ALGEBRAIC G-FUNCTIONS ASSOCIATED TO MATRICES OVER A GROUP-RING

STAVROS GAROUFALIDIS AND JEAN BELLISSARD

ABSTRACT. Given a square matrix with elements in the group-ring of a group, one can consider the sequence formed by the trace (in the sense of the group-ring) of its powers. We prove that the corresponding generating series is an algebraic G-function (in the sense of Siegel) when the group is free of finite rank. Consequently, it follows that the norm of such elements is an exactly computable algebraic number, and their Green function is algebraic. Our proof uses the notion of rational and algebraic power series in non-commuting variables and is an easy application of a theorem of Haiman. Haiman's theorem uses results of linguistics regarding regular and context-free language. On the other hand, when the group is free abelian of finite rank, then the corresponding generating series is a G-function. We ask whether the latter holds for general hyperbolic groups.

CONTENTS

1. Introduction

1.1. **Algebricity of the Green's function for the free group.** Given a group G, consider the group-algebra $\mathbb{Q}[G]$, and define a *trace* map:

$$\text{(1)} \qquad \text{Tr} : \mathbb{Q}[G] \longrightarrow \mathbb{C}, \qquad \text{Tr}(P) = \text{constant term of } P$$

where the constant term is the coefficient of the identity element of G. Let $M_N(R)$ denote the set of N by N matrices with entries in a ring R. We can extend the trace to the algebra $M_N(\mathbb{Q}[G])$ by:

$$\text{(2)} \qquad \text{Tr} : M_N(\mathbb{Q}[G]) \longrightarrow \mathbb{C}, \qquad \text{Tr}(P) = \sum_{j=1}^{N} \text{Tr}(P_{jj}).$$

Definition 1.1. Given $P \in M_N(\mathbb{Q}[G])$, consider the sequence $(a_{P,n})$

$$\text{(3)} \qquad a_{P,n} = \text{Tr}(P^n)$$

and the generating series

$$\text{(4)} \qquad R_P(z) = \sum_{n=0}^{\infty} a_{P,n} z^n.$$

Let F_r denote the free group of rank r.

Theorem 1.1. *The Green's function $R_P(z)$ of every element P of $M_N(\mathbb{Q}[F_r])$ is algebraic.*

Theorem 1.1 appears in the cross-roads of several areas of research:
  (a) operator algebras
  (b) free probability
  (c) linguistics and context-free languages
  (d) non-commutative combinatorics
  (e) mathematical physics

In fact, Woess proves Theorem 1.1 when $N = 1$ using linguistics and context-free languages; see [Woe87, Woe00]. In [Sau03] Sauer also gives a proof using linguistics, with emphasis the rationality of the Novikov-Shubin invariants. Voiculescu proves Theorem 1.1 using the R and S transforms of free probability; see [Voi95, Voi05b]. For additional results using free probability, see [Aom84, CV96] and also [Lal93, Leh99, Leh01].

It is well-known that Theorem 1.1 provides an exact calculation of the norm of $P \in M_N(\mathbb{Q}[F_r]) \subset M_N(L(F_r))$, where $L(F_r)$ denotes the *reduced $\mathbb{C}^*$-algebra* completion of the group-algebra $\mathbb{C}[F_r]$. For a detailed discussion, see the above references.

Our proof of Theorem 1.1 uses the notion of an algebraic function in non-commuting variables and a theorem of Haiman, which itself is based on a theorem of Chomsky-Schützenberger on context-free languages. A by-product of our proof is the fact that the moment generating

series is a matrix of algebraic power series in non-commuting variables (see Proposition 4.4), which is a statement a priori stronger than Theorem 1.1.

An alternative proof of Theorem 1.1 uses methods from functional analysis, and most notably the *Schur complement method* (see below). We will discuss in detail the first proof and postpone the third proof to a later publication. Either proof explains the close relation between the differential properties of the generating function $R_P(z)$ and the word problem in G.

Our aim is to give a proof of algebraicity in the case of the free group, discuss holonomicity in the case of the free abelian group and formulate a question regarding holonomicity for hyperbolic groups. As it turns out, algebricity is well-studied in the above mentioned literature whereas holonomicity is largely absent.

1.2. **Related work.** Our paper was completed in the summer of 2007, and posted on the arxiv `arXiv:0708.4234`. In the fall of 2008, M. Kontsevich brought to the attention of the second author a related earlier paper of Sauer [Sau03] from 2003 that gives a proof of Theorem 1.1 with emphasis on the Novikov-Shubin invariants. Sauer's and our work has been cited by M. Kontsevich in the Arbeitstagung talk Bonn 2011, and (from what we have heard) in other talks too. Theorem 1.1 keeps attracting attention in diverse areas of mathematics. In the summer 2013, C. Kassel informed the second author of related article of Kassel-Reutenauer [KR13] around the theme of Theorem 1.1. Kassel was unaware of Sauer's work and of our work. In view of the interest of Theorem 1.1 and its connections to several branches of mathematics, we were encouraged to submit our article for publication.

2. THE CASE OF THE FREE ABELIAN GROUP

2.1. **Holonomic, algebraic and G-functions.** A priori, $R_P(z)$ is only a formal power series. However, it is easy to see that $(a_{P,n})$ is bounded exponentially by n, which implies that $R_P(z)$ defines an analytic function in a neighborhood of $z = 0$. The paper is concerned with differential/algebraic properties of the function $R_P(z)$. Algebraic and holonomic functions are well-studied objects. Let us recall their definition here.

Definition 2.1. (a) A *holonomic* function $f(z)$ is one that satisfies a linear differential equation with polynomial coefficients. In other words, we have:

$$c_d(z)f^{(d)}(z) + \cdots + c_0(z)f(z) = 0$$

where $c_j(z) \in \mathbb{Q}[z]$ for all $j = 0, \ldots, d$ and $f^{(j)}(z) = d^j/dz^j f(z)$.
(b) An *algebraic* function $f(z)$ is one that satisfies a polynomial equation:

$$Q(f(z), z) = 0$$

where $Q(y, z) \in \mathbb{Q}[y, z]$.

Lesser known to the combinatorics community are G-functions, which originated in the work Siegel on arithmetic problems in elliptic integrals, and transcendence problems in number theory; see [Sie66]. G-functions originate naturally in

   (a) algebraic geometry, related to the regularity properties of the Gauss-Manin connection, see for example [Del70, Kat70, Mal74],
   (b) arithmetic, see for example [And00, Bom81, DGS94],

(c) enumerative combinatorics, as was recently shown in [Gar09].

Definition 2.2. A *G-function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is one which satisfies the following conditions:

(a) for every $n \in \mathbb{N}$, we have $a_n \in \overline{\mathbb{Q}}$,
(b) there exist a constant $C_f > 0$ such that for every $n \in \mathbb{N}$ we have: $|a_n| \leq C_f^n$ (for every conjugate of a_n) and the common denominator of $a_0, \ldots, a_n$ is less than or equal to C_f^n.
(c) $f(z)$ is holonomic.

The next theorem summarizes the analytic continuation and the shape of the singularities of algebraic functions and G-functions. Part (a) follows from the general theory of differential equations (see eg. [Was87]), parts (b) and (d) follow from [CSTU02, Lem.2.2] (see also [DGS94] and [DvdP92]) and (c) follows from a combination of Katz's theorem, Chudnovsky's theorem and André's theorem; see [And00, p.706] and also [CL02].

Theorem 2.1. (a) A holonomic function $f(z)$ can be analytically continued as a multivalued function in $\mathbb{C} \setminus \Sigma_f$ where $\Sigma_f \subset \overline{\mathbb{Q}}$ is the finite set of singular points of $f(z)$.
(b) Every algebraic function $f(z)$ is a G-function.
(c) In a neighborhood of a singular point $\lambda \in \Sigma_f$, a G-function $f(z)$ can be written as a finite sum of germs of the form:

$$(z - \lambda)^{\alpha_\lambda} (\log(z - \lambda))^{\beta_\lambda} h_\lambda(z - \lambda) \tag{5}$$

where $\alpha_\lambda \in \mathbb{Q}$, $\beta_\lambda \in \mathbb{N}$, and h_λ a holonomic G-function.
(d) In addition, $\beta_\lambda = 0$ if $f(z)$ is algebraic.

Remark 2.3. Local expansions of the form (5) are known in the literature as Nilsson series (see [Nil65]), and minimal order linear differential equations that they satisfy are known to be regular singular, with rational exponents $\{a_\lambda\}$ and quasi-unipotent monodromy. For a discussion, see [Kat70, Mal74, Gar09] and references therein.

It is classical and easy to show that the existence of analytic continuation of a function implies the existence of asymptotic expansion of its Taylor series; see for example [Jun31, Com64] and also [CG11, Sec.7] and [Gar09].

Lemma 2.4. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holonomic and analytic at $z = 0$, then the nth Taylor coefficient a_n has an asymptotic expansion in the sense of Poincaré

$$a_n \sim \sum_{\lambda \in \Sigma} \lambda^{-n} n^{-\alpha_\lambda - 1} (\log n)^{\beta_\lambda} \sum_{s=0}^{\infty} \frac{c_{\lambda,s}}{n^s} \tag{6}$$

where Σ_f is the set of singularities of f, $\alpha_\lambda, \beta_\lambda \in \mathbb{Q}$, and $c_{\lambda,s} \in \mathbb{C}$.

2.2. **The case of the free abelian group.** In this section we will summarize what is known about the generating functions $R_P(z)$ when $G = \mathbb{Z}^r$ is the free abelian group or rank r. The next theorem is shown in [Gar09], using Andreé main theorems from [And00]. An alternative proof uses the regular holonomicity of the Gauss-Manin connection and the rationality of its exponents. This was kindly communicated to us by C. Sabbah (see also

[DvdK98]). Holonomicity of $R_P(z)$ also follows from a fundamental result of Wilf-Zeilberger, explained in [Gar09].

Theorem 2.2. [Gar09] *For every $P \in M_N(\mathbb{Q}[\mathbb{Z}^r])$, $R_P(z)$ is a G-function.*

2.3. **A complexity remark.** Given $P \in M_N(\mathbb{Q}[F_r])$ (resp. $P \in M_N(\mathbb{Q}[F_r])$), one may ask for the complexity of a minimal polynomial $Q(y, z) \in \mathbb{Q}[y, z]$ (resp. minimal degree differential operator $D(z, \partial_z) \in \mathbb{Q}\langle z, \partial z \rangle$) so that $Q(R_P(z), z) = 0$ (resp. $D(z, \partial_z)R_P(z) = 0$). One expects that the y-degree of $Q(y, z)$ and the ∂_z-degree of $D(z, \partial_z)$ is exponential in the *complexity* of P, where the latter can be defined to be the degree of P and the maximum of the absolute values of the coefficients of the enrties of P. This prohibits explicit calculations in general.

Acknowledgement. The author wishes to thank K. Dykema, R. Gilman, F. Flajolet, L. Mosher, C. Sabbah and D. Zeilberger for stimulating conversations and F. Lehner and D. Voiculescu for bringing to our attention relevant literature.

3. A THEOREM OF HAIMAN AND A PROOF OF THEOREM 1.1

In [Hai93] Haiman proves the following theorem.

Theorem 3.1. [Hai93] *Let K be a field with a rank 1 discrete valuation v; K_v its completion with respect to the metric induced by v. Let $f(x_1, \ldots, x_r, y_1, \ldots, y_r)$ be a rational power series over K in non-commuting indeterminants. Any coefficient of $f(x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1})$ converging over K_v is algebraic over K.*

Letting $K = \mathbb{Q}(z)$, and $K_v = \mathbb{Q}((z))$ the ring of formal Laurent series in z, and considering the element $(1 - zP)^{-1}$, where $P \in \mathcal{M}_N(\mathbb{Q}[F_r])$, gives an immediate proof of Theorem 1.1.

In the next section we will give a detailed description of Haiman's argument which exhibits a close relation to linguistics, as well as an obstruction to generalizing Theorem 1.1 to groups other than the free group.

4. ALGEBRAIC AND RATIONAL FUNCTIONS IN NONCOMMUTING VARIABLES

4.1. **Rational, algebraic and holonomic functions in one variable.** In this section all functions will be analytic in a neighborhood of $z = 0$. Let $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ denote respectively the set of rational, algebraic and holonomic functions, analytic at $z = 0$. Let $\mathbb{Q}[[z]]$ denote the set of formal power series in z. Using the injective Taylor series map around $z = 0$, we will consider $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ as subsets of $\mathbb{Q}[[z]]$:

$$(7) \qquad \mathbb{Q}_0^{\mathrm{rat}}(z) \subset \mathbb{Q}_0^{\mathrm{alg}}(z) \subset \mathbb{Q}_0^{\mathrm{hol}}(z) \subset \mathbb{Q}[[z]].$$

$\mathbb{Q}[[z]]$ has *two* multiplications:

- the usual multiplication of formal power series

$$(8) \qquad (\sum_{n=0}^{\infty} a_n z^n) \cdot (\sum_{n=0}^{\infty} b_n z^n) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) z^n.$$

- The *Hadamard product*:

$$(9) \qquad (\sum_{n=0}^{\infty} a_n z^n) \circledast (\sum_{n=0}^{\infty} b_n z^n) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

With respect to the usual multiplication, $\mathbb{Q}[[z]]$ is an algebra and $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ are subalgebras. In case two power series are convergent in a neighborhood of zero, so is their Hadamard product. Hadamard, Borel and Jungen studied the analytic continuation and the singularities of the Hadamard product of two functions; see [Bor98, Jun31]. Their method used an integral representation of the Hadamard product, and a deformation of the contour of integration; see [Jun31, Fig.2,p.303]. Let us summarize these classical results.

Theorem 4.1. *(a) If f and g are rational, so is $f \circledast g$.*
(b) If f is rational and g is algebraic, then $f \circledast g$ is algebraic.
(c) If f and g are holonomic (resp. regular holonomic with rational exponents), so is $f \circledast g$.
(d) If f and g are algebraic, then $f \circledast g$ is not necessarily algebraic.

For a proof, see Thm.7, Thm.8, Theorem E and the example of p.298 from [Jun31].

4.2. **Rational and algebraic functions in noncommuting variables.** In this section we discuss a generalization of the previous section to non-commuting variables. Let X be a finite set, and let X^* denote the free monoid on X. In other words, X consists of the set of all words in X, including the empty word e. Let $\mathbb{Q}\langle X \rangle$ (resp. $\mathbb{Q}\langle\langle X \rangle\rangle$) denote the algebra of polynomials (resp. formal power series) in non-commuting variables. In [Sch62], Schützenberger defines the notion of a *rational* and an *algebraic* power series in non-commuting variables. Let $\mathbb{Q}^{\mathrm{rat}}\langle X \rangle$ and $\mathbb{Q}^{\mathrm{alg}}\langle X \rangle$ denote the sets of rational (resp. algebraic) power series. Then, we have an inclusion:

$$(10) \qquad \mathbb{Q}^{\mathrm{rat}}\langle X \rangle \subset \mathbb{Q}^{\mathrm{alg}}\langle X \rangle \subset \mathbb{Q}\langle\langle X \rangle\rangle.$$

$\mathbb{Q}\langle\langle X \rangle\rangle$ has two multiplications:

- the usual multiplication of formal power series in non-commuting variables:

$$(11) \qquad (\sum_{w \in X^*} a_w w) \cdot (\sum_{w \in X^*} b_w w) = \sum_{w \in X^*} (\sum_{w',w'':w'w''=w} a_{w'} b_{w''}) w.$$

- The Hadamard product:

$$(12) \qquad (\sum_{w \in X^*} a_w w) \circledast (\sum_{w \in X^*} b_w w) = \sum_{w \in X^*} a_w b_w w.$$

With respect to the usual multiplication, $\mathbb{Q}\langle\langle X \rangle\rangle$ is a non-commutative algebra and $\mathbb{Q}^{\mathrm{rat}}\langle X \rangle$ and $\mathbb{Q}^{\mathrm{alg}}\langle X \rangle$ are subalgebras. We have the following analogue of Theorem 4.1.

Theorem 4.2. [Sch62, Pro.2.2] *(a) If $f \in \mathbb{Q}^{\mathrm{rat}}\langle X \rangle$ and $g \in \mathbb{Q}^{\mathrm{rat}}\langle X \rangle$, then $f \circledast g \in \mathbb{Q}^{\mathrm{rat}}\langle X \rangle$.*
(b) If $f \in \mathbb{Q}^{\mathrm{rat}}\langle X \rangle$ and $g \in \mathbb{Q}^{\mathrm{alg}}\langle X \rangle$, then $f \circledast g \in \mathbb{Q}^{\mathrm{alg}}\langle X \rangle$.

Remark 4.1. The notion of rational and algebraic functions works for an arbitrary ring $\mathcal{R}$ of characteristic zero, instead of $\mathbb{Q}$. Theorem 4.2 is still valid.

4.3. **Proof of Theorem 1.1.** Let F_r denote the free group of rank r with generating set $\{u_1, \ldots, u_r\}$, and

$$X = \{x_1, \ldots, x_r, \overline{x}_1, \ldots, \overline{x}_r\}.$$

Consider the monoid map:

(13) $$\pi : X^* \longrightarrow F_r, \qquad \pi(x_i) = u_i, \qquad \pi(\overline{x}_i) = u_i^{-1}.$$

The kernel $\mathrm{Ker}(\pi)$ of π is the set of those words in X which reduce to the identity under the relations $x_i \overline{x}_i = \overline{x}_i x_i = e$. Let

(14) $$\Delta = \sum_{w \in \mathrm{Ker}(\pi)} w \ \in \mathbb{Q}\langle\langle X \rangle\rangle.$$

The next proposition is attributed to Chomsky-Schützenberger by Haiman. For a proof, see [Hai93, Sec.3].

Proposition 4.2. [NM63] Δ *is algebraic.*

The map π has a right inverse (that satisfies $\pi \circ \iota = I_{F_r}$):

(15) $$\iota : F_r \longrightarrow X$$

defined by mapping a reduced word in u_i to a corresponding word in X. For every $f \in \mathbb{Q}[F_r]$ we have a key relation between trace and Hadamard product:

(16) $$\mathrm{Tr}(f) = \phi(\iota(f) \circledast \Delta)$$

where ϕ is a $\mathbb{Q}$-linear map defined by:

(17) $$\phi : \mathbb{Q}\langle X \rangle \longrightarrow \mathbb{Q}, \qquad \phi(w) = 1 \qquad \text{for } w \in X^*.$$

Now, fix $P \in M_N(\mathbb{Q}[F_r])$. Let Δ_N denote the N by N matrix with entries equal to Δ, and $\mathcal{R} = \mathbb{Q}(z)$. Let

$$P_z = z\iota(P) \in M_N(\mathcal{R}\langle X \rangle), \qquad P_z^* = \sum_{n=0}^{\infty} P_z^n \in M_N(\mathcal{R}\langle\langle X \rangle\rangle).$$

Notice that P_z^* is well-defined since P_z has no z-constant term.

Lemma 4.3. We have:

(18) $$P_z^* \in M_N(\mathcal{R}^{\mathrm{rat}}\langle X \rangle).$$

Proof. P_z^* satisfies the matrix equation

$$(1 - P_z)P_z^* = I$$

with entries in $\mathcal{R}\langle X \rangle$. $\square$

Lemma 4.3, together with Propositions 4.2 and part (b) of 4.2 imply the following result, which we can think as a noncommutative analogue of Theorem 1.1.

Proposition 4.4. *For every $P \in M_N(\mathbb{Q}[F_r])$, we have:*

$$(19) \qquad \sum_{n=0}^{\infty} z^n (\iota(P))^n \circledast \Delta_N \in M_N(\mathcal{R}^{\mathrm{alg}}\langle X \rangle).$$

Consider the abelianization ring homomorphism:

$$(20) \qquad \psi : \mathcal{R}\langle\langle X \rangle\rangle \longrightarrow \mathcal{R}[[X]]$$

where $\mathcal{R}[[X]]$ is the formal power series ring in commuting variables. Haiman proves the following:

Proposition 4.5. [Hai93, Prop.3.3] *If $f \in \mathcal{R}^{\mathrm{alg}}\langle X \rangle$, then $\psi(f)$ is algebraic over $\mathcal{R}(X)$.*

It follows that $\psi(P_z^* \circledast \Delta_N) \in M_N(\mathcal{R}^{\mathrm{alg}}(X))$. Consider now the subalgebra $\mathcal{R}^{\mathrm{conv}}[[X]]$ of $\mathcal{R}[[X]]$ that contains all elements of the form

$$\sum_{w \in X^*} a_w w$$

where $a_w \in z^{l(w)}\mathbb{Q}[[z]]$, where $l(w)$ denotes the length of w. Then, we can define an algebra map:

$$(21) \qquad \phi_z : \mathcal{R}^{\mathrm{conv}}[[X]] \longrightarrow \mathbb{Q}[[z]], \qquad \phi_z(w) = 1 \qquad \text{for } x \in X.$$

Haiman shows that if $f \in \mathcal{R}^{\mathrm{alg}}(X) \cap \mathcal{R}^{\mathrm{conv}}[[X]]$, then $\phi_z(f) \in \mathbb{Q}^{\mathrm{alg}}$. To state our final conclusion, we define for $1 \le i, j \le N$, the sequence $(a_{P,n}^{ij})$ by

$$a_{P,n}^{ij} = \mathrm{Tr}((P^n)_{ij})$$

and the matrix of generating series $A_P(z) \in M_N(\mathbb{Q}[[z]])$ by:

$$(A_P(z))_{ij} = \sum_{n=0}^{\infty} a_{P,n}^{ij} z^n.$$

Lemma 4.6. *We have:*

$$(22) \qquad (\phi_z \circ \psi)(P_z^* \circledast \Delta_N) = A_P(z).$$

Thus, $A_P(z) \in M_N(\mathbb{Q}_0^{\mathrm{alg}}(z))$.

Proof. Equation (22) follows from Equation (16). The conclusion follows from the above discussion. $\square$

Thus, the entries of $A_P(z)$ are algebraic functions, convergent at $z = 0$. Since by definition we have:

$$R_P(z) = \sum_{i=1}^{N} (A_P(z))_{ii}$$

it follows that $R_P(z) \in \mathbb{Q}_0^{\mathrm{alg}}(z)$. This completes the proof of Theorem 1.1. $\square$

5. Some Linguistics

5.1. **Regular and context-free languages.** Haiman's proof uses the key Proposition 4.2 from linguistics. Let us recall some concepts from this field. See for example [BR88, Lin97, Yan98] and references therein. Given a finite set X (the alphabet), a language L is a collection of words in X. In other words, $\mathcal{L} \subset X^*$. The *generating series* F_L of a language is:

$$F_{\mathcal{L}} = \sum_{w \in L} w \in \mathbb{Q}\langle\langle X \rangle\rangle.$$

It follows that for two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ we have:

$$F_{\mathcal{L}_1 \cap \mathcal{L}_2} = F_{\mathcal{L}_1} \circledast F_{\mathcal{L}_2}.$$

A language L is called *rational* (resp. *context-free*) iff $F_L \in \mathbb{Q}^{\mathrm{rat}}(X)$ (resp. $F_L \in \mathbb{Q}^{\mathrm{alg}}(X)$). In this context, Theorem 4.2 takes the following form:

Theorem 5.1. [NM63] *(a) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are rational languages, so is $\mathcal{L}_1 \cap \mathcal{L}_2$.*
(b) If $\mathcal{L}_1$ is rational and $\mathcal{L}_2$ is (unambiguous) context-free, then $\mathcal{L}_1 \cap \mathcal{L}_2$ is (unambiguous) context-free.

It was pointed out to us independently by D. Zeilberger and F. Flajolet that the above theorem essentially proves Theorem 1.1.

5.2. **Some questions.** Let us end this short paper with some questions. Despite the similarity in their statements and the multitude of proofs, Theorems 1.1 and 2.2 have different assumptions, different proofs and different conclusions.

Consider a generating set X for a group G such that every element of G can be written as a word in X with nonnegative exponents. Given X and G, let $\mathcal{L}_X$ denote the set of all words in X that map to the identity in G. Deciding membership in $\mathcal{L}_X$ is the *word problem* in G.

Definition 5.1. A group G has *context-free word problem* if it has a generating set X such that the language $\mathcal{L}_X$ is *context-free*.

The proof of Theorem 1.1 applies to groups with a context-free word problem. Muller-Schupp classified those groups. In [MS83] Muller-Schupp prove that G has context-free word problem iff G has a free finite-index subgroup.

On the other hand, if G is the fundamental group of a hyperbolic manifold of dimension not equal to 2, then G does not have a free finite-index subgroup.

Thus, the linguistics proof of Theorem 1.1 does not apply to the case of hyperbolic groups in dimension three. Neither does it apply to the case of $\mathbb{Z}^r$ since the latter does not have context-free word problem.

Question 5.2. If P is a hyperbolic group and $P \in M_N(\mathbb{Q}[G])$, is it true that $R_P(z)$ is a G-function?

The question may be relevant to low dimensional topology, when one tries to compute the ℓ^2-torsion of a hyperbolic manifold using Luecke's theorem; [Lüc94]. In that case, the matrix P comes from Fox (free differential) calculus of a presentation of the fundamental group G of the hyperbolic manifold. See also [DL09].

Question 5.3. Given $P \in M_N(\mathbb{Q}[F_r])$, consider the abelianization $P^{ab} \in M_N(\mathbb{Q}[\mathbb{Z}^r])$, and the G-functions $R_P(z)$ and $R_{P^{ab}}(z)$. How are the singularities of $R_P(z)$ and $R_{P^{ab}}(z)$ related?

Question 5.4. What is a holonomic function in non-commuting variables?

6. A FUNCTIONAL ANALYSIS INTERPRETATION OF THEOREM 1.1

The present paper is focusing on results and techniques inspired by algebra, non-commutative algebraic combinatorics. However it is worth mentioning that Theorem 1.1 has applications to problems coming from functional analysis, spectral theory, and the spectrum of Schrödinger operators. For instance, the Schrödinger equation describing the electron motion in a d-dimensional periodic crystal, can be well approximated by the difference equation on a lattice of same dimension. The corresponding operator can be seen as an element of the group ring of $\mathbb{Z}^d$. The function $R_P(z)$ defined previously is noting but the diagonal element of the resolvent and is used to compute the spectral measure, through the Charles de la Vallée Poussin theorem. There are instances for which, this operator is better approximated by the free group analog. For instance the *retracable path approximation* was used by Brinkman and Rice [BR71] in 1971 to treat the effect of spin-orbit coupling in the Hall effect, while it was used in [BFZ94] to compute the electronic Density of States when the electron is submitted to a random magnetic field. The same operator, seen as an element of the free group ring, is used to describe various infinite dimension approximations. The seminal work of Georges and Kotliar [GK92] used this free group approximation to give the first model known with a *Mott-Hubbard* transition.

Another domain in which the Theorem 1.1 may apply is the Voiculescu Theory of *Free Probability* [Voi05b, Voi05a]. The so-called R-*transform* used to treat the convolution of free random variables, is also based upon the Schur complement formula. In particular the free central limit theorem asserts that a sum of identically distributed free random variable obey the semicircle law, is a special case of the present result.

Besides the two proofs of Theorem 1.1 discussed in this paper, the algebraic character of $R_P(z)$ can also be deduced from the used of the Schur complement method [Sch18]. This is what makes the free group approximation so attractive to theoretical physicists. This method, also known under the name of *Feshbach method* [Fes58, Hernd, Fes67] is used in many domains of Physics, Quantum Chemistry, Solid State Physics, Nuclear Physics, to reduce the Hilbert space to a finite dimensional one and make the problem amenable to numerical calculations. However, very few Mathematical Physicists have paid attention to the fact that algebraicity or holonomy can give rise to results concerning the explicit computation of the spectral radius, or more generally, to the band edges, of the Hamiltonian they consider. This later problem is known to be notably hard with other methods.

For the benefit of the reader, we include some history of that method. The Schur complement method [Sch18] is widely used in numerical analysis under this name, while Mathematical Physicists prefer the reference to Feshbach [Fes58]. In Quantum Chemistry, the common reference is Feshbach-Fano [Fan35] or Feshbach-Löwdin [L62]. This method is used in various algorithms in Quantum Chemistry (*ab initio* calculations), in Solid State Physics (the muffin tin approximation, LMTO) as well as in Nuclear Physics. The formula used above is found in the original paper of Schur [Sch18, p.217].

The formula has been proposed also by an astronomer Tadeusz Banachiewicz in 1937, even though closely related results were obtained in 1923 by Hans Boltz and in 1933 by Ralf Rohan [PS05]. Applied to the Green function of a selfadjoint operator with finite rank perturbation, it becomes the Kreĭn formula [Kre46].

Let us end this section with a small dictionary that compares our notions with those in physics.

$H \in M_N(\mathbb{Q}[F_r])$	Hamiltonian
$1/(z - H)$	resolvant
$1/zR_H(1/z)$	trace of the resolvant
$\mathrm{Tr}(H^n)$	nth moment of H

References

[And00]   Yves André, *Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité*, Ann. of Math. (2) **151** (2000), no. 2, 705–740.

[Aom84]   Kazuhiko Aomoto, *Spectral theory on a free group and algebraic curves*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), no. 2, 297–318.

[BFZ94]   A. Barelli, R. Fleckinger, and T. Ziman, *Localization of two-dimensional electrons with uniform and random fluxes*, EPL (Europhysics Letters) **27** (1994), no. 7, 531.

[Bom81]   E. Bombieri, *On G-functions*, Recent progress in analytic number theory, Vol. 2 (Durham, 1979), Academic Press, London, 1981, pp. 1–67.

[Bor98]   E. Borel, *Sur les singularités des séries de Taylor*, Bull. Soc. Math. France **26** (1898), 238–248.

[BR71]   W.F. Brinkman and T.M. Rice, *Hall effect in the presence of strong spin-disorder scattering.*, Physical Review B (Solid State) **4** (1971), no. 5, 1566 – 1571.

[BR88]   Jean Berstel and Christophe Reutenauer, *Rational series and their languages*, EATCS Monographs on Theoretical Computer Science, vol. 12, Springer-Verlag, Berlin, 1988.

[CG11]   Ovidiu Costin and Stavros Garoufalidis, *Resurgence of the Kontsevich-Zagier series*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 1225–1258.

[CL02]   Antoine Chambert-Loir, *Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky)*, Astérisque (2002), no. 282, Exp. No. 886, viii, 175–209, Séminaire Bourbaki, Vol. 2000/2001.

[Com64]   L. Comtet, *Calcul pratique des coefficients de Taylor d'une fonction algébrique*, Enseignement Math. (2) **10** (1964), 267–270.

[CSTU02]   Olivier Cormier, Michael F. Singer, Barry M. Trager, and Felix Ulmer, *Linear differential operators for polynomial equations*, J. Symbolic Comput. **34** (2002), no. 5, 355–398.

[CV96]   Pierre-Alain Cherix and Alain Valette, *On spectra of simple random walks on one-relator groups*, Pacific J. Math. **175** (1996), no. 2, 417–438, With an appendix by Paul Jolissaint.

[Del70]   Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970.

[DGS94]   Bernard Dwork, Giovanni Gerotto, and Francis J. Sullivan, *An introduction to G-functions*, Annals of Mathematics Studies, vol. 133, Princeton University Press, Princeton, NJ, 1994.

[DL09]   Oliver T. Dasbach and Matilde N. Lalin, *Mahler measure under variations of the base group*, Forum Math. **21** (2009), no. 4, 621–637.

[DvdK98]   J. J. Duistermaat and Wilberd van der Kallen, *Constant terms in powers of a Laurent polynomial*, Indag. Math. (N.S.) **9** (1998), no. 2, 221–231.

[DvdP92]   Bernard M. Dwork and Alfred J. van der Poorten, *The Eisenstein constant*, Duke Math. J. **65** (1992), no. 1, 23–43.

[Fan35]   Ugo Fano, *Absorption spectrum of the noble gases near the limit of the arc spectrum*, Nuovo Cimento **12** (1935), 154–161.

[Fes58]   H. Feshbach, *Unified theory of nuclear reactions.*, Annals of Physics (New York) (U.S.) (1958), 357.

[Fes67]   ———, *The unified theory of nuclear reactions. iii. overlapping resonances.*, Ann. Phys. (N. Y.), 43: 410-20(July 1967) (1967), no. 3.

[Gar09]   Stavros Garoufalidis, *G-functions and multisum versus holonomic sequences*, Adv. Math. **220** (2009), no. 6, 1945–1955.

[GK92]   A.a Georges and G.b Kotliar, *Hubbard model in infinite dimensions*, Physical Review B **45** (1992), no. 12, 6479–6483 (English), cited By (since 1996)369.

[Hai93]   Mark Haiman, *Noncommutative rational power series and algebraic generating functions*, European J. Combin. **14** (1993), no. 4, 335–339.

[Hernd]   Feshbach Herman, *A unified theory of nuclear reactions. ii.*, Annals of Physics **19** (n.d.), 287 – 313.

[Jun31]   R. Jungen, *Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence*, Comment. Math. Helv. **3** (1931), no. 1, 266–306.

[Kat70]   Nicholas M. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 39, 175–232.

[KR13]   Christian Kassel and Christophe Reutenauer, *Algebraicity of the zeta function associated to a matrix over a free group algebra*, 2013, arXiv:1303.3481, Preprint.

[Kre46]   M. Krein, *Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m)*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **52** (1946), 651–654.

[Lal93]   Steven P. Lalley, *Finite range random walk on free groups and homogeneous trees*, Ann. Probab. **21** (1993), no. 4, 2087–2130.

[Leh99]   Franz Lehner, *Computing norms of free operators with matrix coefficients*, Amer. J. Math. **121** (1999), no. 3, 453–486.

[Leh01]   ———, *On the computation of spectra in free probability*, J. Funct. Anal. **183** (2001), no. 2, 451–471.

[Lin97]   Peter Linz, *An introduction to formal languages and automata / peter linz.*, Sudbury, Mass. : Jones and Bartlett Publishers, c1997., 1997.

[Lüc94]   Wolfgang Lück, L^2-*torsion and* 3-*manifolds*, Low-dimensional topology (Knoxville, TN, 1992), Conf. Proc. Lecture Notes Geom. Topology, III, Int. Press, Cambridge, MA, 1994, pp. 75–107.

[L62]   PerOlov Lwdin, *Studies in perturbation theory. iv. solution of eigenvalue problem by projection operator formalism*, Journal of Mathematical Physics **3** (1962), no. 5, 969–982.

[Mal74]   Bernard Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 405–430 (1975).

[MS83]   David E. Muller and Paul E. Schupp, *Groups, the theory of ends, and context-free languages*, J. Comput. System Sci. **26** (1983), no. 3, 295–310.

[Nil65]   Nils Nilsson, *Some growth and ramification properties of certain integrals on algebraic manifolds*, Ark. Mat. **5** (1965), 463–476 (1965).

[NM63]   Chomsky N. and Schtzenberger M.P., *The algebraic theory of context-free languages*.*, Studies in Logic and the Foundations of Mathematics **35** (1963), no. Computer Programming and Formal Systems, 118 – 161.

[PS05]   S. Puntanen and G. P. H. Styan, *Historical introduction: Issai schur and the early development of the schur complement.*, NUMERICAL METHODS AND ALGORITHMS **4** (2005), 1 – 16.

[Sau03]   Roman Sauer, *Power series over the group ring of a free group and applications to Novikov-Shubin invariants*, High-dimensional manifold topology, World Sci. Publ., River Edge, NJ, 2003, pp. 449–468.

[Sch18]   J. Schur, *ber potenzreihen, die im innern des einheitskreises beschrnkt sind.*, Journal fr die reine und angewandte Mathematik **1918** (1918), no. 148, 122.

[Sch62]   M. P. Schützenberger, *On a theorem of R. Jungen*, Proc. Amer. Math. Soc. **13** (1962), 885–890.

[Sie66]   Carl L. Siegel,
           " Uber einige Anwendungen diophantischer Approximationen (1929), Gesammelte Abhandlungen,
           Band I, 1966, pp. 209–266.
[Voi95]   Dan Voiculescu, *Operations on certain non-commutative operator-valued random variables*,
           Astérisque (1995), no. 232, 243–275, Recent advances in operator algebras (Orléans, 1992).
[Voi05a]   _____, *Aspects of free probability*, XIVth International Congress on Mathematical Physics, World
           Sci. Publ., Hackensack, NJ, 2005, pp. 145–157.
[Voi05b]   _____, *Free probability and the von Neumann algebras of free groups*, Rep. Math. Phys. **55**
           (2005), no. 1, 127–133.
[Was87]   Wolfgang Wasow, *Asymptotic expansions for ordinary differential equations*, Dover Publications
           Inc., New York, 1987, Reprint of the 1976 edition.
[Woe87]   Wolfgang Woess, *Context-free languages and random walks on groups*, Discrete Math. **67** (1987),
           no. 1, 81–87.
[Woe00]   _____, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol.
           138, Cambridge University Press, Cambridge, 2000.
[Yan98]   Song Y. Yan, *An introduction to formal languages and machine computation*, World Scientific
           Publishing Co. Inc., River Edge, NJ, 1998.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
http://www.math.gatech.edu/~stavros
   E-mail address: stavros@math.gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA,
http://www.math.gatech.edu/~jeanbel
   E-mail address: jeanbel@math.gatech.edu