A GENERATING SERIES FOR MURAKAMI-OHTSUKI-YAMADA GRAPH EVALUATIONS

STAVROS GAROUFALIDIS AND ROLAND VAN DER VEEN

Abstract. Murakami-Ohtsuki-Yamada introduced an evaluation of certain oriented planar trivalent graphs with colored edges. This evaluation plays a key role in the evaluation of the colored HOMFLY polynomial of a link in 3-space and its Khovanov-Rozansky categorification. Our goal is is to give a generating series formula for the evaluation of MOY graphs, which may be useful in categorification, and in the study of q-holonomicity of the colored HOMFLY polynomial.

Contents

1. Introduction

1.1. **The colored HOMFLY polynomial and its recursion.** The HOMFLY polynomial of a framed oriented link L in 3-space is a powerful link invariant which takes values in the ring

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$\mathbb{Z}[q^{1/2}, a^{1/2}, (q^{1/2}-q^{-1/2})^{-1}]$ and when specialized to $a=q^N$, it recovers the $\mathfrak{sl}_N$ invariant of the link, colored by the fundamental representation. The HOMFLY polynomial has a colored version $W_{L,\lambda}(q,a)$ which depends on a partition λ for each component of L [ML03, MM08]. Roughly, $W_{L,\lambda}(q,a)$ is the HOMFLY of a universal linear combination of cables of the link L, where each component colored by a partition λ is a cabled as many times as the number of boxes of λ [AM98]. When suitably normalized, the colored HOMFLY polynomial takes values in the ring $\mathbb{Z}[q^{1/2}, a^{1/2}]$.

In [MOY98], Murakami-Ohtsuki-Yamada gave a formula for the colored HOMFLY polynomial in terms of evaluations of some planar, trivalent, oriented colored graphs (in short, MOY graphs). A key property of a MOY graph and its evaluation is that it takes values in $\mathbb{N}[q^{1/2}]$. The non-negativity of the coefficients of those evaluations play an important role in categorification program developed by Khovanov-Rozansky [KR08].

The colored HOMFLY polynomial appears in physics literature in relation to the large N limit of $\mathrm{U}(N)$ Chern-Simons theory and its string dualities [LMV00]. Aganagic-Vafa conjectured that the colored HOMFLY polynomial of a knot, colored by the symmetric powers of the fundamental representation, satisfies a recursion relation with coefficients in $\mathbb{Z}[q^{1/2}, a^{1/2}]$ [AVa]. The operator form of such a recursion is a polynomial in four variables q, a, M and L where $LM = qML$ and all other variables commute. A further refinement of such an operator by adding a fifth variable t, related to the categorification of the colored Khovanov-Rozansky Homology has been proposed by Gukov et al [DGR06]. Several flavors of this so-called super-polynomial with fascinating properties have recently been conjectured in the physics literature. For a survey article that summarizes recent developments, see [GS] and [AVb].

On the mathematics side, it was observed by the first author that q-holonomicity of the colored HOMFLY polynomial (thought of as a function of a partition with a fixed number of rows) follows from q-holonomicity of the evaluations of the MOY graphs (thought of as a function of their colors) [Garb]. This observation was our primary motivation to study evaluations of MOY graphs using generating series, much in the spirit of spin networks and their evaluations [GvdV13]. In a future publication, we will apply our results to deduce the q-holonomicity of the MOY graph evaluations.

1.2. **A generating series for the classical evaluation of MOY graphs.** A MOY *graph* Γ is a planar trivalent graph Γ with oriented edges, without sinks or sources. It may contain multiple edges and loops, as well as components with no edges. A *coloring* γ of a MOY graph Γ is a *flow* $\gamma : E(\Gamma) \to \mathbb{N}$, i.e., an assignment of a natural number to each edge such that at each vertex, the sum of the numbers of the incoming edges equal to the sum of the numbers of the outgoing edges. An example of a MOY graph and its coloring is shown in Figure 1.

For a positive natural number N, Murakami-Ohtsuki-Yamada [MOY98] define the evaluation $\langle \Gamma, \gamma \rangle_N(q) \in \mathbb{N}[q^{\pm 1/2}]$. Consider the *classical evaluation* $\langle \Gamma, \gamma \rangle_N(1) \in \mathbb{N}$ and its generating series

$$F^{\mathrm{class}}_{\Gamma,N}(w) = \sum_{\gamma} \langle \Gamma, \gamma \rangle_N(1) w^{\gamma} \in \mathbb{N}[w]\,,$$

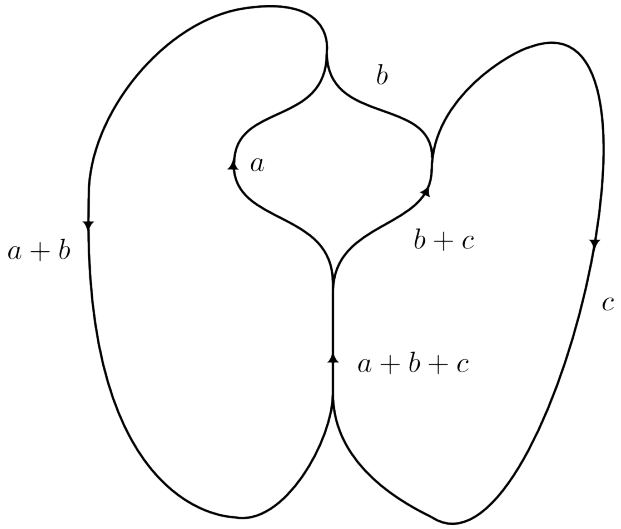

Figure 1. A MOY graph and its coloring by arbitrary natural numbers a, b, c.

where $w = (w_e)_{e\in E(\Gamma)}$, and all variables commute. As usual, if $\gamma : E(\Gamma) \to \mathbb{N}$, we denote $w^\gamma = \prod_{\gamma\in E(\Gamma)} w_e^{\gamma(e)}$. The classical evaluation has been studied by Lobb-Zentner and Grant [LZ, Gra] in relation to moduli space of $\mathfrak{sl}_N$ representations of the complements of MOY graphs.

A *cycle* of Γ is a 2-regular subgraph of Γ such that each component has a consistent orientation. Let $\mathcal{C}(\Gamma)$ denote the set of cycles of Γ. The *classical cycle polynomial* is given by

$$P_\Gamma^{\mathrm{class}}(w) = \sum_{C\in\mathcal{C}(\Gamma)} w^C \in \mathbb{N}[w]\,.$$

Our first result identifies the generating series $F_{\Gamma,N}^{\mathrm{class}}$ with the N-th power of the classical cycle polynomial.

Theorem 1.1. *We have:*

$$F_{\Gamma,N}^{\mathrm{class}}(w) = \left(P_\Gamma^{\mathrm{class}}(w)\right)^N\,.$$

1.3. **A generating series for the $\mathfrak{sl}_N$ evaluation of MOY graphs.** To extend Theorem 1.1 to MOY graph evaluations, we need to introduce the corresponding generating series and the cycle polynomial. These are series in sets of q-commuting variables (z, Z) (for the generating series) and x (for the cycle polynomial).

To each vertex v of a MOY graph Γ, we denote the three adjacent half-edges (i.e., flags) by (v, l), (v, m) and (v, r) with the convention of Figure 2. We also assign six ordered variables to v:

$$z_{v,l} < z_{v,m} < z_{v,r} < Z_{v,r} < Z_{v,m} < Z_{v,l} \tag{1}$$

which commute except in the following instance

$$z_{v,r}z_{v,l} = q^{-\frac{1}{4}} z_{v,l}z_{v,r}, \qquad Z_{v,r}Z_{v,l} = q^{-\frac{1}{4}} Z_{v,l}Z_{v,r} \tag{2}$$

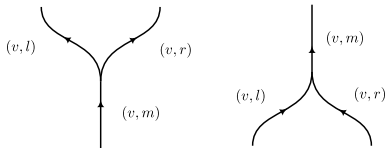

Figure 2. A MOY graph and its coloring for arbitrary natural numbers a,b,c.

Fix a total ordering $<$ of the set of vertices $V(\Gamma)$ of Γ. Together with (1), this gives a total ordering of the variables z, Z where if $v < w$ then $z_{v,s}, Z_{v,s'} < z_{w,t}, Z_{w,t'}$ for all $s, s', t, t', \in \{l, m, r\}$.

Likewise, we consider a set of q-commuting variables $x = (x_C)_{C \in \mathcal{C}(\Gamma)}$, one for each cycle of Γ. The commutation relations for the cycle variables are expressed in terms of the following intersection product (skew-symmetric form) on the set $\mathcal{C}(\Gamma)$.

$$\langle C, C' \rangle = \frac{1}{2} \left(\#\{v \in V(\Gamma) | (v,l) \in C, (v,r) \in C'\} - \#\{v \in V(\Gamma) | (v,r) \in C, (v,l) \in C'\} \right)$$

In terms of this product we define

$$x_C x_{C'} = q^{\langle C, C' \rangle} x_{C'} x_C \,.$$

There is a well-defined monomial homomorphism map

$$\mu(x_C) = z^C Z^C \,. \tag{3}$$

which satisfies $\mu(x_C x_{C'}) = \mu(x_C)\mu(x_{C'})$.

The *generating series* of the MOY evaluations of Γ is defined by

$$F_{\Gamma,N}(q, z, Z) = \sum_{\gamma} \langle \Gamma, \gamma \rangle_N(q) z^\gamma Z^\gamma \in \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle\langle z, Z \rangle\rangle \,,$$

Here the monomials z^γ, Z^γ are understood to be in their standard ordering.

The *cycle polynomial* of Γ is defined in terms of the *rotation number* $\mathrm{rot}(C)$ of a cycle C. If C is connected and oriented counter-clockwise then $\mathrm{rot}(C) = 1$, if it is oriented clock-wise, then $\mathrm{rot}(C) = -1$. For a general cycle C, $\mathrm{rot}(C)$ is the sum of the rotation numbers of its connected components. The cycle polynomial is then

$$P_\Gamma(q, a, x) = \sum_{C \in \mathcal{C}(\Gamma)} (a^{-\frac{1}{2}} q^{\frac{1}{2}})^{\mathrm{rot}(C)} x_C \in \mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}]\langle x \rangle \,. \tag{4}$$

Finally, we need to introduce a q-version of the N-th power appearing in Theorem 1.1. In analogy with the q-Pochhammer symbol, we define

$$(P_\Gamma(q, a, x), q)_N = \prod_{k=0}^{N-1} P_\Gamma(q, a, q^{k\mathrm{rot}(C)} x_C) \in \mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}]\langle x \rangle \,.$$

We can now state our theorem.

Theorem 1.2. *For every MOY graph Γ and natural number N we have:*

$$F_{\Gamma,N}(q, z, Z) = \mu\left((P_\Gamma(q, q^N, x), q)_N \right) \,.$$

Since $\mu\left((P_\Gamma(1,1,x),1)_N\right) = (P_\Gamma^{\text{class}}(zZ))^N$, Theorem 1.1 follows from Theorem 1.2 when $q=1$. By $P_\Gamma^{\text{class}}(zZ)$ we mean the classical cycle polynomial where we set $w_e = z_{h_1} z_{h_2} Z_{h_1} Z_{h_2}$ if edge e is the union of the half-edges h_1, h_2.

1.4. **A generating series for the HOMFLY evaluation of MOY graphs.** In [Garb, Lem.2.2] it was shown that given a MOY graph (Γ,γ) there exists $\langle\Gamma,\gamma\rangle(q,a) \in \mathbb{Q}(q^{1/2},a^{1/2})$ such that for every positive natural number N, we have:

$$\langle\Gamma,\gamma\rangle(q,q^N) = \langle\Gamma,\gamma\rangle_N(q)\,.$$

Consider the generating series

$$F_\Gamma(q,a,z,Z) = \sum_\gamma \langle\Gamma,\gamma\rangle(q,a) z^\gamma Z^\gamma$$

and the rings

$$\mathcal{R} = \mathbb{Z}[[q^{\frac{1}{2}}]][a^{-\frac{1}{2}},a^{\frac{1}{2}}] \qquad \mathcal{R}_+ = \mathbb{Z}[[q^{\frac{1}{2}}]][a^{+\frac{1}{2}}] \qquad \mathcal{R}_- = \mathbb{Z}[[q^{\frac{1}{2}}]][a^{-\frac{1}{2}}]\,.$$

We say that a MOY graph Γ is *positive* if the rotation number of every nonempty cycle is positive. In that case, $F_\Gamma(q,a,z,Z) \in \mathcal{R}\langle\langle z,Z\rangle\rangle$.

Theorem 1.3. *Assume that Γ is positive. Then we have:*

(5)
$$F_\Gamma(q,a,z,Z)\mu\left((P_\Gamma(q,a^{-1},x),q)_\infty\right) = \mu\left((P_\Gamma(q,a,x),q)_\infty\right)$$

(6)
$$F_\Gamma(q,q^2a,z,Z) = \mu\left(P_\Gamma(q^{-1},a,x)_\infty\right) F_\Gamma(q,a,z,Z)\mu\left(P_\Gamma(q,a^{-1},x)_\infty\right)$$

where

$$(P_\Gamma(q,a,x),q)_\infty \in \mathcal{R}_-\langle\langle x\rangle\rangle \qquad \mu\left((P_\Gamma(q,a,x),q)_\infty\right) \in \mathcal{R}\langle\langle z,Z\rangle\rangle$$

Remark 1.4. Equation (5) is reminiscent to the 3D index of a tetrahedron introduced by Dimofte-Gaiotto-Gukov [DGG] and further studied by [Gara, Eqn.B.1] and [GHRS].

Remark 1.5. Although non-positive MOY graphs exist (see for instance the example in Section 5.2), the colored HOMFLY polynomial of a link L is a linear combination, with q-proper hypergeometric coefficients, of the evaluation of positive MOY graphs. Indeed, choose a braid β whose closure is L and the closure is chosen so that all strands rotate counter-clockwise. Then, Equation on p.341 of [MOY98] replaces each crossing with a linear combination of positive MOY graphs.

2. MOY graphs and their evaluation

In this section we recall the evaluation of a MOY graph given by [MOY98].

2.1. **States.** For a MOY graph Γ, let $V(\Gamma)$, $E(\Gamma)$, $H(\Gamma)$ and $\mathcal{C}(\Gamma)$ denote its set of vertices, edges, half-edges and cycles. For a fixed positive integer N, define the N element set

$$A_N = \{-\frac{N-1}{2}, \dots, \frac{N-3}{2}, \frac{N-1}{2}\}$$

A *state* is a function $\sigma : \mathcal{C}(\Gamma) \to 2^{A_N}$ where 2^X denotes the set of subsets of X, with the additional requirement that if C and C' both contain the same edge then $\sigma(C) \cap \sigma(C') = \emptyset$. For all states we require $\sigma(\emptyset) = \emptyset$. A state σ gives rise to functions

$$\sigma : H(\Gamma) \to 2^{A_N}, \qquad \sigma : E(\Gamma) \to 2^{A_N}$$

defined by $\sigma(h) = \bigcup_{C:h\in C} \sigma(C)$ and $\sigma(e) = \bigcup_{C:e\in C} \sigma(C)$. σ induces a flow $|\sigma|$ on the graph Γ defined by $|\sigma|(e) = |\sigma(e)|$ for $e \in E(\Gamma)$. For a cycle C we define $|\sigma|(C) = |\sigma(C)|$. Finally, given a state σ define

$$\mathrm{rot}(\sigma) = \sum_{C\in\mathcal{C}(\Gamma)} \mathrm{rot}(C) \sum_{x\in\sigma(C)} x$$

2.2. **Definition of the MOY evaluation.** The MOY invariant of (Γ, γ), denoted by $\langle \Gamma, \gamma\rangle_N(q)$ is given by

$$\langle \Gamma, \gamma\rangle_N(q) = \sum_{\sigma:|\sigma|=\gamma} q^{\mathrm{rot}(\sigma)} \prod_{v\in V(\Gamma)} \mathrm{wt}(v;\sigma)\,.$$

Here define the *weight* by $\mathrm{wt}(v;\sigma) = q^{\frac{1}{4}(R(v;\sigma)-L(v;\sigma))}$. In this formula

$$\begin{aligned} L(v;\sigma) =& |\{(a,b) \in \sigma(v,l) \times \sigma(v,r) | a > b\}| \\ R(v;\sigma) =& |\{(a,b) \in \sigma(v,l) \times \sigma(v,r) | a < b\}| \end{aligned}$$

Note that in their original paper Murakami, Ohtsuki and Yamada worked with slightly different definitions: their concept of a state was tied to edges instead of cycles, but the two definitions are equivalent. Also their vertex weights were introduced as $\mathrm{wt}(v;\sigma) = q^{\frac{|\sigma(v,l)||\sigma(v,r)|-2L(v;\sigma)}{4}}$ which coincides with our definition above.

3. Proofs

In this section we present the proofs of theorems 1.2 and 1.3. As mentioned in the introduction theorem 1.1 follows directly from Theorem 1.2 by setting $q = 1$.

3.1. **Proof of Theorem 1.2.** We start with the product $\mu\left((P_\Gamma(q, q^N, x), q)_N\right)$ on the right hand side of the equation and show that after applying the μ map and ordering the variables we get the generating function $F_{\Gamma,N}(q, z, Z)$.

First we rewrite the product in a more symmetric fashion as follows:

$$(P_\Gamma(q, q^N, x), q)_N = \prod_{j\in A_N} \left(\sum_{C\in\mathcal{C}(\Gamma)} q^{j\,\mathrm{rot}(C)} x_C \right)$$

Next we need to recognize that the monomials in the expansion of the latter product are in bijection with the states σ. Denote by m_σ the monomial corresponding to state σ. It is defined as

$$m_\sigma = \prod_{j\in A_N} \prod_{C|j\in\sigma(C)} x_C \,.$$

Conversely any monomial in the expanded product looks like $\prod_{j\in A_N} x_{C_j}$. This monomial corresponds to the state σ defined by $\sigma(C) = \{j : C_j = C\}$. Summarizing we can say that the j-th factor in the product corresponds to the choice of which cycle to label by $j \in A_N$ in creating a state.

From the formula $\mathrm{rot}(\sigma) = \sum_{j\in A_N} \sum_{C:j\in\sigma(C)} \mathrm{rot}(C)$ it then follows that

$$(P_\Gamma(q, q^N, x), q)_N = \sum_\sigma q^{\mathrm{rot}(\sigma)} m_\sigma \,.$$

Our next task is to apply the monomial map μ and bring the monomials $\mu(m_\sigma)$ into the canonical order $<$ of the z, Z variables. We claim that the necessary q-commutations produce exactly coefficient

$$\prod_{v\in V(\Gamma)} \mathrm{wt}(v;\sigma) = q^{\frac{1}{4}\sum_v (R(v;\sigma)-L(v;\sigma))}$$

The $R(v;\sigma)$ terms come from applying $Z_{v,l}Z_{v,r} = q^{\frac{1}{4}} Z_{v,r}Z_{v,l}$ and the $L(v;\sigma)$ terms come from applying $z_{v,r}z_{v,l}q^{-\frac{1}{4}} z_{v,l}z_{v,r}$. The claim now follows from the bijection between the states σ and the monomials m_σ because it shows that the following situations (a) and (b) are equivalent:

(a) We have a pair of elements $(j_L, j_R) \in \sigma(C_l) \times \sigma(C_r)$ and a pair of cycles $C_l, C_r \in \mathcal{C}(\Gamma)$ such that the half-edge $(v, l) \in C_l$ and $(v, r) \in C_r$.
(b) The monomial m_σ contains the factor x_{C_l} in the j_l-th place and x_{C_r} in the j_r-th place. Moreover μ_{C_l} includes $z_{v,l}Z_{v,l}$ and μ_{C_r} includes $z_{v,r}Z_{v,r}$.

More concretely, in the graph part (a) the pair (j_l, j_r) contributes 1 to $L(v;\sigma)$ if $j_l > j_r$ and 1 to $R(v;\sigma)$ otherwise. In the monomial part (b) the case $j_l > j_r$ means that z_r comes before z_l in the product so to bring it into canonical order we need to commute the two and pick up a term $q^{-\frac{1}{4}}$. The case $j_l < j_r$ means we need to commute the upper case variables only and pick up a term $q^{\frac{1}{4}}$. To summarize we have now shown that

$$\mu(m_\sigma) = \prod_{v\in V(\Gamma)} \mathrm{wt}(v;\sigma) z^{|\sigma|} Z^{|\sigma|}$$

where the latter monomials are in canonical order. Therefore

$$\mu\left((P_\Gamma(q, q^N, x), q)_N\right) = \mu\left(\sum_\sigma q^{\mathrm{rot}(\sigma)} m_\sigma\right) = \sum_\sigma q^{\mathrm{rot}(\sigma)} \prod_{v\in V(\Gamma)} \mathrm{wt}(v;\sigma) z^{|\sigma|} Z^{|\sigma|}$$

$$= \sum_\gamma \sum_{\sigma:|\sigma|=\gamma} q^{\mathrm{rot}(\sigma)} \prod_{v\in V(\Gamma)} \mathrm{wt}(v;\sigma) z^\gamma Z^\gamma = F_{\Gamma,N}(q, z, Z)$$

which concludes the proof of Theorem 1.2. □

4. Proof of theorem 1.3

For part (a) we set $a = q^N$, multiply both sides of the equality in Theorem 1.2 from the right by $\mu(P_\Gamma(q, a, q^{N\mathrm{rot}}x), q)_\infty$ to obtain

$$F_\Gamma(q, a, z, Z)\mu\left((P_\Gamma(q, a, q^{N\mathrm{rot}}x), q)_\infty\right) = \mu\left((P_\Gamma(q, a, x), q)_\infty\right)$$

Since

$$P_\Gamma(q, a, q^{N\mathrm{rot}}x) = \sum_{c\in C}(a^{\frac{1}{2}}q^{\frac{1}{2}})^{\mathrm{rot}(c)}x_c = P_\Gamma(q, a^{-1}, x)$$

we see that

$$(P_\Gamma(q, a, q^{N\mathrm{rot}}x), q)_\infty = (P_\Gamma(q, a^{-1}, x), q)_\infty$$

We have now proven that for all N and $a = q^N$:

$$F_\Gamma(q, a, z, Z)\mu\left((P_\Gamma(q, a^{-1}, x), q)_\infty\right) = \mu\left((P_\Gamma(q, a, x), q)_\infty\right)$$

Therefore the equality holds for all a. This completes the proof of the first equality in the theorem.

For part (b) we start by writing down the statement of part (a):

$$F_\Gamma(q, a, z, Z)\mu\left((P_\Gamma(q, a^{-1}, x), q)_\infty\right) = \mu\left((P_\Gamma(q, a, x), q)_\infty\right) \tag{7}$$

Replacing a by q^2a yields

$$F_\Gamma(q, q^2a, z, Z)\mu\left((P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty\right) = \mu\left((P_\Gamma(q, q^2a, x), q)_\infty\right) \tag{8}$$

This can be simplified since $P_\Gamma(q, q^2a, x) = P_\Gamma(q, a, q^{-\mathrm{rot}}x)$ and $P_\Gamma(q, q^{-2}a^{-1}, x) = P_\Gamma(q, a^{-1}, q^{\mathrm{rot}}x)$ So

$$(P_\Gamma(q, q^2a, x), q)_\infty = P_\Gamma(q^{-1}, a, x)(P_\Gamma(q, a, x), q)_\infty \tag{9}$$

And

$$P_\Gamma(q, a^{-1}, x)(P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty = (P_\Gamma(q, a^{-1}, x), q)_\infty \tag{10}$$

Substituting equation (9) into equation (8) we get:

$$F_\Gamma(q, q^2a, z, Z)\mu\left((P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty\right) = \mu\left(P_\Gamma(q^{-1}, a, x)\right)\mu\left((P_\Gamma(q, a, x), q)_\infty\right) \tag{11}$$

Multiplying equation (7) from the left by $\mu\left(P_\Gamma(q^{-1}, a, x)\right)$ and substituting Equation (10) gives:

$$\begin{aligned}\mu\left(P_\Gamma(q^{-1}, a, x)\right)F_\Gamma(q, a, z, Z)\mu\left(P_\Gamma(q, a^{-1}, x)\right)\mu\left((P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty\right)\\ = \mu\left(P_\Gamma(q^{-1}, a, x)\right)\mu\left((P_\Gamma(q, a, x), q)_\infty\right)\end{aligned} \tag{12}$$

Combining Equations (11) and (12) gives:

$$\begin{aligned}F_\Gamma(q, q^2a, z, Z)\mu\left((P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty\right)\\ = \mu\left(P_\Gamma(q^{-1}, a, x)\right)F_\Gamma(q, a, z, Z)\mu\left(P_\Gamma(q, a^{-1}, x)(P_\Gamma(q, q^{-2}a^{-1}, x), q)_\infty\right)\end{aligned} \tag{13}$$

Finally multiplying by the inverse of $\mu(P_\Gamma(q,q^{-2}a^{-1},x),q)_\infty$ from the right we find:

$$F_\Gamma(q,q^2a,z,Z) = \mu\left(P_\Gamma(q^{-1},a,x)\right) F_\Gamma(q,a,z,Z)\mu\left(P_\Gamma(q,a^{-1},x)\right)$$

This concludes the proof of Theorem 1.3. □

5. Examples

5.1. **Unknot.** For the unknot our theorems specialize to the binomial theorem and its q-generalization.

Let the unknot O be oriented counter-clockwise. We have no vertices and one single edge. This means we have exactly two cycles: the empty cycle and the cycle C that is the whole unknot.

Therefore $P_O^{class} = 1 + w_C$ and so Theorem 1.1 states that

$$\sum_\gamma \langle O,\gamma\rangle w^\gamma = F_{O,N}^{class}(w) = (P_O^{class}(w))^N = (1+w_C)^N$$

which is consistent with the binomial theorem and the classical evaluation

$$\langle O,\gamma\rangle_N(1) = \binom{N}{\gamma}$$

Next, according to [MOY98] the quantum evaluation is

$$\langle O,\gamma\rangle_N(q) = \begin{bmatrix} N \\ \gamma \end{bmatrix}$$

where we are using the symmetric quantum binomial defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!} \quad [n]! = \prod_{j=1}^{n} \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

Theorem 1.2 now becomes a q-analogue of the binomial Theorem. First note that the empty set has rotation number 0 an the cycle C rotation number 1, whose cycle variable we call x_C. The empty set has cycle variable 1. Since there is only one single half-edge and no vertex we will use the commuting variables z and Z for it, so $\mu(x_C) = zZ$. In this notation we have

$$P_O = 1 + a^{-\frac{1}{2}} q^{\frac{1}{2}} x_C$$

Theorem 1.2 now states that the generating series of evaluations

$$F_{O,N}(q,z,Z) = \sum_\gamma \langle O,\gamma\rangle(q) z^\gamma Z^\gamma = \sum_{\gamma\in\mathbb{N}} \begin{bmatrix} N \\ \gamma \end{bmatrix} (zZ)^\gamma$$

equals the following Pochhammer product:

$$\mu(P_O(q,q^N,x),q)_N = \mu \prod_{k=0}^{N-1} (1 + q^{\frac{1-N}{2}+k} x) = (-q^{\frac{1-N}{2}} zZ, q)_N$$

The unknot is a positive so Theorem 1.3 applies, where $(P_O(q,a,x),q)_\infty = (-q^{\frac{1}{2}}a^{-\frac{1}{2}}x,q)_\infty$ In addition, all variables commute so we may write the theorem as

$$F_O(q,a,z,Z) = \mu\left(\frac{(P_O(q,a,x),q)_\infty}{(P_O(q,a^{-\frac{1}{2}},x),q)_\infty}\right) = \frac{(-q^{\frac{1}{2}}a^{-\frac{1}{2}}zZ,q)_\infty}{(-q^{\frac{1}{2}}a^{\frac{1}{2}}zZ,q)_\infty}\,.$$

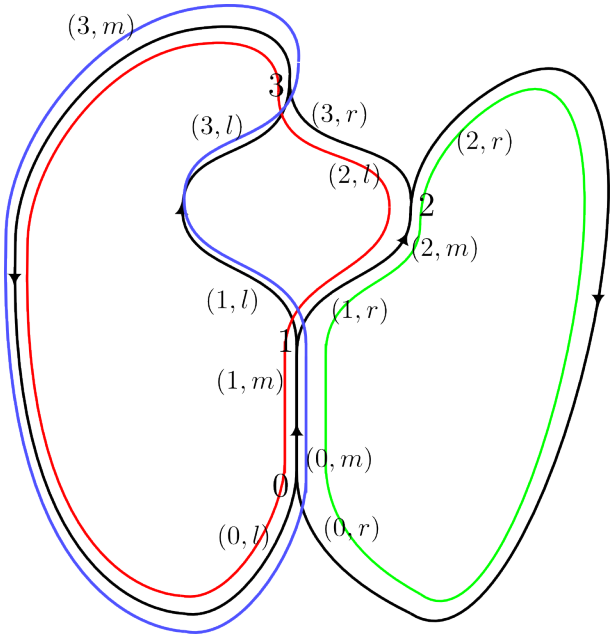

Figure 3. The tetrahedron graph where we have indicated the names of the vertices and half edges. The three non-empty cycles are indicated in red, green and blue.

5.2. **Tetrahedron.** The tetrahedron graph Γ shown in Figure 3 has four cycles, $\emptyset$ and three non-empty cycles C_r C_g and C_b with classical cycle variables w_r w_g and w_b. If $\langle \Gamma, a, b, c\rangle_N(q)$ is the evaluation of the tetrahedron graph γ labeled with natural numbers a, b, c as indicated in Figure 1 then Theorem 1.1 states:

$$\sum_{a,b,c\in\mathbb{N}} \langle \Gamma, a, b, c\rangle(1) w_r^b w_g^c w_b^a = (1 + w_r + w_g + w_b)^N$$

in accordance to the following direct evaluation, see [MOY98] and the multinomial theorem.

$$\langle \Gamma, a, b, c\rangle_N(1) = \binom{N}{a, b, c, N-a-b-c}$$

The q-analogue of this formula is:

$$\langle \Gamma, a, b, c\rangle_N(q) = \begin{bmatrix} N \\ a, b, c, N-a-b-c \end{bmatrix}$$

To see how Theorem 1.2 fits these numbers in a generating function, first note that the rotation numbers of the cycles are $\mathrm{rot}(C_r) = \mathrm{rot}(C_b) = 1$ and $\mathrm{rot}(C_g) = -1$. The corresponding cycle variables x_r, x_g and x_b can be expressed by the μ map as follows (with the the variables in the canonical order):

$$\begin{aligned}
\mu(x_\emptyset) &= 1\\
\mu(x_r) &= z_{0,l}z_{0,m}z_{1,m}z_{1,r}z_{2,l}z_{2,m}z_{3,r}z_{3,m}Z_{3,M}Z_{3,R}Z_{2,M}Z_{2,L}Z_{1,R}Z_{1,M}Z_{0,M}Z_{0,L}\\
\mu(x_g) &= z_{0,l}z_{0,m}z_{1,l}z_{1,m}z_{3,l}z_{3,m}Z_{3,M}Z_{3,L}Z_{1,M}Z_{1,L}Z_{0,M}Z_{0,L}\\
\mu(x_b) &= z_{0,m}z_{0,r}z_{1,m}z_{1,r}z_{2,m}z_{2,r}Z_{2,R}Z_{2,M}Z_{1,R}Z_{1,M}Z_{0,R}Z_{0,M}
\end{aligned}$$

The intersection numbers of the cycles are: $\langle x_r, x_g\rangle = 1$ $\langle x_r, x_b\rangle = -1$ $\langle x_g, x_b\rangle = -1$. From this it follows that $x_r x_g = q x_g x_r$, $x_r x_b = q^{-1} x_b x_r$ and $x_g x_b = q^{-1} x_b x_g$. This may also be checked to follow from applying the μ map and the commutation relations for the half-edge variables.

Next

$$P_\Gamma(q, a, x) = 1 + a^{-\frac{1}{2}} q^{\frac{1}{2}} x_r + a^{\frac{1}{2}} q^{-\frac{1}{2}} x_g + a^{-\frac{1}{2}} q^{\frac{1}{2}} x_b$$

The generating function is equal to $F_{\Gamma,N}(q, z, Z) =$

$$\sum_{a,b,c} \langle \Gamma, a, b, c\rangle_N(q) z_{0,l}^{a+b} z_{0,m}^{a+b+c} z_{0,r}^{c} z_{1,l}^{a} z_{1,m}^{a+b+c} z_{1,r}^{b+c} z_{2,l}^{b} z_{2,m}^{b+c} z_{2,r}^{c} z_{3,l}^{a} z_{3,m}^{a+b} z_{3,r}^{b} \times$$

$$Z_{3,r}^{b} Z_{3,m}^{a+b} Z_{3,l}^{a} Z_{2,r}^{c} Z_{2,m}^{b+c} Z_{2,l}^{b} Z_{1,r}^{b+c} Z_{1,m}^{a+b+c} Z_{1,l}^{a} Z_{0,r}^{c} Z_{0,m}^{a+b+c} Z_{0,l}^{a+b}$$

By Theorem 1.2 this equals

$$\mu(P_\Gamma(q, q^N, x), q)_N = \mu \prod_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} (1 + q^j x_r + q^{-j} x_g + q^j x_b)$$

The tetrahedron graph Γ in our example is not positive, $\mathrm{rot}(C_g) = -1$, so Theorem 1.3 does not apply. However the green cycle may be turned into a positive cycle by moving its ends to the left.

For several reasons this example may still be too simple in that there are no relations between the cycles. For more complicated graphs, monomials can usually be written as a product of cycles in multiple ways. Also a simple closed form evaluation in terms of q-binomials is generally not to be expected.

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References

[AM98] A. K. Aiston and H. R. Morton, *Idempotents of Hecke algebras of type A*, J. Knot Theory Ramifications **7** (1998), no. 4, 463–487.

[AVa] Mina Aganagic and Cumrun Vafa, *Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots*, arXiv:1204.4709, Preprint 2004.

[AVb] ———, *Topological Strings, D-Model, and Knot Contact Homology*, arXiv:1304.5778, Preprint 2013.

[DGG] Tudor Dimofte, Davide Gaiotto, and Sergei Gukov, *3-manifolds and 3d indices*, arXiv:1112.5179, Preprint 2011.

[DGR06] Nathan M. Dunfield, Sergei Gukov, and Jacob Rasmussen, *The superpolynomial for knot homologies*, Experiment. Math. **15** (2006), no. 2, 129–159.

[Gara] Stavros Garoufalidis, *The 3D index of an ideal triangulation and angle structures*, arXiv:1208.1663, Preprint 2012.

[Garb] ———, *The colored HOMFLY polynomial is q-holonomic*, arXiv:1211.6388, Preprint 2012.

[GHRS] Stavros Garoufalidis, Craig D. Hodgson, Hyam Rubinstein, and Henry Segerman, 1-*efficient triangulations and the index of a cusped hyperbolic 3-manifold*, arXiv:1303.5278, Preprint 2013.

[Gra] Jonathan Grant, *The moduli problem of Lobb and Zentner and the coloured* $\mathfrak{sl}(N)$ *graph invariant*, arXiv:1212.4511, Preprint 2012.

[GS] Sergei Gukov and Ingmar Saberi, *Lectures on knot homology and quantum curves*, arXiv:1211.6075, Preprint 2012.

[GvdV13] Stavros Garoufalidis and Roland van der Veen, *Asymptotics of classical spin networks*, Geom. Topol. **17** (2013), no. 1, 1–37, With an appendix by Don Zagier.

[KR08] Mikhail Khovanov and Lev Rozansky, *Matrix factorizations and link homology*, Fund. Math. **199** (2008), no. 1, 1–91.

[LMV00] José M. F. Labastida, Marcos Mariño, and Cumrun Vafa, *Knots, links and branes at large N*, J. High Energy Phys. (2000), no. 11, Paper 7, 42.

[LZ] Andrew Lobb and Raphael Zentner, *The quantum* $\mathfrak{sl}(n)$ *graph invariant and a moduli space*, arXiv:1204.5372, Preprint 2012.

[ML03] Hugh R. Morton and Sascha G. Lukac, *The Homfly polynomial of the decorated Hopf link*, J. Knot Theory Ramifications **12** (2003), no. 3, 395–416.

[MM08] H. R. Morton and P. M. G. Manchón, *Geometrical relations and plethysms in the Homfly skein of the annulus*, J. Lond. Math. Soc. (2) **78** (2008), no. 2, 305–328.

[MOY98] Hitoshi Murakami, Tomotada Ohtsuki, and Shuji Yamada, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. (2) **44** (1998), no. 3-4, 325–360.

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