_j(x,0)$ for $t < s$.

Now fix an $m \in \{1,\dots,d\}$, and choose $\phi_{m,0}$ such that $\exp(\phi'_{m,0}(x)) = \lambda_m(x)$, where $\lambda_1(x),\dots,\lambda_d(x)$ are the eigenvalues of (8). Since (8) is regular, it follows that the roots $\lambda_1(x),\dots,\lambda_d(x)$ of the characteristic polynomial $P(x,\lambda)$ never collide, and never vanish for $x \in I$. Thus,

$$0 \neq \left(\lambda \frac{d}{d\lambda} P(x,\lambda)\right)_{\lambda=\lambda_m(x)} = \sum_{j=0}^{d} j a_j(x,0)\lambda_m^j(x)$$

for all $x \in I$. Thus, after we choose $\phi_{m,0}$, it follows that we can find functions $\phi_{m,s}$ for $s \geq 0$ that satisfy the above hierarchy. Moreover, for every $m$, the sequence of functions $\phi_{m,s}$ is uniquely determined by the above hierarchy and the initial conditions $\phi_{m,s}(0) = 0$. $\square$

4.1. **An alternative formal series.** In this section we present an alternative, and slightly more general form, of formal series. In case of regular $\epsilon$-difference equations this alternative form will not be needed. However, when eigenvalues collide or vanish, one must use this alternative form of formal series. Thus, in the present paper we will not use this alternative form of formal series, and the reader may skip this section.

**Definition 4.5.** An *alternative formal series* $\tilde{\psi}(x,\epsilon)$ is one of the form

$$\tilde{\psi}(x,\epsilon) = \exp\left(\epsilon^{-1}\phi(x)\right)\sum_{s=0}^{\infty}\phi_s(x)\epsilon^s \tag{19}$$

where $\phi, \phi_s \in C^\infty(I)$ are smooth functions for all $s$, and $\phi_0(x) \neq 0$ for all $x \in I$.

In the remainder of this subsection, we will refer to alternative formal series simply by formal series.

Note that if $\tilde{\psi}(x,\epsilon)$ is a formal series, so is $\tilde{\psi}(x+j\epsilon,\epsilon)$ for any $j \in \mathbb{Z}$, where the latter may be defined using the Taylor series of $\phi_s(x+j\epsilon)$ and $\phi(x+j\epsilon)$ around $x$. It follows that

$$\tilde{\psi}(x+j\epsilon,\epsilon) \quad = \quad \exp\left(\epsilon^{-1}\phi(x)\right)(\phi_0(x) + (\phi^{(1)}(x)\phi_0(x) + \phi_0^{(1)}(x) + \phi_1(x))\epsilon + \dots).$$

Moreover, if $\tilde{\psi}(x,e)$ is a formal series, then the ratio $\tilde{\psi}(x+\epsilon,\epsilon)/\tilde{\psi}(x,\epsilon)$ lies in the ring $C^\infty(I)[[\epsilon]]$. This follows from (20) and the following computation, valid for every $j \in \mathbb{N}$:

$$\begin{aligned}
\tilde{\psi}(x+j\epsilon,\epsilon) &= \exp(\epsilon^{-1}\phi(x+j\epsilon))(\phi_0(x+j\epsilon) + \phi_1(x+j\epsilon)\epsilon + O(\epsilon^2)) \\
&= \exp\left(\epsilon^{-1}\phi(x) + \phi^{(1)}(x)j + \frac{\phi^{(2)}(x)}{2}j^2\epsilon + O(\epsilon^2)\right)\left(\phi_0(x) + (\phi_0^{(1)}(x)j + \phi_1(x))\epsilon + O(\epsilon^2)\right) \\
&= \tilde{\psi}(x,\epsilon)\exp(\phi^{(1)}(x)j)\left(1 + \left(\frac{\phi^{(2)}(x)}{2}j^2 + \frac{\phi_0^{(1)}(x)}{\phi_0(x)}\right)\epsilon + O(\epsilon^2)\right)
\end{aligned}$$

In analogy with Lemma 4.2, this implies that

**Lemma 4.6.** *$C^\infty(I)[[\epsilon]]$ is a finite difference ring, under the map $x \to x+\epsilon$.*

**Definition 4.7.** We say that a formal series $\tilde{\psi}$ of (19) is a *formal series solution* to (8) iff

$$\frac{1}{\tilde{\psi}(x,\epsilon)}\sum_{j=0}^{d} a_j(x,\epsilon)\tilde{\psi}(x+j\epsilon,\epsilon) = 0 \in C^\infty(I)[[\epsilon]]. \tag{20}$$

It is easy to see that if $\tilde{\psi}$ is a formal solution to (8), then the leading term $\phi$ satisfies the equation

$$\exp(\phi'(x)) = \lambda(x) \tag{21}$$

where $\lambda(x)$ is an eigenvalue of (8).

In analogy with Proposition 4.4, we have the following:

**Proposition 4.8.** *Assume that* (8) *is regular. Then,* (8) *has $d$ unique formal series solutions $\tilde{\psi}_1,\dots,\tilde{\psi}_d$ with leading terms corresponding to the eigenvalues of* (8).

## 5. Proof of Theorem 3

In this section we prove Theorem 3. The strategy is to

(a) prove that there exists a solution $\psi_1$ with the stated properties where $\lambda_1(x)$ is an eigenvalue with maximum magnitude.
(b) use this solution $\psi_1$ to reduce Theorem 3 to the case of a $\epsilon$-difference equation of degree one less than the original one.
(c) prove that the constructed set of solutions is a locally fundamental set.

Without loss of generality, we will assume that the eigenvalues of (8) satisfy the inequality:

$$|\lambda_1(x)| \geq |\lambda_2(x)| \geq \cdots \geq |\lambda_d(x)|$$

for all $x \in I$. Otherwise, we can partition $I$ into subintervals where this is true.

5.1. **Existence of a solution corresponding to the eigenvalue of maximum magnitude.** Consider first a formal solution $\tilde{\psi}_1$ of (8) given in Proposition 4.4, which satisfies (15) and (17). Consider the smooth functions $\phi_{1,s} \in C^\infty(I)$ of (17).

The proof of the following lemma (due to Borel in case $\phi_{1,s}$ are constant functions, for all $s$) can be found in [GG, Lemma 2.5]:

**Lemma 5.1.** *There exists a smooth function $\hat{\Phi}_1 \in C^\infty(I \times [0, \epsilon_0])$ such that we have (uniformly in $x \in I$):*

$$\hat{\Phi}_1(x, \epsilon) \sim_{\epsilon \to 0} \sum_{j=0}^{\infty} \phi_{1,s}(x)\epsilon^s.$$

Now, consider the unique solution $\psi_1$ of (8) with initial conditions

$$\psi_1(k\epsilon, \epsilon) = \exp(\epsilon^{-1}\hat{\Phi}_1(k\epsilon, \epsilon)) \quad \text{for} \quad k = 0, \dots, d-1$$

and for small enough $\epsilon$, where without loss of generality, we assume that $I = [0, b]$.

Of course, for large $k$ it may not be true that $\psi_1(k\epsilon, \epsilon) = \exp(\epsilon^{-1}\hat{\Phi}_1(k\epsilon, \epsilon))$. The next proposition estimates the error, uniformly with respect to $k$:

**Proposition 5.2.** *There exists an $\epsilon' > 0$ and constants $C_s$ such that for all $(k\epsilon, \epsilon) \in \Delta_{\epsilon', I}$ and all $s \in \mathbb{N}$, we have (uniformly in $k$):*

$$\left| \frac{\psi_1(k\epsilon, \epsilon)}{\exp(\epsilon^{-1}\hat{\Phi}_1(k\epsilon, \epsilon))} - 1 \right| < C_s \epsilon^s \tag{22}$$

*Proof.* Let us make a change of variables:

$$\theta = \frac{\psi_1}{\hat{\psi}_1}, \tag{23}$$

where

$$\hat{\psi}_1(x, \epsilon) = \exp\left(\epsilon^{-1}\hat{\Phi}_1(x, \epsilon)\right).$$

We will show that for a fixed $s_0$, and for every $s$ there exists a constant $C_s$ such that for all $(k\epsilon, \epsilon) \in \Delta_{\epsilon_0, I}$ we have:

$$|\theta(k\epsilon, \epsilon) - 1| < C_s \epsilon^{s+1-s_0}, \tag{24}$$

Since $\psi_1$ satisfies (8), it follows that $\theta$ satisfies

$$\sum_{j=0}^{d} b_j(k\epsilon, \epsilon)\theta((k+j)\epsilon, \epsilon) = 0 \tag{25}$$

where

$$b_j(x, \epsilon) = a_j(x, \epsilon)\frac{\hat{\psi}_1(x + j\epsilon, \epsilon)}{\hat{\psi}_1(x, \epsilon)}.$$

It is easy to see that

- $b_j(x,\epsilon) \in C^\infty(I \times [0,\epsilon_0])$, has uniform (with respect to $x$) $\epsilon$-asymptotic expansion as in (9),
- $b_{j,s}(x,0) = a_j(x,0)\lambda_1^j(x)$,
- and since $\tilde{\psi}$ is a formal series solution to (8) and $\hat{\Phi}_1$ is given by Lemma 5.1, if follows that for every $s$ we have:

$$\sum_{j=0}^{d} b_j(x,\epsilon) = O(\epsilon^s) \tag{26}$$

The characteristic polynomial of (25) has roots $\mu_m(x) := \lambda_m(x)/\lambda_1(x)$ for $j = 1,\dots,d$. If (8) is regular, so is (25). Notice that $\lambda_1(x)$ vanishes nowhere since (25) is regular.

We now show (24). Let us write the difference equation (25) in matrix form

$$\Theta((k+1)\epsilon,\epsilon) = A(k\epsilon,\epsilon)\Theta(k\epsilon,\epsilon) \tag{27}$$

where

$$\Theta(x,\epsilon) = \begin{pmatrix} \theta(x,\epsilon) \\ \theta(x+\epsilon,\epsilon) \\ \theta(x+2\epsilon,\epsilon) \\ \dots \\ \dots \\ \theta(x+(d-1)\epsilon,\epsilon) \end{pmatrix} \qquad A(x,\epsilon) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ -c_0(x,\epsilon) & -c_1(x,\epsilon) & -c_2(x,\epsilon) & \dots & -c_{d-1}(x,\epsilon) \end{pmatrix}$$

and $c_j(x,\epsilon) = b_j(x,\epsilon)/b_d(x,\epsilon)$. Iterating, we obtain that

$$\Theta(k\epsilon,\epsilon) = A(k\epsilon,\epsilon)A((k-1)\epsilon,\epsilon)\dots A(\epsilon,\epsilon)\Theta(0,\epsilon)$$

for all $k \geq 1$, where $\Theta(0,\epsilon) = \mathbb{1}$, a column vector with all entries equal to 1.

Equation (26) gives:

$$A(k\epsilon,\epsilon)\mathbb{1} = \mathbb{1} + \epsilon^s \mathbb{E}_k(\epsilon) \tag{28}$$

where $\|\mathbb{E}_k(\epsilon)\| < C$ uniformly in $k$ and $\epsilon$. Feeding in the above equation, we obtain:

$$\Theta(k\epsilon,\epsilon) = \mathbb{1} + \epsilon^s \mathbb{E}_k(\epsilon) + \epsilon^s \sum_{j=1}^{k-1} A(k\epsilon,\epsilon)A((k-1)\epsilon,\epsilon)\dots A((j+1)\epsilon,\epsilon)\mathbb{E}_j(\epsilon). \tag{29}$$

Now, let us look at the roots $\mu_1(x,\epsilon),\dots,\mu_d(x,\epsilon)$ of

$$\sum_{j=0}^{d} b_j(x,\epsilon)\mu^j = 0.$$

Since $\max_j \sup_{x\in I} |\mu_j(x,0)| = 1$, it follows that

$$\max_j \sup_{x\in I} |\mu_j(x,\epsilon)| \leq 1 + C\epsilon.$$

Since $k\epsilon$ lies in $I$, a compact set, Lemma 3.2 and Equation (29) imply that

$$\|\Theta_k(\epsilon) - \mathbb{1}\| \leq kC'_s \epsilon^{s+1} \leq C_s \epsilon^{s-s_0+1} \tag{30}$$

for all $k\epsilon \in I$, where $s_0 = 1$. This completes the proof of (24).

Equations (24) and (22) differ in the presence of $s_0$. It is easy to see that if $f$ is a function such that for a fixed $s_0$ and any $s \in \mathbb{N}$ we have:

$$|f(\epsilon) - \sum_{t=0}^{s+s_0} C_t \epsilon^t| < D_s \epsilon^{s+1},$$

then

$$|f(\epsilon) - \sum_{t=0}^{s} C_t \epsilon^t| < (D_s + |C_{s+1}| + \dots |C_{s+s_0}|)\epsilon^{s+1}.$$

This observation shows that (24) implies (22) and concludes the proof of the Proposition. □

**Proposition 5.3.** (a) *There exists a smooth function* $\Phi_1 \in C^\infty(I \times [0,\epsilon'])$ *such that*

(a) *For all* $(k\epsilon, \epsilon) \in \Delta_{\epsilon', I}$, *we have:*

$$\psi_1(k\epsilon, \epsilon) = \exp\left(\epsilon^{-1}\Phi_1(k\epsilon, \epsilon)\right),$$

(b) $\Phi_1$ *has an asymptotic expansion (uniform with respect to* $x$*):*

$$\Phi_1(x, \epsilon) \sim_{\epsilon \to 0} \exp\left(\epsilon^{-1}\sum_{s=0}^\infty \phi_{1,s}(x)\epsilon^s\right)$$

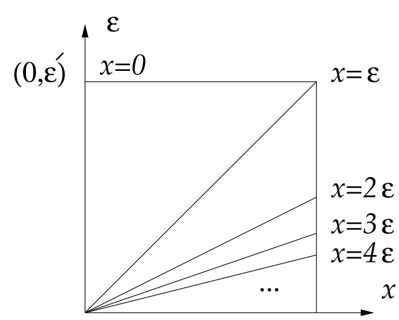
*where* $\phi_{1,s}$ *are as in Lemma 5.1. As a result, we have an asymptotic expansion (uniform with respect to* $k$*):*

$$\psi_1(k\epsilon, \epsilon) \sim_{\epsilon \to 0} \exp\left(\epsilon^{-1}\sum_{s=0}^\infty \phi_{1,s}(k\epsilon)\epsilon^s\right).$$

*Proof.* Consider the change of variables $\theta$ as in (23).

Due to our choice of initial conditions it follows that for every fixed $k = 0, \dots, d-1$, the function $\epsilon \to \theta(k\epsilon, \epsilon)$ is smooth. Using this and the smoothness of the coefficients of (8), it follows that for every fixed $k$, the function $\epsilon \to \theta(k\epsilon, \epsilon)$ is smooth.

So far, the function $\theta$ is defined on $\Delta^{\mathrm{seg}}_{\epsilon', I}$:




which is a union of line segments in a rectangle $I \times [0,\epsilon']$, and satisfies (22).

The complement of these line segments in $[0,\epsilon']$ consists of an infinite union of open triangles, together with the horizontal segment $I \times 0$. We can smoothly interpolate $\theta$ inside these open triangles so that it is defined on $I \times (0, \epsilon']$ and

$$|\theta(x,\epsilon) - 1| < C_s \epsilon^s \tag{31}$$

for all $(x, \epsilon) \in I \times (0, \epsilon']$ and all $s \in \mathbb{N}$.

Let us extend $\theta$ to $I \times [0,\epsilon']$ by defining $\theta(x, 0) = 1$ for all $x \in I$.

We claim that $\theta$ is smooth on $I \times [0, \epsilon']$. We need only to check this at the points $(x, 0)$ for $x \in I$. This follows easily from (31). For example, to check continuity at $(x, 0)$, consider a sequence $(x_n, \epsilon_n)$ such that $\lim_{n\to\infty}(x_n, \epsilon_n) = (x, 0)$. Then, (31) for $s = 1$ implies that $|\theta(x_n, \epsilon_n) - 1| < C_s\epsilon_n \sim_{n\to\infty} 0$, thus $\theta$ is continuous at $(x, 0)$. Using (31) for $s+1$ it follows that $\partial^s/\partial\epsilon^s\theta(x,\epsilon)|_{\epsilon=0} = 0$ for all $s > 0$, and we find that $\theta$ has an $\epsilon$-asymptotic expansion (uniform with respect to $x$):

$$\theta(x, \epsilon) \sim_{\epsilon\to 0} 1$$

Restricting further $\epsilon'$ if needed, we may assume that $|\theta(x, \epsilon)| > 0$ for all $(x, \epsilon) \in I \times [0, \epsilon']$; in other words $\log\theta(x, \epsilon)$ makes sense for all $(x, \epsilon) \in I \times [0, \epsilon']$.

Now, we can finish the proof of Proposition 5.3.

Let us define

$$\Phi_1(x, \epsilon) = \hat{\Phi}_1(x, \epsilon) + \epsilon \log\theta(x, \epsilon)$$

Then, (23) implies (a). Since $\theta(x, \epsilon)$ is asymptotic to 1 (uniformly on $x$), it follows that $\Phi_1(x, \epsilon)$ is asymptotic to $\hat{\Phi}_1(x, \epsilon)$. Using the asymptotic of $\hat{\Phi}_1$ given by Lemma 5.1, (b) follows. □

5.2. **A reduction to an $\epsilon$-difference equation of smaller degree.** We will now prove Theorem 3 by induction on the degree $d$ of the $\epsilon$-difference equation. For $d = 1$, it follows from Proposition 5.3. The inductive step is the next Proposition.

**Proposition 5.4.** *Assume that Theorem 3 holds for regular $\epsilon$-difference equations of degree less than $d$. Then it holds for regular $\epsilon$-difference equations of degree $d$.*

*Proof.* Consider a $\epsilon$-difference equation (8) of degree $d$. We will use the solution $\psi_1$ of it constructed in Proposition 5.3 to reduce it to an equivalent equation of degree $d-1$, and an inhomogeneous $\epsilon$-difference equation of degree 1.

Consider the dependent change of variables

$$\theta = \frac{\phi}{\psi_1} \tag{32}$$

This is well-defined since $\psi_1$ is nowhere zero. Then, $\phi$ satisfies (8) iff $\theta$ satisfies

$$\sum_{j=0}^{d} b_j(k\epsilon, \epsilon)\theta((k+j)\epsilon, \epsilon) = 0 \tag{33}$$

where

$$b_j(x, \epsilon) = a_j(x, \epsilon)\frac{\psi_1(x + j\epsilon, \epsilon)}{\psi_1(x, \epsilon)}.$$

The characteristic polynomials of (8) and (33) are related by

$$P_{(33)}(\lambda) = \lambda_1(x)^d P_{(8)}(\lambda/\lambda_1(x)).$$

As in the proof of Proposition 5.3, it is easy to see that (33) is a regular $\epsilon$-difference equation. Moreover, it is easy to see that Theorem 3 holds for (8) iff it holds for (33). Indeed, check that the change of variables given by (32) preserves the asymptotics of the solutions of (8) and (33).

Thus, it suffices to work with (33). In that case, $\theta = 1$ is a solution of (33), since $\psi$ is a solution of (8). It follows that

$$\sum_{j=0}^{d} b_j(x, \epsilon) = 0. \tag{34}$$

(Compare this with (26)). Let us define

$$\zeta(k\epsilon, \epsilon) = \theta((k+1)\epsilon, \epsilon) - \theta(k\epsilon, \epsilon). \tag{35}$$

Then, we get that $\zeta$ is a solution of the $\epsilon$-difference equation

$$\sum_{j=0}^{d-1} c_j(k\epsilon, \epsilon)\zeta((k+j)\epsilon, \epsilon) = 0 \tag{36}$$

where

$$c_s(x, \epsilon) = \sum_{j=s+1}^{d} b_j(x, \epsilon).$$

The characteristic equations of (33) and (36) are related by

$$\sum_{j=0}^{d} b_j(x, 0)\lambda^j = (\lambda - 1)\sum_{j=0}^{d-1} c_j(x, 0)\lambda^j$$

Since $c_0(x, \epsilon) = \sum_{j=1}^{d} b_j(x, \epsilon) = -b_0(x, \epsilon)$ (by (34)) and $c_d(x, \epsilon) = b_d(x, \epsilon)$, the same arguments of Proposition 5.3 imply that (36) is regular, assuming that (33) is regular.

By the induction hypothesis, it follows that (36) satisfies Theorem 3.

For the remainder of this section, fix a solution $\zeta$ of (36) which satisfies the properties of Theorem 3. In other words, $\zeta$ satisfies (33) and $\zeta(k\epsilon,\epsilon)=\exp(\epsilon^{-1}Z(k\epsilon,\epsilon))$ where $Z$ is a smooth function with uniform (with respect to $x$) asymptotics:

$$Z(x,\epsilon) \sim_{\epsilon\to 0} \sum_{s=0}^\infty Z_s(x)\epsilon^s.$$

**Lemma 5.5.** *There exists a formal solution*

$$\tilde\theta(x,\epsilon)=\exp(\epsilon^{-1}\Theta(x,\epsilon))$$

*of* (36) *such that*

$$\Theta(x,\epsilon)=\sum_{s=0}^\infty \Theta_s(x)\epsilon^s.$$

*Proof.* We need to solve the formal power series Equation

$$\exp\left(\epsilon^{-1}\sum_{s=0}^\infty \Theta_s(x+\epsilon)\epsilon^s\right)-\exp\left(\epsilon^{-1}\sum_{s=0}^\infty \Theta_s(x)\epsilon^s\right)=\exp\left(\epsilon^{-1}\sum_{s=0}^\infty Z_s(x)\epsilon^s\right)$$

for $\Theta$ in terms of $Z$. Using the Taylor expansion $\Theta_0(x+\epsilon)=\Theta_0(x)+\Theta_0'(x)\epsilon+O(\epsilon^2)$ it is easy to see that the above equation equals to

$$\exp\left(\epsilon^{-1}\Theta_0(x)+O(1)\right)-\exp\left(\epsilon^{-1}\Theta_0(x)+O(1)\right)=\exp\left(\epsilon^{-1}Z_0(x)+O(1)\right)$$

from which follows that $\Theta_0=Z_0$. Dividing the equation by $\exp(\epsilon^{-1}\Theta_0)$ we get an equation in formal power series with nonnegative powers of $\epsilon$. Moreover, the coefficient of $\epsilon^s$ in that power series (for $s\geq 0$) equals to

$$(\exp(\Theta_0'(x))-1)\exp(\Theta_{s+1}(x))H_s(x)$$

where $H_s(x)$ is a function of $\Theta_j$ and $Z_j$ for $j=1,\dots,s$. Since $\exp(\Theta_0'(x))=\exp(Z_0'(x))$ is an eigenvalue of (36), it is never equal to 1.

This and induction prove the lemma. □

**Lemma 5.6.** (a) There exists a solution to the equation

$$\zeta(k\epsilon,\epsilon)=\theta((k+1)\epsilon,\epsilon)-\theta(k\epsilon,\epsilon) \tag{37}$$

for $\theta$ in terms of $\zeta$ with appropriate initial condition.
(b) For all $(k\epsilon,\epsilon)\in\Delta'_{\epsilon',I}$ we have:

$$\zeta(k\epsilon,\epsilon)\sim_{\epsilon\to 0}\exp(Z(k\epsilon,\epsilon))$$

where

$$\exp(Z(x,\epsilon))=\exp(\Theta(x+\epsilon,\epsilon))-\exp(\Theta(x,\epsilon))$$

*Proof.* Fix $\epsilon>0$ and let $I=[a,b]$. Consider a natural number $k$ such that $k\epsilon\in I$ and $(k+1)\epsilon\in I$. This is equivalent to $k_1\leq k\leq k_2$ where $k_1$ and $k_2$ are natural numbers that depend on $\epsilon$ and $I$, although we do not indicate this in our notation.

Then equation (37) implies that

$$\begin{aligned}
\zeta(k_1\epsilon,\epsilon) &= \theta((k_1+1)\epsilon,\epsilon)-\theta(k_1\epsilon,\epsilon)\\
\zeta((k_1+1)\epsilon,\epsilon) &= \theta((k_1+2)\epsilon,\epsilon)-\theta((k_1+1)\epsilon,\epsilon)\\
\dots &= \dots\\
\zeta((k_2-1)\epsilon,\epsilon) &= \theta(k_2\epsilon,\epsilon)-\theta((k_2-1)\epsilon,\epsilon)
\end{aligned}$$

Summing up, we obtain that

$$\theta(k_2\epsilon,\epsilon)=\theta(k_1\epsilon,\epsilon)+\sum_{j=k_1}^{k_2-1}\zeta(j\epsilon,\epsilon).$$

Choose initial conditions so that $\theta(k_1\epsilon,\epsilon)=\exp(Z(k_1\epsilon,\epsilon))$. This completes part (a).

Part (b) follows by a telescoping calculation. □

To finish the proof of Proposition 5.4, it suffices to show that the solution $\zeta$ of Lemma 5.6 is asymptotic to the formal solution $\tilde{\theta}$ of Lemma 5.5.

Since

$$\zeta(k\epsilon,\epsilon) \sim_{\epsilon\to 0} \exp(Z(k\epsilon,\epsilon))$$

and

$$\exp(Z(k\epsilon,\epsilon)) = \exp(\Theta((k+1)\epsilon,\epsilon)) - \exp(\Theta(k\epsilon,\epsilon))$$

it follows by the definition of $\theta$ given in Lemma 5.6 and by a telescopic sum, that:

$$\begin{aligned}
\theta(k\epsilon,\epsilon) &= \theta(k_1\epsilon,\epsilon) + \sum_{j=k_1}^{k-1} \zeta(j\epsilon,\epsilon) \\
&\sim_{\epsilon\to 0} \theta(k_1\epsilon,\epsilon) + \sum_{j=1}^{k-1} (\exp(\Theta((j+1)\epsilon,\epsilon)) - \exp(\Theta(j\epsilon,\epsilon))) \\
&= \theta(k_1\epsilon,\epsilon) + \exp(\Theta(k\epsilon,\epsilon)) - \exp(\Theta(k\epsilon,\epsilon)) \\
&= \exp(\Theta(k\epsilon,\epsilon))
\end{aligned}$$

This concludes the proof of Proposition 5.4. □

5.3. **The solutions form a locally fundamental set.** Let us summarize what we have obtained so far.

Consider a partition $I = \cup_{p\in P} I_p$ of $I = [x_0, x_P]$ given by $I_p = [x_p, x_{p+1}]$ for $p = 1, \dots, P-1$, and consider a permutation $\sigma_p$ of $\{1, \dots, d\}$ such that

$$|\lambda_{\sigma_p(1)}(x)| \geq |\lambda_{\sigma_p(2)}(x)| \geq \cdots \geq |\lambda_{\sigma_p(d)}(x)| \qquad \text{for all} \qquad x \in I_p.$$

We have constructed solutions smooth functions $\Phi_m$ for $m = 1, \dots, d$ with asymptotic expansion given by (10) and (11).

Let us define

$$\psi_m(x,\epsilon) = \exp\left(\epsilon^{-1}\Phi_m(x,\epsilon)\right), \tag{38}$$

where $\Phi_m$ are smooth functions with asymptotic expansions as in Equations (10) and (11).

Moreover, we have shown that for every interval $I_p$ (as in the discussion prior to Theorem 3),

$$\{\psi_1(k\epsilon,\epsilon), \dots, \psi_d(k\epsilon,\epsilon)\}$$

is a set of solutions of (8) when $k\epsilon \in I_p$.

Fix a solution $\psi(k\epsilon,\epsilon)$ of (8) and an interval $I_p$. The following lemma certainly implies that $\{\psi_1, \dots, \psi_d\}$ is a locally fundamental set of solutions of (8). This concludes the proof of Theorem 3. □

In addition, the next lemma motivates the definition of a regular solution, given in the following section.

**Lemma 5.7.** (a) Fix $\psi$ and $I_p$ as above. Then, there exist smooth functions $c_m^p$ such that

$$\psi(k\epsilon,\epsilon) = c_1^p(\epsilon)\psi_{\sigma_p(1)}(k\epsilon,\epsilon) + \cdots + c_d^p(\epsilon)\psi_{\sigma_p(d)}(k\epsilon,\epsilon) \tag{39}$$

for all $k\epsilon \in I_p$.

(b) Moreover, for every $p$ and $m = 1, \dots, d$ we have

$$c_m^p(\epsilon) = \frac{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon\epsilon - \epsilon,\epsilon)}{\psi_{\sigma_p(m)}(x_p^\epsilon\epsilon,\epsilon)}\gamma_m^p(\epsilon) \tag{40}$$

for some smooth functions $\gamma_m^p$, with the understanding that $\psi_{\sigma_{-1}(m)} = 1$. Here $[x]$ is the *largest integer smaller than* $x$, and

$$x_p^\epsilon = \left[\frac{x_p}{\epsilon}\right] + 1.$$

*Proof.* Without loss of generality, let us assume that $\sigma_p(j) = j$ for $j = 1, \dots, d$. Equation (39) is a linear equation in $c^p_m$, with solutions

$$c^p_m(\epsilon) = \frac{\det W_m(x^\epsilon_p, \epsilon)}{\det W(x^\epsilon_p, \epsilon)}$$

where $I_p = [x_p, x_{p+1}]$,

$$W(x,\epsilon) = \begin{pmatrix} \psi_1(x,\epsilon) & \dots & \psi_m(x,\epsilon) & \dots & \psi_d(x,\epsilon) \\ \psi_1(x+\epsilon,\epsilon) & \dots & \psi_m(x+\epsilon,\epsilon) & \dots & \psi_d(x+\epsilon,\epsilon) \\ \dots & \dots & \dots & \dots & \dots \\ \psi_1(x+(d-1)\epsilon,\epsilon) & \dots & \psi_m(x+(d-1)\epsilon,\epsilon) & \dots & \psi_d(x+(d-1)\epsilon,\epsilon) \end{pmatrix}$$

and

$$W_m(x,\epsilon) = \begin{pmatrix} \psi_1(x,\epsilon) & \dots & \psi(x,\epsilon) & \dots & \psi_d(x,\epsilon) \\ \psi_1(x+\epsilon,\epsilon) & \dots & \psi(x+\epsilon,\epsilon) & \dots & \psi_d(x+\epsilon,\epsilon) \\ \dots & \dots & \dots & \dots & \dots \\ \psi_1(x+(d-1)\epsilon,\epsilon) & \dots & \psi(x+(d-1)\epsilon,\epsilon) & \dots & \psi_d(x+(d-1)\epsilon,\epsilon) \end{pmatrix}$$

where the $\psi$'s are in the $m$th column of $W_m$.

We will show that for small enough $\epsilon$, $W(x,\epsilon)$ is nonsingular.

Using Equations (38), (10) and (11), it follows that

$$\begin{aligned} \frac{1}{\psi_1(x,\epsilon)\dots\psi_d(x,\epsilon)} \det(W(x,\epsilon)) &= \det(\exp(j\phi'_m(x))) + O(\epsilon) \\ &= \det(\lambda_m(x)^j) + O(\epsilon) \\ &= \pm\prod_{i\neq j}(\lambda_i(x) - \lambda_j(x)) + O(\epsilon) \end{aligned}$$

uniformly in $x$, where $\lambda_m$ are the eigenvalues of (8). Since (8) is regular, its eigenvalues never collide.

Similarly, using Equation (38), we have:

$$\frac{1}{\psi_1(x^\epsilon_p\epsilon,\epsilon)\dots\widehat{\psi_m}(x^\epsilon_p\epsilon,\epsilon)\dots\psi_d(x^\epsilon_p\epsilon,\epsilon)} \det(W_m(x^\epsilon_p\epsilon,\epsilon)) = \det B_{m,0}(x^\epsilon_p\epsilon,\epsilon) + O(\epsilon)$$

where

$$B_{m,j}(x,\epsilon) = \begin{pmatrix} \lambda_1(x)^{-j} & \dots & \psi(x,\epsilon) & \dots & \lambda_d(x)^{-j} \\ \lambda_1(x)^{1-j} & \dots & \psi(x+\epsilon,\epsilon) & \dots & \lambda_d(x)^{1-j} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_1(x)^{d-1-j} & \dots & \psi(x+(d-1)\epsilon,\epsilon) & \dots & \lambda_d(x)^{d-1-j} \end{pmatrix}$$

where the $\psi$'s are in the $m$th column of $B_{m,j}$. Thus,

$$c^p_m(\epsilon) = \frac{1}{\psi_m(x^\epsilon_p\epsilon,\epsilon)} \frac{\det(B_{m,0}(x^\epsilon_p\epsilon,\epsilon))}{\prod_{j\neq m}\lambda_j(x^\epsilon_p\epsilon) - \lambda_m(x^\epsilon_p\epsilon)} + O(\epsilon). \tag{41}$$

The idea now is to move the recursion relation backwards $d$ times. Using the solution $\psi(k\epsilon,\epsilon)$ for $k\epsilon \in I_{p-1}$ will allow us to compute the smooth functions $c^p_m$.

In detail, consider the matrix $B_{m,0}(x^\epsilon_p\epsilon,\epsilon)$ and move the recursion relation backwards once. Using Equation (27) and the fact that the the $j$th column of $B_{m,j}(x^\epsilon_p,\epsilon)$ for $j \neq m$ is an eigenvector of $A(x^\epsilon_p\epsilon,\epsilon)$ (up to $O(\epsilon)$) with eigenvalue $\lambda_j(x^\epsilon_p\epsilon)$, it follows that

$$B_{m,0}(x^\epsilon_p\epsilon,\epsilon) = A(x^\epsilon_p\epsilon,\epsilon)B_{m,1}(x^\epsilon_p\epsilon - \epsilon,\epsilon) + O(\epsilon).$$

Iterating $d-1$ more times, it follows that

$$B_{m,0}(x^\epsilon_p\epsilon,\epsilon) = A(x^\epsilon_p\epsilon,\epsilon)\dots A(x^\epsilon_p\epsilon - (d-1)\epsilon,\epsilon)B_{m,d}(x^\epsilon_p\epsilon - d\epsilon,\epsilon) + O(\epsilon).$$

Since $x_p^\epsilon \epsilon - d\epsilon \in I_{p-1}$, it follows (as in the computation of $W_m(x,\epsilon)$ above) that:

$$\begin{aligned}
\det(B_m(x_p^\epsilon \epsilon - d\epsilon, \epsilon)) &= \psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon, \epsilon) \\
&\quad \det \begin{pmatrix} \lambda_1(x_p^\epsilon \epsilon - \epsilon)^{1-d} & \dots & \frac{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - d\epsilon,\epsilon)}{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon,\epsilon)} & \dots & \lambda_d(x_p^\epsilon \epsilon - \epsilon)^{1-d} \\
\lambda_1(x_p^\epsilon \epsilon - \epsilon)^{2-d} & \dots & \frac{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon + (1-d)\epsilon,\epsilon)}{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon,\epsilon)} & \dots & \lambda_d(x_p^\epsilon \epsilon - \epsilon)^{2-d} \\
\dots & \dots & \dots & \dots & \dots \\
\lambda_1(x_p^\epsilon \epsilon - \epsilon)^{d-d} & \dots & \frac{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon,\epsilon)}{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon,\epsilon)} & \dots & \lambda_d(x_p^\epsilon \epsilon - \epsilon)^{d-d}
\end{pmatrix} \\
&= \psi_{\sigma_{p-1}(m)}(x_p^\epsilon \epsilon - \epsilon, \epsilon) \\
&\quad \prod_{i \neq j, j \neq m} (\lambda_i^{-1}(x_p^\epsilon \epsilon - \epsilon) - \lambda_j^{-1}(x_p^\epsilon \epsilon - \epsilon)) \\
&\quad \prod_{i \neq m} (\lambda_i^{-1}(x_p^\epsilon \epsilon - \epsilon) - \lambda_{\sigma_{p-1}(m)}^{-1}(x_p^\epsilon \epsilon - \epsilon)) + O(\epsilon).
\end{aligned}$$

This, together with Equation (41) proves (40). □

## 6. Regular solutions and their asymptotics

In this section we discuss regular solutions of $q$ and $\epsilon$-difference equations and their asymptotics.

### 6.1. Regular solutions to $\epsilon$-difference equations.

According to Lemma 5.7, a solution $\psi$ to (8) determines a collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$, where

$$S_p = \{m \in \{1, \dots, d\} | \gamma_m^p \neq 0\}.$$

**Definition 6.1.** Fix a collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$. We say that a solution $\psi$ of a regular equation (8) is $S$-*regular* iff for every $p \in \mathcal{P}$ we have:

- $\gamma_m^p = 0$ if $m \notin S_p$.
- $\gamma_m^p$ are *not flat* at 0 for all $m \in S_p$. That is, some derivative of $\gamma_m^p$ at $\epsilon = 0$ does not vanish.
- For every $p \in \mathcal{P}$ there exists an element $\eta(p) \in S_p$ such that
$$|\lambda_{\eta(p)}(x)| > |\lambda_j(x)| \quad \text{for all} \quad j \in S_p - \{\eta(p)\}, \quad x \in \text{Interior}(I_p).$$
In other words, in the interior of the interval $I_p$, and among the eigenvalues $\lambda_j(x)$ for $j \in S_p$, there is a unique eigenvalue of strictly maximum magnitude.

We will say that a solution to (3) is *regular* if it is $S$-regular for some $S$.

### 6.2. Asymptotics of regular solutions of $\epsilon$-difference equations.

*Proof.* (of Theorem 4) Let $\psi$ be an $S$-regular solution to (8). Let us assume that $S = \{1, \dots, d\}$, and $|\lambda_1(x)| > |\lambda_m(x)|$ for $m = 2, \dots, d$ and all $x$ in the interior Interior($I$) of the closed interval $I = [0, b]$. Fix an $x \in I$.

Then, we have:

$$\psi(x, \epsilon) = c_1(\epsilon)\psi_1(x, \epsilon) + \dots + c_d(\epsilon)\psi_d(x, \epsilon).$$

for $x = k\epsilon$, where $c_1(\epsilon) = c_1 \epsilon^{n_1} + O(\epsilon^{n_1+1})$, and $c_1 \neq 0$.

Then, we have:

$$\psi(x, \epsilon) = c_1(\epsilon)\psi_1(x, \epsilon) \left( 1 + \sum_{m=2}^{d} \frac{c_j(\epsilon)}{c_1(\epsilon)} \frac{\psi_m(x, \epsilon)}{\psi_1(x, \epsilon)} \right). \tag{42}$$

Recall from Theorem 3 that

$$\begin{aligned}
\psi_m(x, \epsilon) &= \exp(\epsilon^{-1}\Phi_m(x, \epsilon)) \\
\Phi_m(x, \epsilon) &= \Phi_{m,0}(x) + O(\epsilon) \\
\Phi'_{m,0}(x) &= \log \lambda_m(x).
\end{aligned}$$

Thus,

$$\operatorname{Re}(\Phi_{m,0})'(x) = \operatorname{Re}(\log \lambda_m(x)) = \log|\lambda_m(x)|.$$

Combined with $|\lambda_m(x)| < |\lambda_1(x)|$ for $m \geq 2$, and $\Phi_{m,0}(0) = 0 = \Phi_{1,0}(0)$, it follows that

$$\operatorname{Re}(\Phi_{m,0})(x) < \operatorname{Re}(\Phi_{1,0})(x)$$

for all $x \in I$ and

$$\operatorname{Re}(\Phi_{1,0})(x) = \int_0^x \log|\lambda_1(t)|dt.$$

Therefore,

$$\lim_{\epsilon\to 0^+}\left(1 + \sum_{m=2}^{d} \frac{c_j(\epsilon)}{c_1(\epsilon)}\frac{\psi_m(x,\epsilon)}{\psi_1(x,\epsilon)}\right) = 1.$$

and

$$\lim_{\epsilon\to 0^+} \epsilon \log|c_1(\epsilon)| = 0.$$

Thus, Equation (42) implies that

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \epsilon \log|\psi(x,\epsilon)| &= \lim_{\epsilon\to 0^+} \epsilon \log|\psi_1(x,\epsilon)| \\
&= \lim_{\epsilon\to 0^+} \epsilon \log|\exp(\epsilon^{-1}\Phi_m(x,\epsilon))| \\
&= \lim_{\epsilon\to 0^+} \epsilon \log|\exp(\epsilon^{-1}\Phi_m(x,0) + O(1))| \\
&= \operatorname{Re}(\Phi_{1,0}(x)) \\
&= \int_0^x \log|\lambda_1(t)|dt
\end{aligned}$$

The result follows.

In the general case, we partition the interval $I_p$ as in the discussion prior to Theorem 3 and repeat the above proof using Equation (43). The result follows. □

6.3. **Asymptotics of regular solutions of $q$-difference equations.** First, we need to define what is a regular solution to a $q$-difference equation.

Consider a solution $\psi$ of a $q$-difference equation and a partition of $S^1$ as in Section 1.8. Then, at each interval $I_p$, we can write the solution as a linear combination of fundamental solutions, as in Equation (4). Let $S_p$ be the indexing set of the fundamental solutions that we use in each interval $I_p$; see (6).

Suppose that the partition of $S^1$ is given by $I_p = [e^{2\pi i x_p}, e^{2\pi i x_{p+1}}]$ for $p = 0, \dots, P-1$, with $q_0 = 1$, $q_P = e^{2\pi i}$.

Then, with $q = e^{2\pi i\epsilon}$ it turns out that for every $p$ and $m = 1, \dots, d$ we have

$$c_m^p(q) = \frac{\psi_{\sigma_{p-1}(m)}(x_p^\epsilon\epsilon - \epsilon, \epsilon)}{\psi_{\sigma_p(m)}(x_p^\epsilon\epsilon, \epsilon)}\gamma_m^p(q) \tag{43}$$

for some smooth functions $\gamma_m^p$, with the understanding that $\psi_{\sigma_{-1}(m)} = 1$.

**Definition 6.2.** Fix a collection $S = \{S_p | p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$. We say that a solution $\psi$ of a regular equation (3) is *$S$-regular* iff for every $p \in \mathcal{P}$ we have:

- $\gamma_m^p = 0$ if $m \notin S_p$.
- $\gamma_m^p$ are *not flat* at 0 for all $m \in S_p$. That is, some derivative of $\gamma_m^p$ at $\epsilon = 0$ does not vanish.
- For every $p \in \mathcal{P}$ there exists an element $\eta(p) \in S_p$ such that

  $$|\lambda_{\gamma(p)}(v)| > |\lambda_j(v)| \quad \text{for all} \quad j \in S_p - \{\gamma(p)\}, \quad v \in \operatorname{Interior}(I_p).$$

  In other words, in the interior of the interval $I_p$, and among the eigenvalues $\lambda_j(v)$ for $j \in S_p$, there is a unique eigenvalue of strictly maximum magnitude.

We will say that a solution to (3) is *regular* if it is $S$-regular for some $S$.

*Proof.* (of Theorem 2) It follows from Equation (14) of Lemma 2.7 and Theorem 4. □

## 7. Applications to Quantum Topology

7.1. **The $A$-polynomial of a knot and its noncommutative version.** In this section we discuss general features of $q$-difference equations for the colored Jones function of a knot.

The coefficients of the $q$-difference equations are rational functions of $q$ and $q^n = Q$. In order to simplify the typesetting, we will give the $q$-difference equation

$$\sum_{j=0}^{d} b_j(q^n, q) f(n+j) = 0$$

in operator form

$$Pf = 0 \tag{44}$$

where

$$P = \sum_{j=0}^{d} b_j(Q, q) E^j$$

and where the operators $E$, $Q$ and $q$, act on a discrete function $f : \mathbb{N} \to \mathbb{Z}[q^{\pm}]$ by

$$(qf)(n) = qf(n) \qquad (Qf)(n) = q^n f(n) \qquad (Ef)(n) = f(n+1).$$

Note that $q$ commutes with $Q$ and $E$, and that $EQ = qQE$.

It follows by definition that the characteristic polynomial $\mathrm{ch}P(v,\lambda)$ of (44) is obtained from $P$ by setting $q = 1$, replacing $(E, Q)$ by $(\lambda, v)$. In other words, we have:

$$\mathrm{ch}P(v,\lambda) = \sum_{j=0}^{d} b_j(v,1)\lambda^j.$$

In [Ga1], the first author showed that the colored Jones function $J_K$ of a knot $K$ satisfies an essentially unique smallest degree $q$-difference equation $P_K J_K = 0$ where the coefficients $a_j(u,v)$ of $P_K$ are rational functions of $u$ and $v$ with rational coefficients.

In [Ga1], the operator $P_K$ was called the *non-commutative $A$-polynomial* of $K$.

In [Ga1], it was conjectured that:

**Conjecture 1.** *(AJ Conjecture) Up to a multiplication by a polynomial in $v$, we have*

$$\mathrm{ch}P_K(\lambda, v) = A_K(L, M)|_{(L,M^2)=(\lambda,v)}$$

*where $A_K$ is the $A$-polynomial of $K$, defined by* [CCGLS].

The $A$-polynomial of $K$ parametrizes the moduli space of characters of $\mathrm{SL}_2(\mathbb{C})$ representations of $\pi_1(S^3 - K)$, restricted to the boundary torus $\partial M$. The $A$-polynomial of a knot is an important ingredient to the Geometrization of the knot complement and its Dehn fillings.

The $A$-polynomial $A_K$ of a knot $K$ in $S^3$ satisfies symmetries, which we will list here, and refer to [CCGLS] and [CL] for proofs.

- **(S1)** It has integer coefficients and even powers of $M$, that is $A_K(L, M) \in \mathbb{Z}[L, M^2]$.
- **(S2)** It is reciprocal, that is, $A_K(L^{-1}, M^{-1}) = \pm L^k M^l A(L, M)$.
- **(S3)** It is tempered, that is the edge polynomials of its Newton polygon are cyclotomic.
- **(S4)** It specializes to
$$A_K(L, 1) = \pm(L-1)^{n_+}(L+1)^{n_-}$$
for some integers $n_\pm$.
- **(S5)** $L - 1$ is always a factor of $A_K$, that corresponds to $U(1)$ representations.

If the colored Jones function of a knot was an $S$-regular solution to a regular $q$-difference equation, and if the AJ Conjecture were true, then it follows that for every $\alpha \in [0,1]$ we have:

$$\lim_{n\to\infty} \frac{\log|J_K(n)(e^{\frac{2\pi i\alpha}{n}})|}{n} = \sigma^J_{S,K}(\alpha) = \sigma^A_{S,K}(\alpha) \tag{45}$$

where $\sigma^A_{S,K}$ is the $A$-*entropy* of a knot, defined as follows.

**Definition 7.1.** For a knot $K$ in $S^3$, let $L_j(t)$ for $j = 1, \dots, d$ denote the roots of the equation

$$A_K(L_j(t), e^{it/2}) = 0$$

for $t \in [0, 2\pi]$, where $d$ is the $L$-degree of $A_K$. Fix a partition $\cup_{p\in\mathcal{P}} I_p$ of $[0, 2\pi]$ by closed intervals with nonoverlapping interiors and a permutation $\sigma_p$ of the set $\{1, \dots, d\}$ such that

$$|L_{\sigma_p(1)}(t)| \geq |L_{\sigma_p(2)}(t)| \geq \dots \geq |L_{\sigma_p(d)}(t)| \qquad \text{for all} \qquad t \in I_p.$$

For every collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$, we can define the $S$-*entropy*

$$\sigma^A_{S,K} : [0, 1] \to \mathbb{R}$$

by

$$\sigma^A_{S,K}(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log \chi_S(\alpha t) dt,$$

where $\chi_S : [0, 1] \to \mathbb{R}$ is defined by

$$\chi_S(t) = \max_{j\in S_p} |L_{\sigma_p(j)}(t)| \qquad \text{if} \qquad t \in I_p.$$

It is natural to ask how the $A$-entropy of a hyperbolic knot (evaluated at $\alpha = 1$) compares to the Hyperbolic Volume. The answer to this question is essentially contained in work of D. Boyd, [Bo], which we quote without proof here. We urge the reader to look in [Bo] for beautiful and suggestive calculations.

Boyd introduced and studied another invariant of a knot, the *Mahler measure*

$$m_K = \int_{S^1\times S^1} \log |A_K(x, y)| dxdy$$

Using Jensen's formula, and the symmetry (S3), it follows that

$$2\pi m_K = \sum_{j=0}^{d} \int_0^{2\pi} \log^+ |L_j(t)| dt$$

where $\log^+(a) = \log \max\{1, a\}$.

Using (S2), it follows that $1/L_j(t)$ is an eigenvalue for every eigenvalue $L_j(t)$.

More generally, among the roots $L_j(t)$ there is a distinguished one, $L_1$, corresponding to the discrete faithful representation when $t = 2\pi$. Let $L_d(t) = 1$ denote the eigenvalue corresponding to the $U(1)$ representations. Boyd informs us that for 2-bridge knots $K$ (in particular, for the $3_1$ and $4_1$ knots), it is true that

$$\sigma^A_{\{1,d\},K}(1) = \mathrm{vol}_K.$$

It follows by (S4) that the eigenvalues collide at $t = 0$. Moreover, Boyd informs us that for 2-bridge knots there exists a $t_0 \in (0, \pi)$ such that $|L_j(t)| = 1$ for all $j$ and all $t : t_0 < t < \pi$.

Thus, if we want to apply Theorem 1 to the GHVC, we need to deal with irregular $q$-difference equations. We will discuss this topic in detail in a later publication.

Meanwhile, let us discuss some examples, taken from [Ga1].

7.2. **Examples: The $3_1$ and $4_1$ knots.** In this section we discuss in detail $q$-difference equation of the the colored Jones function of the two simplest knots, namely the trefoil $3_1$ and the figure eight $4_1$. The former is not hyperbolic, and the latter is.

In [Ga1], the first author computed that the colored Jones function $J_{3_1}$ (resp. $J_{4_1}$) satisfies the second (resp. third) order $q$-difference equation

$$P_{3_1} J_{3_1} = 0 \qquad \text{resp.} \qquad P_{4_1} J_{4_1} = 0$$

where the noncommutative $A$-polynomials $P_{3_1}$ and $P_{4_1}$ are given by:

$$\begin{aligned}
P_{3_1} &= \frac{q^3Q^2(q^2-q^2Q)}{q^3-q^4Q^2} \\
&+\frac{(q-q^2Q)(q+q^2Q)(q^4-q^5Q+q^6Q^2-q^7Q^2-q^7Q^3+q^8Q^4)}{q^2Q(q-q^4Q^2)(q^3-q^4Q^2)}E \\
&+\frac{-1+q^2Q}{Q(q-q^4Q^2)}E^2 \\
P_{4_1} &= \frac{q^5Q(-q^3+q^3Q)}{(q^2+q^3Q)(-q^5+q^6Q^2)} \\
&-\frac{(q^2-q^3Q)(q^8-2q^9Q+q^{10}Q-q^9Q^2+q^{10}Q^2-q^{11}Q^2+q^{10}Q^3-2q^{11}Q^3+q^{12}Q^4)}{q^5Q(q+q^3Q)(q^5-q^6Q^2)}E \\
&+\frac{(-q+q^3Q)(q^4+q^5Q-2q^6Q-q^7Q^2+q^8Q^2-q^9Q^2-2q^{10}Q^3+q^{11}Q^3+q^{12}Q^4)}{q^4Q(q^2+q^3Q)(-q+q^6Q^2)}E^2 \\
&+\frac{q^4Q(-1+q^3Q)}{(q+q^3Q)(q-q^6Q^2)}E^3
\end{aligned}$$

If we wish, we may clear denominators in $P_{3_1}$ and $P_{4_1}$. It follows that the characteristic polynomials are given by:

$$\begin{aligned}
\mathrm{ch}P_{3_1}(L,M) &= -\frac{(L-1)(L+M^3)}{M(1+M)} \\
\mathrm{ch}P_{4_1}(L,M) &= \frac{(L-1)(L-LM-M^2-2LM^2-L^2M^2-LM^3+LM^4)}{M(1+M)^2}
\end{aligned}$$

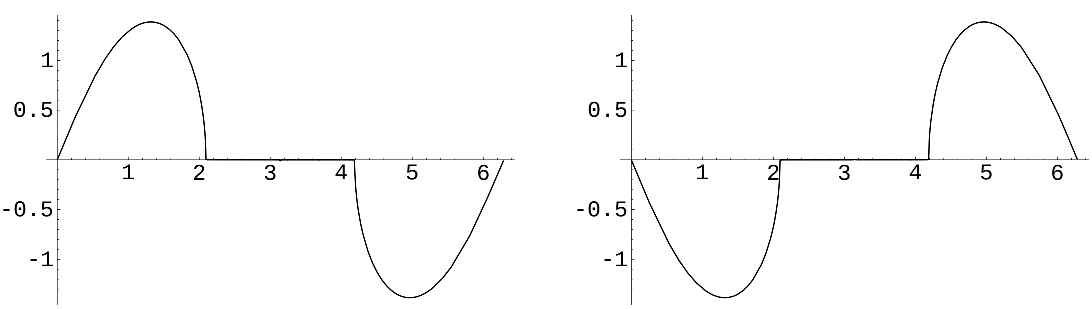
Inspection shows that $P_{3_1}$ and $P_{4_1}$ are not regular. Nevertheless, let us try to compute the $S$-entropy.

For the case of $3_1$, we have $|L_j(t)|=1$ for $j=1,2,3$ and in this case

$$\sigma^A_{S,3_1}(1) = \mathrm{vol}_{3_1} = 0$$

for all $S$.

For $4_1$ knot, we have 3 eigenvalues $L_1(t)$, $L_2(t)=1/L_1(t)$ and $L_3(t)=1$. Assuming appropriate choices for the branches of the eigenvalues, the plot of $\log|L_1(t)|$ and $\log|L_2(t)|=-\log|L_1(t)|$ for $t\in[0,2\pi]$ is given by:




It follows that

$$\sigma^A_{\{1,3\},4_1}(1) = \mathrm{vol}(4_1) = 2.029883 \qquad \sigma^A_{\{1,2,3\},4_1}(1) = 2\mathrm{vol}_{4_1} = 4.05977.$$

Since the HVC is true for the $4_1$ knot, it suggests that the colored Jones function lies in a strictly smaller subspace of the vector space of solutions to the $q$-difference equation $P_K J_K = 0$. Using work of Murakami [Mu], one can figure out exactly the selection principle; that is which locally fundamental solutions contribute to the colored Jones function.

Note that the associated $q$-difference equation of the $4_1$ knot has the following features: of *collision*, *resonance* and *vanishing*:

- The eigenvalues collide at $t=0$ (since $L_1(0)=L_1(0)=-1$), at $t=\pi/3$ (since $L_1(\pi/3)=L_1(\pi/3)=-1$) and by symmetry at $t=2\pi/3$ and $t=2\pi$.
- There is resonance on the interval $[\pi/3, 2\pi/3]$ where all three eigenvalues have equal magnitude.
- There is vanishing of the coefficients at $t=\pi/2$ (since the denominator $M+1$ of the coefficients is singular at $t=\pi/2$).

Moreover, there is an additional difficult problem of *selection principle*.

We plan to study these problems in later publications.

## References


[B1] G. Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Amer. Math. Soc. **9** (1908) 219–231.

[B2] ______, *Formal theory of irregular difference equations*, Acta Math. **54** (1930) 205–246.

[BT] ______ and W. Trjitzinsky, *Analytic theory of singular difference equations*, Acta Math. **60** (1932) 1–89.

[Bo] W. Boyd, *Mahler's measure and invariants of hyperbolic manifolds*, Number theory for the millennium, I (Urbana, IL, 2000), 127–143, A K Peters, Natick, 2002.

[CC] O. Costin, and R. Costin, *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. Math. Anal. 27 (1996), no. 1, 110–134.

[CCGLS] D. Cooper, D, M. Culler, H. Gillet, D. Long and P. Shalen, *Plane curves associated to character varieties of* 3-*manifolds*, Invent. Math. **118** (1994) 47–84.

[CL] ______ and D. Long, *Remarks on the A-polynomial of a knot*, J. Knot Theory Ramifications **5** (1996) 609–628.

[DM] P. Deift and K. McLaughlin, *A continuum limit of the Toda lattice*, Mem. Amer. Math. Soc. **131** (1998), no. 624.

[GL1] S. Garoufalidis and T.T.Q. Le, *The colored Jones function is q-holonomic* Geom. and Topology **9** (2005) 1253–1293.

[GL2] ______ and ______, *Asymptotics of the colored Jones function*, preprint 2005 `math.GT/0508100`.

[GL3] ______ and ______, *Gevrey asymptotics in quantum topology*, preprint 2005.

[Ga1] ______, *On the characteristic and deformation varieties of a knot*, Proceedings of the CassonFest, Geometry and Topology Monographs **7** (2004) 291–309.

[Ga2] ______, *Difference and differential equations for the colored Jones function*, preprint 2003 `math.GT/0306229`.

[Ga3] ______, *Gevrey asymptotics of difference equations with a parameter*, preprint 2005.

[GG] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Text in Mathematics **14** Springer-Verlag, 1973.

[GLR] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Academic Press, NY, 1982.

[Gr] G. Green, *On the motion of waves in a variable canal of small depth and width*, Trans. Cambridge Philos. Soc. **6** (1837) 457–462.

[Gu] S. Gukov, *Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial*, Comm. Math. Phys. **255** (2005) 577–627.

[K] R. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Modern Phys. Lett. A **39** (1997) 269–275.

[L] J. Liouville, *Sur le développement des fonctions ou parties de fonctions en séries*, J. Math. Pures Appl. **2** (1837) 16–35.

[J] V. Jones, *Hecke algebra representation of braid groups and link polynomials*, Annals Math. **126** (1987) p. 335–388.

[MM] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001) 85–104.

[Mu] ______, *Some limits of the colored Jones polynomials of the figure-eight knot*, Kyungpook Math. J. **44** (2004) 369–383.

[O] F. Olver, *Asymptotics and special functions*, Reprint. AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997.

[vPS] M. van der Put and M. Singer, *Galois theory of difference equations*, Lecture Notes in Mathematics, **1666** Springer-Verlag 1997.

[Th] W. Thurston, *The geometry and topology of 3-manifolds*, 1979 notes, available from MSRI.

[Wa] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Reprint of the 1976 edition. Dover Publications, New York, 1987.

[WZ] J. Wimp, and D. Zeilberger, *Resurrecting the asymptotics of linear recurrences*, J. Math. Anal. Appl. **111** (1985) 162–176.

[Wi] ______, *Computation with recurrence relations*, Applicable Mathematics Series, Pitman, Boston 1984.

[Wt] E.Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Physics. **121** (1989) 360–376.

[Z] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990) 321–368.

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